

Time-delayed conjugate coupling in dynamical systems

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Abstract. We study the effect of time-delay when the coupling between nonlinear systems is “conjugate”, namely through dissimilar variables. This form of coupling can induce anomalous transitions such as the emergence of oscillatory dynamics between regimes of amplitude death and oscillation death. The specific cases of coupled Landau-Stuart oscillators as well as a predator-prey model system with cross-predation are discussed. The dynamical behaviour is analyzed numerically and the regions corresponding to different asymptotic states are identified in parameter space.

1 Introduction

Natural systems generally interact over a wide range of time-scales and with a variety of coupling scenarios. Interactions between systems can be modelled through different forms of the coupling, which can be linear or nonlinear, one-way or bidirectional, through similar or dissimilar variables, and so on. One of the most interesting collective states that results as a consequence of such interactions is that of complete synchronization when the variables of the interacting systems become identical. Other forms of synchrony are known, and these have also been extensively studied over the past few decades [1].

Another intriguing emergent phenomenon in coupled oscillator systems is that of oscillation quenching, when the interacting oscillators cease oscillation altogether. Two types of oscillation quenching are known, amplitude death (AD) and oscillation death (OD). The underlying mechanisms of these phenomena are crucially different: Amplitude death appears as a result of the stabilization of an already existing steady state that is unstable in the absence of coupling [2] while oscillation death is a result of breaking a symmetry of the system through a pitchfork bifurcation, giving rise to two additional branches [3]. OD can be interpreted as a background mechanism of

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cellular differentiation [4,5] whereas AD is used in control applications where suppression of unwanted oscillations is desired, e.g., in laser application [6] and neuronal system [7]. In addition, numerous recent publications have focussed on examining the similarity between the quenching of oscillatory states (i.e., AD and OD) and the Turing instability in spatially extended systems [8–13]; this is an active area of current research on coupled oscillators.

Time-delay in the interaction is an important consideration. In many physical systems, delay is inevitable since the information flow between individual units is not instantaneous. The effect of time-delay has been widely studied in the context of both synchronization and amplitude death over the past two decades [14–16]. It has also been noted that when the coupling between dynamical systems involves time-delay, there can be a transition between two different synchronization states: this is termed as the phase-flip transition (PFT), and is a feature of broad relevance. The phase flip has been observed in regimes of amplitude–death as well as in regions of periodic, quasi-periodic, and chaotic dynamics [17] and in a variety of systems ranging from coupled lasers to coupled ecosystems [18,19].

The coupling of systems through dissimilar variables – which we term “conjugate” coupling – has been investigated in earlier work [20,21]. Such coupling arises naturally in many experimental systems such as coupled semiconductor lasers [6], where the light emitted from one laser diode is monitored by photo-diode detector whose alternating voltage signal is amplified or reduced and fed into the other. This has also been studied by Feudel and her colleagues in the dynamics of two predator–prey systems that are coupled via cross predation, in which each predator consumes prey in both systems, thereby stabilizing the food web to a new equilibrium [22]. Conjugate coupling induces the amplitude death [20] as well as phase-flip transition in coupled oscillators [23], while it also enhance the synchronization in coupled system [24,25]. We note in passing that coupling dissimilar variables can also induce rhythmogenesis, namely also cause systems to resume oscillation [26,27] by driving them out of an amplitude death situation.

In this paper, we investigate the dual effects of conjugate coupling and time-delay in nonlinear oscillators. In this coupling scheme, the interaction between the dynamical systems is maintained through cross diffusion terms, both of which include time-delay. Interestingly, we find that this type of interaction in dynamical systems not only shows the transition between synchronization states but also shows the transition between quenching states with changing coupling strength for time–delay. The present studies are carried out for limit cycle oscillator and ecological model but the results are applicable more generally.

2 Limit-cycle oscillators

We first study the effect of time-delay conjugate coupling in a pair of coupled Stuart–Landau oscillators. For simplicity we take both oscillators to identical but these coupled with the conjugate (dissimilar) variable, with time-delay. The oscillators are distinguished by subscripts, $i, j = 1, 2$, and the equations of motion are

$$\begin{aligned}\dot{x}_i &= P_i x_i - \omega y_i + k[y_j(t - \tau) - x_i], \\ \dot{y}_i &= P_i y_i + \omega x_i + \alpha k[x_j(t - \tau) - y_i]\end{aligned}\quad (1)$$

where $P_i = 1 - x_i^2 - y_i^2$ and $i \neq j$. ω is the intrinsic frequency of each oscillator. The parameter k (≥ 0) controls the conjugate coupling strength and τ (≥ 0) is the propagation time-delay. The parameter α ($0 \leq \alpha \leq 1$) determines the nature of the coupling

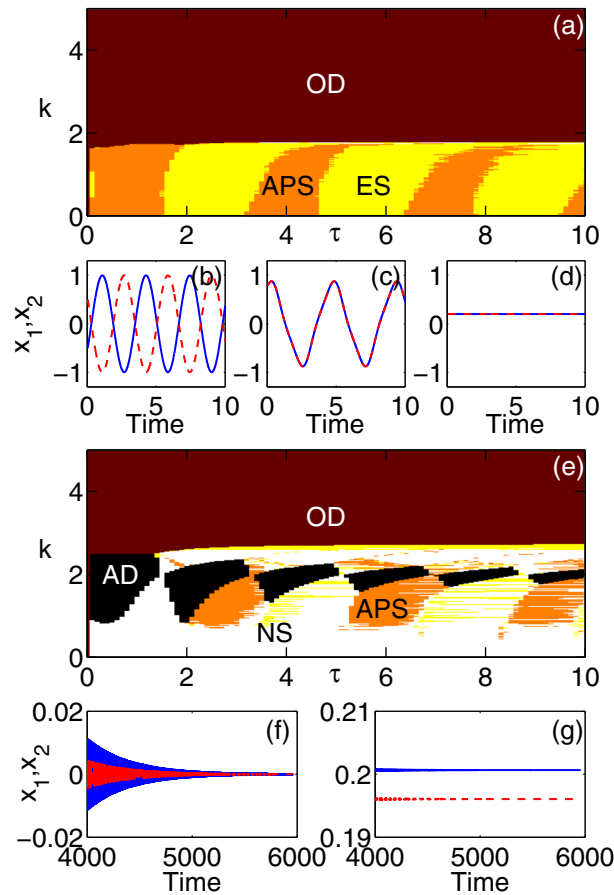


Fig. 1. (a) Schematic phase-diagram as a function of the parameters $k - \tau$ for delay conjugate coupling in Stuart-Landau oscillators, equation (1) for $\alpha = 0$. The trajectories, $x_{1,2}$ in different regions of parameters space are shown as a function of time in (b) for anti-phase synchronization (APS) at $\tau = 4$ and $k = 1$, (c) for exact synchronization (ES) at $\tau = 4$ and $k = 1.5$, for oscillation death (OD) at $\tau = 4$ and $k = 2.5$. The schematic phase-diagram as a function of $k - \tau$ when $\alpha = 1$ in the same system. The trajectories shown correspond to (f) amplitude death (AD) for $\tau = 3$ and $k = 2$, and (g) oscillation death (OD) at $\tau = 3$ and $k = 3$.

between the x and y components and introduces asymmetry in the coupling function. Such a parameter has the potential to control, and can be easily implemented in practical situations that require that fluctuations should be stabilized through AD and OD. We consider the two extreme cases $\alpha = 0$, namely scalar coupling and $\alpha = 1$ which is non-scalar (symmetric) coupling [28]. In the absence of delay, i.e., $\tau = 0$ the phenomena of AD and OD have been discussed in some detail earlier [20]. Other values of α may also be considered.

To study the time-delay in conjugate coupling of Stuart-Landau systems equation (1), we first obtain the phase diagram in parameters plane $k - \tau$ in Figure 1 at fixed $\omega = 2$. The various dynamical states that emerge in the coupled system are indicated. In order to compute Figure 1 we use several order parameters. Consider the time averages $\langle\langle x_i \rangle\rangle_t$, namely the time-average of x_i for each of the oscillators and their mean $\bar{x} = \langle\langle x_i \rangle\rangle_j$ and variance $\rho = \sigma^2(\langle x_i \rangle_t)$. We also consider the

variance of amplitude, $\sigma^2(x_i)$, of individual oscillators and its mean $\bar{\sigma} = \langle \sigma^2(x_i) \rangle_j$ over both the oscillators; this will be zero for all non-oscillating states. Note that for AD, where both oscillators settle at $(x_i, y_i) = (0, 0)$ has $\bar{x} = \bar{\sigma} = \rho = 0$. However for OD where both oscillators settle at either (i) identical fixed points $\bar{x} > 0$ and $\bar{\sigma} = \rho = 0$, or (ii) different fixed points, in which case $\bar{\sigma} = 0$ and $\bar{x}, \rho > 0$ which are newly created.

In order to distinguish the different type of dynamics in the oscillatory regime, we consider the phase difference $\Delta\phi = |\phi_j - \phi_i|/\pi$, where $\phi_i = \arctan(y_i/x_i)$. Complete or in-phase synchronization is characterized by $\Delta\phi \approx 0$ while anti-phase synchronization (APS) has $\Delta\phi \approx 1$. The third possibility is that the coupled systems are in a non-synchronized state (NS), where $0 < \Delta\phi < \pi$. The different regimes are characterized by different thresholds for these various measures (and whose specific values do not affect the qualitative nature of the results).

We first study the effect of time-delay with scalar conjugate coupling ($\alpha = 0$) in Stuart-Landau oscillators, equation (1), for intrinsic frequency $\omega = 2$. The different dynamical states are shown schematically in Figure 1. At $\tau = 0$, when coupling strength k is increased the coupled oscillators go to the exact synchronization (ES) state which is followed by oscillation death with a homogeneous steady-state (HSS). However for time-delay τ , the coupled systems interact with each other at different times scale and show different type of synchronized states both in and anti-phase synchronization (APS). Typical trajectories are shown in Figure 1b and c for APS and ES respectively. When the coupling strength is further increased, say to $k > 2$, coupled oscillators show a transition from the oscillatory state to oscillation death. Figure 1d shows the transient trajectories of OD. Note that for $\tau > 0$ and small coupling strength k , we observe islands of exact and anti-phase synchronization, similar to references [14–16]. In this regime the coupled oscillators show sharp jumps from exact synchronization state to anti-phase synchronization state and vice-versa with time-delay: this is similar to the phase-flip transition in delay in directly coupled systems [20].

When the coupling is non-scalar i.e., $\alpha = 1$ in equation (1), on the other hand, time-delay appears to have a very different effect, and the schematic phase diagram is shown in Figure 1e. At $\tau = 0$, oscillators with conjugate coupling are *not* synchronized for weak coupling strength. For higher strength ($k > 1$) the coupled systems show a transition from the oscillatory state to amplitude death, where both the oscillators dynamics settle down to the fixed point at the origin. Further increase of coupling ($k > 2.3$) shows a transition from amplitude to oscillation death state. These transitions have been studied in detail in [20] for zero time-delay.

For time-delay however, we observe a series of separated islands of AD in parameters plane for lower values of coupling. The area of the islands decrease as a function of the delay and we note that a transition takes place from an oscillatory state to AD, followed by rythmogenesis, which is then followed by oscillation death state with higher coupling (see Fig. 1e). Trajectories of $x_{1,2}$ variables in these various states are shown in Figures 1f and 1g at different combinations of τ and k ; see the figure caption.

One difference between conjugate coupling without time-delay ($\tau = 0$), as studied in Reference [20], and with time-delay (Fig. 1e) is the induction of non-synchronized oscillatory states between the phases of AD and OD. Disorder is in some sense *induced* between the regimes of ordered AD and OD; this is very like the so-called anomalous transition [29, 30] as has been observed for mismatched coupled oscillators in the absence of delay [29] as well as with time-delay [30]. It may be noted (in contrast to previous studies) that the effect occurs here for identical systems due to the fact that time-delay coupling has been introduced in conjugate variables.

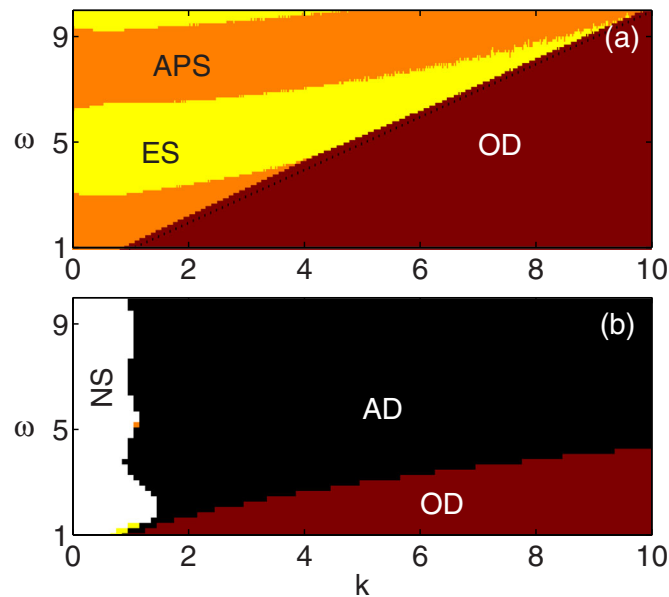


Fig. 2. Phase diagram in parameters space $\omega - k$ for (a) $\alpha = 0$ and (b) $\alpha = 1$ at time-delay $\tau = 1$.

We have also studied the transition from oscillatory state to oscillation quenching as well as between the quenching states for fixed time-delay ($\tau = 1$) as a function of $\omega - k$ as shown in Figures 2a and 2b for $\alpha = 0$ and $\alpha = 1$ respectively. For non-scalar case $\alpha = 0$, the coupled system shows dynamics different from what was observed in Figure 1. The critical coupling value for oscillation death is shifted to the higher values with increasing frequency (see Fig. 2a) when the coupling strength is increased. Note that for scalar coupling, the coupled system is always unsynchronized at weak coupling. For lower frequency, the coupled system always shows transition from oscillatory state to oscillation death. As we increase the frequency ω , the system undergoes amplitude death and remains in that state even when coupling is increased (see Fig. 2b). One of the noticeable effects of delay in contrast with [20] is the introduction of a non-synchronized region between AD and OD. Furthermore, the islands of AD shrink in size with increasing time-delay.

3 Delayed cross-predation

In order to see the effect of time-delay in conjugate-coupled ecological systems, we consider predator-prey systems that can exhibit oscillatory behavior without any coupling. We study the Truscott-Brindley model which was introduced to model the dynamics of phytoplankton and zooplankton in the ocean [31]. This model mimics the emergence of a plankton bloom using the concept of excitable systems. In case of coupling between such systems the predators can be considered as mostly feeding on their respective preys. These are coupled by introducing a cross-predation term, so that one predator can also consume prey of the other system. Such cross-predation corresponds precisely to conjugate coupling of two oscillating systems. In practice, cross-predation presents an option of an alternative food resource for each

of the predators and time-delay, corresponding to the typical time taken for predator migration between the systems, can play an important role.

Following [22], the predator-prey equations with conjugate coupling under time-delay can be written as

$$\begin{aligned}
 \dot{x}_1 &= rx_1(1-x_1) - y_1 \left(\frac{x_1^2}{\nu^2 + x_1^2} \right) - ky_2(t-\tau) \left(\frac{x_1^2}{\nu^2 + x_1^2} \right) \\
 \dot{y}_1 &= a \left(\frac{x_1^2}{\nu^2 + x_1^2} - \omega \right) + \rho ky_1 \left(\frac{x_2(t-\tau)^2}{\nu^2 + x_2(t-\tau)^2} \right) \\
 \dot{x}_2 &= rx_2(1-x_2) - y_2 \left(\frac{x_2^2}{\nu^2 + x_2^2} \right) - ky_1(t-\tau) \left(\frac{x_2^2}{\nu^2 + x_2^2} \right) \\
 \dot{y}_2 &= a \left(\frac{x_2^2}{\nu^2 + x_2^2} - \omega \right) + \rho ky_2 \left(\frac{x_1(t-\tau)^2}{\nu^2 + x_1(t-\tau)^2} \right)
 \end{aligned} \tag{2}$$

where x and y represent the prey and predator populations. The growth rate of the prey is represented by a logistic growth function, with the maximum growth rate r . Predation by the predator is represented by a Holling type III function, ν is the half saturation constant for the predator which governs how quickly the maximum predation rate is achieved when the prey population increases. The constant model the maximum growth rate of the predator which is in general a product of ingestion rate with a yield factor (less than 1) accounting for the fact that not all of the ingested prey are converted into predator biomass. Here, we have assumed that there is no inter-species competition of the two prey species, each of them can grow to its own carrying capacity in the absence of predation. Both prey species are consumed by the predator following a Holling type III functional response, but the cross-predation terms are scaled with preference factors k . The parameter ρ again account for the fact that not all consumed prey are converted into the biomass of the predator.

In the absence of time-delay, this system namely equation (2) has been extensively studied in [22]. Setting the parameters $r = 0.43$, $\nu = 0.053$, $\omega = 0.56$, and $a = 0.05$ (to make contact with earlier results) we examine the dynamics as a function of delay. Results are shown in Figure 3a which depicts the different dynamical regimes observed in the system for different $\tau - k$. When $\tau = 0$ and $k = 0$, the uncoupled systems exhibit periodic oscillations. As the coupling strength k is increased from 0, we observe a regime of multistability when both periodic and stationary behaviour coexist [22]. In this regime, the observed different dynamical behaviour are exact synchronization, anti-phase synchronization, unsynchronized state (chaotic dynamics) and steady states. Further increasing of the coupling strength $k > 0.21$, the dynamics of coupled predator-prey systems goes to oscillation death (HSS), where both predator populations settle down to the same but newly created steady states. For higher coupling strength ($k > 0.78$) there is a transition from a homogeneous steady state to a non-homogeneous steady state (IHSS).

At $k > 0.98$ the coupled system again shows a transition from the IHSS to a homogeneous steady state. However in the presence of time-delay, the region of multistability dominates. Increasing time-delay and the coupling strength k results in a phase-flip transition, when the predator population shown a transition from anti-phase to the exact synchronization state. The anti-phase dynamics of predator populations for $\tau = 10$ is shown in Figure 3b, while the exact synchronization state at $\tau = 50$ shown in Figure 3c for coupling strength $k=0.1$. For longer time-delays, the multistability regime reappears for higher coupling (see Fig. 3a). On the other hand, the regions of stationary state behavior (both HSS and IHSS) shrink with time-delay and disappear above a critical value (here $\tau=20$). Representative time-series of the

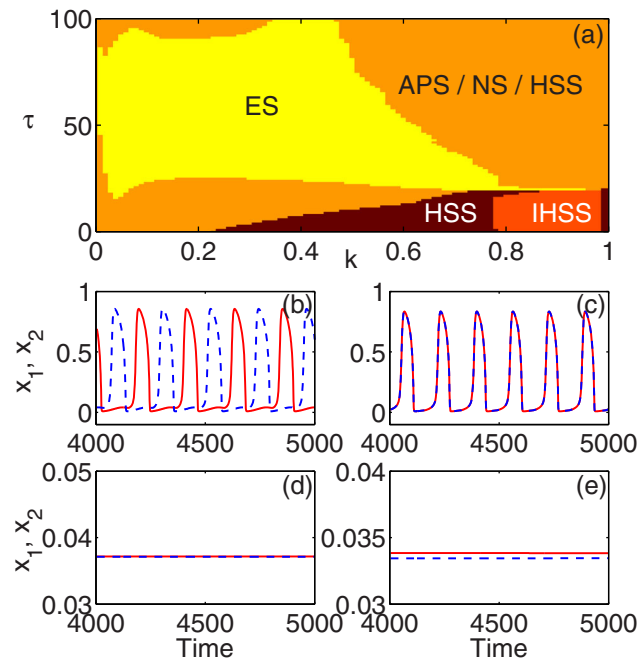


Fig. 3. Different dynamical regimes of the time-delay conjugate coupled predator-prey systems, equation (2) as a function of τ and k . Regions having multistability are shown (marked ES/APS/NS/HSS) while exact synchronization and oscillations death are labeled by ES and HSS or IHSS respectively. The time series of predator populations showing (b) APS at $\tau = 10$, (c) ES at $\tau = 50$ for oscillations at fixed coupling strength of $k = 0.1$. The time series from the regime of (d) HSS at $k = 0.75$ and (e) IHSS at $k = 0.95$ for oscillation death regimes at fixed $\tau = 5$ are shown below.

predator population coupling strength $k = 0.75$ and $k = 0.95$ are shown in Figures 3d and 3e at fixed time delay $\tau = 5$.

Compared to the case of no time-delay [22] the main difference is that there is a region of NS or desynchronization (along with coexisting APS & HSS attractors) e.g., at higher value of coupling, and when the delay is increased the dynamics goes from OD to NS. This is similar to the anomalous effect discussed for equation (1) at $\alpha = 1$. It is clear that this is a consequence of the time-delay in the conjugate coupling.

4 Concluding remarks

In earlier work we have shown [20, 21] that coupling via dissimilar variables mimics, in some sense, time-delay coupling. The effect of additional time-delay therefore has unexpected effects. We have examined some simple such systems here by incorporating time-delay in conjugate coupling. The specific systems we have made application to are model oscillators that have been studied earlier although without time-delay. The Landau-Stuart system is one that has been studied extensively, and the model of cross predation that has been introduced earlier [22] provides a natural context for conjugate coupling. Comparison with earlier studies shows that there are both similarities and differences in the dynamics when the two effects are combined.

In particular, we see that this form of coupling scheme can induce an anomalous transition from amplitude or oscillation death to a state of desynchronized motion,

even when the coupled oscillators are identical; such a transition has not been reported earlier. Both time-delay and conjugate coupling arise naturally in the interactions between dynamical systems. The effects reported here – in particular the existence of both oscillation and amplitude death, as well as regimes of asynchrony – will be of interest in the context of control, and are deserving of further study.

It is a pleasure to dedicate this article to Prof. Ulrike Feudel, friend and colleague for almost two decades. Over the years our groups have collaborated widely and enriched our research programmes. In addition, AS thanks IMSc for Postdoctoral Fellowship while MDS and AP acknowledge the financial support from DST, Government of India who have also supported RR through the award of the JC Bose National Fellowship.

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