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Transient chaos in multidimensional Hamiltonian system with weak dissipation

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Abstract. The dynamics of two coupled twist maps with weak dissipation is studied. The calculation of Lyapunov exponents is used to analyze the structure of the action plane of the system. The chaotic transient dynamics is revealed for extremely small values of dissipation by calculation of finite-time Lyapunov exponents. The stagger-and-step method is used to obtain the chaotic saddle and it is found that it is similar to the Arnold web.

1 Introduction

Although traditionally the study of nonlinear dynamical systems mainly deals with the long-time stable behavior, nowadays it is known that the transient process from initial state to the attractor can demonstrate a lot of interesting phenomena. The transient processes are studied in both classical model systems such as the logistic map [1], the Henon map [2], Rössler system [3] and in a number of dynamical models for a wide range of phenomena in physics, biology, chemistry, economics, engineering, and even social sciences [4–9]. The situation when the dynamics on finite time is chaotic while the attractor is regular is usually refered as chaotic transient process. One of the earliest studies of this phenomenon was made in [13] where the existence of the chaotic behavior which persists for a long but finite time in the Lortenz system was shown. Later similar phenomena were observed in a number of systems (see [10–12, 14] for some examples). It seems to be of most interest and is usually due to non-attracting invariant sets with chaotic dynamics (chaotic saddles) in the phase space [15–17] and the complex structure of its stable and unstable manifolds. The permanent chaos (chaotic attractor) can be regarded as a limit of transient chaos when the average lifetime of the underlying chaotic set becomes infinite. So the transient chaos is more common and possibly richer phenomenon than permanent chaos and can be regarded as a kind of metastable state.

In this work we consider transient chaos that occurs in the system with Arnold web [18] with weak dissipation. Arnold web is typical for non-integrable Hamiltonian

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systems with more than two degrees of freedom (DOF) [16–20] the system of intersecting the resonance (stochastic) layers, which form a dense web. The diffusion along this web (usually called Arnold diffusion) is possible for arbitrary small values of non-integrable perturbation unlike the systems with two DOF. In fact, the systems with weak dissipation form a special class of dynamical systems which are intermediate between conservative and dissipative ones. The peculiarities of dynamics that typical for systems with weak dissipation are studied in a number of papers [21–25], basically, they are about the coexistence of a large number of attractors and very long transient processes. Also it is known that chaos in the non-integrable conservative systems can be observed for almost any parameters, but typically in a very narrow region of the phase space [20,26,27]. In contrast, in dissipative systems chaos appears only in the specific range of parameters but the basin of the chaotic regime usually is rather large. So systems with weak dissipation are expected to demonstrate some intermediate behavior; the chaotic transient process is one of expected phenomena.

2 Two coupled twist maps

Let's consider the system of two coupled twist maps (1), that have been investigated in [28,29]. In these works the authors have been provided numerical evidence of global diffusion occurring in slightly perturbed integrable Hamiltonian systems and have been shown that even if a system is suffciently close to be integrable, global diffusion occurs on a set with peculiar topology, the so-called Arnold web, and is qualitatively different from Chirikov diffusion, occurring in more perturbed systems.

$$\begin{cases} \varphi'_1 = \varphi_1 + I_1, \\ I'_1 = I_1 + \varepsilon \dfrac{df}{d\varphi_1}(\varphi_1 + I_1, \varphi_2 + I_2) \end{cases} \\ \begin{cases} \varphi'_2 = \varphi_2 + I_2, \\ I'_2 = I_2 + \varepsilon \dfrac{df}{d\varphi_2}(\varphi_1 + I_1, \varphi_2 + I_2) \end{cases} \tag{1}$$

where $f(\varphi_1, \varphi_2) = 1/(\cos(\varphi_1) + \cos(\varphi_2) + 4)$. For small ε this system can be interpreted as the Poincaré map of the 3 DOF system with Hamiltonian: $H_\varepsilon = \frac{I_1^2}{2} + \frac{I_2^2}{2} + I_3 + \varepsilon f(\varphi_1, \varphi_2)$ by the section plane $\varphi_3 = const$. So parameter ε can be interpreted both as the coupling amplitude for coupled twist maps and the non-integrable perturbation amplitude for the Hamiltonian system (1).

The structure of the action plane (I_1,I_2) seems to be the most informative because for integrable case the actions completely determine the orbit, so the investigation of the action plane is in fact similar to the investigation of the parameter plane. Also as for a Hamiltonian system frequencies are determined as $\omega_i = \frac{\partial H}{\partial I_i}$, it can be assumed that the frequencies are proportional (in the system (1) equal) to the actions.

Resonance conditions for system (1) can be written as $k_1 I_1 + k_2 I_2 + 2\pi k_3 = 0$, where k_1, k_2, k_3 are integers, the straight lines on the Figure 1 mark some of them.

To investigate the system (1) we numerically calculate the largest Lyapunov exponent using Benettin algorithm [30] and plot its values on the action plane (Fig. 2). The resonance (Arnold) web is clearly seen for coupling amplitude $\varepsilon = 0.3$ (Fig. 2a) as the regions with largest values of Lyapunov exponent (dark lines) because the chaotic regimes (stochastic layers) are situated along the resonances. With the growth of coupling amplitude ε the Lyapunov values outside the resonance layers increases (Figs. 2b and 2c) and only few resonance lines can be seen. It seems rather natural as ε is a perturbation amplitude.

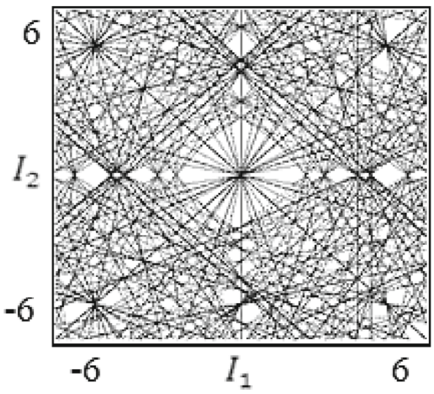

Fig. 1. Resonance lines corresponding to the resonance relations $k_1 I_1 + k_2 I_2 + 2\pi k_3 = 0$ (k_1, k_2, k_3 are integers).

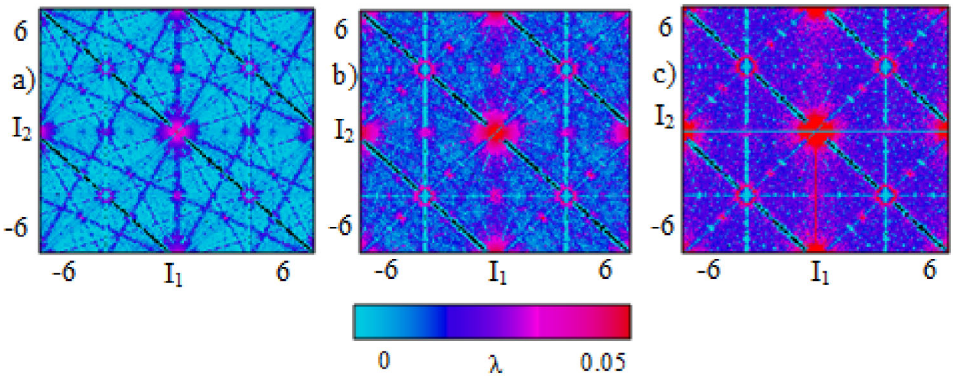

Fig. 2. Maps of dynamical regimes in the action plane of (1) at different values ε: (a) $\varepsilon = 0.3$; (b) $\varepsilon = 0.6$; (c) $\varepsilon = 0.8$. Lyapunov exponent values are indicated on the color palette. The realization length N=1000 (without transient process).

Let's introduce the linear dissipation in the system (1) as follows:

$$\begin{cases} \varphi'_1 = \varphi_1 + I_1 \\ I'_1 = \alpha I_1 + \varepsilon \dfrac{df}{d\varphi_1}(\varphi_1 + I_1, \varphi_2 + I_2) \end{cases}$$
$$\begin{cases} \varphi'_2 = \varphi_2 + I_2 \\ I'_2 = \alpha I_2 + \varepsilon \dfrac{df}{d\varphi_2}(\varphi_1 + I_1, \varphi_2 + I_2) \end{cases} \tag{2}$$

where $f(\varphi_1, \varphi_2) = 1/(\cos(\varphi_1) + \cos(\varphi_2) + 4)$.

The determinant of the Jacobi matrix for (2) is equal to α^2, so the system is conservative for $\alpha = 1$ and dissipative for $\alpha < 1$.

3 The chaotic transient process

When we introduce the small dissipation in the system (2), regular attractors occur, but the transition process can be rather long. Let's study the transition process in

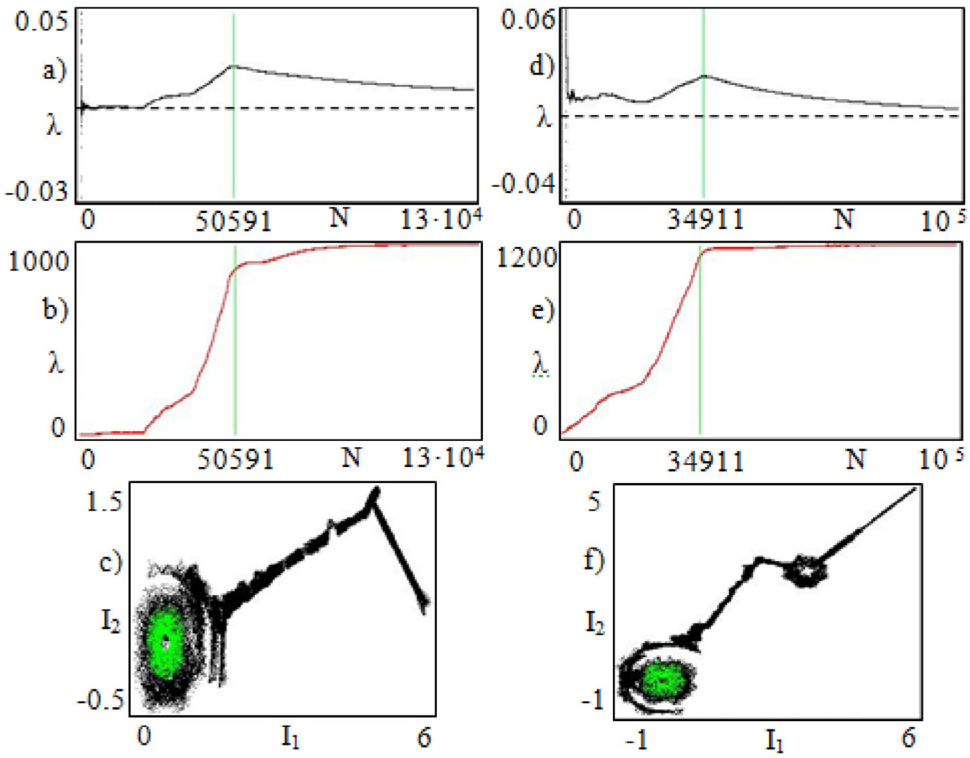

Fig. 3. Dependencies of finite time Lyapunov exponent (a,d) and the Lyapunov sum (b,e) on the realization length N and corresponding phase portraits (c,f) for the system (2) with $\alpha = 0.9999$, $\varepsilon = 0.6$. The orbits start from the points marked 1 (left column) and 2 (right column) on Figure 6a.

more details. We calculated the dependence of a finite time Lyapunov exponent (the Benettin algorithm again was used) on the realization length using for calculating a set of orbits starting from different initial points. Typical plots are shown in Figures 3 and 4 for different small dissipation levels ($\alpha = 0.9999$ for Fig. 3 and $\alpha = 0.999$ for Fig. 4). We can see two (except first approx. 1000 iterations where the dynamics strongly dependes on initial point) stages on each plot: during the first Lyapunov exponent remains definitely positive while during the second it monotonically decreases to zero. It seems natural to refer the first part as the chaotic transient process and the second as a regular transient process. Numerically, we define the global (except the initial 1000 iterations) maximum of local Lyapunov exponent as a border between these two stages, it is marked by vertical line on Figures 3 and 4. On the phase portraits (Figs. 3c, 3f; Figs. 4c, 4f) the chaotic transition process is marked with black points and the regular transition process is marked with green points. One can see that during the chaotic process the representative points move mostly along the resonance lines while during the regular process the points move near the attractor. Also one can see that a time period with the linear growth of Lyapunov sum exist on every plot which means that the points move along the chaotic set in the phase space. These results show that there is a set with chaotic behavior situated along the resonance lines in the action plane which results in the chaotic transient process.

Figure 5 shows the values of largest Lyapunov exponents on the action plane of system under investigation. 10000 iterations were used to calculate Lyapunov

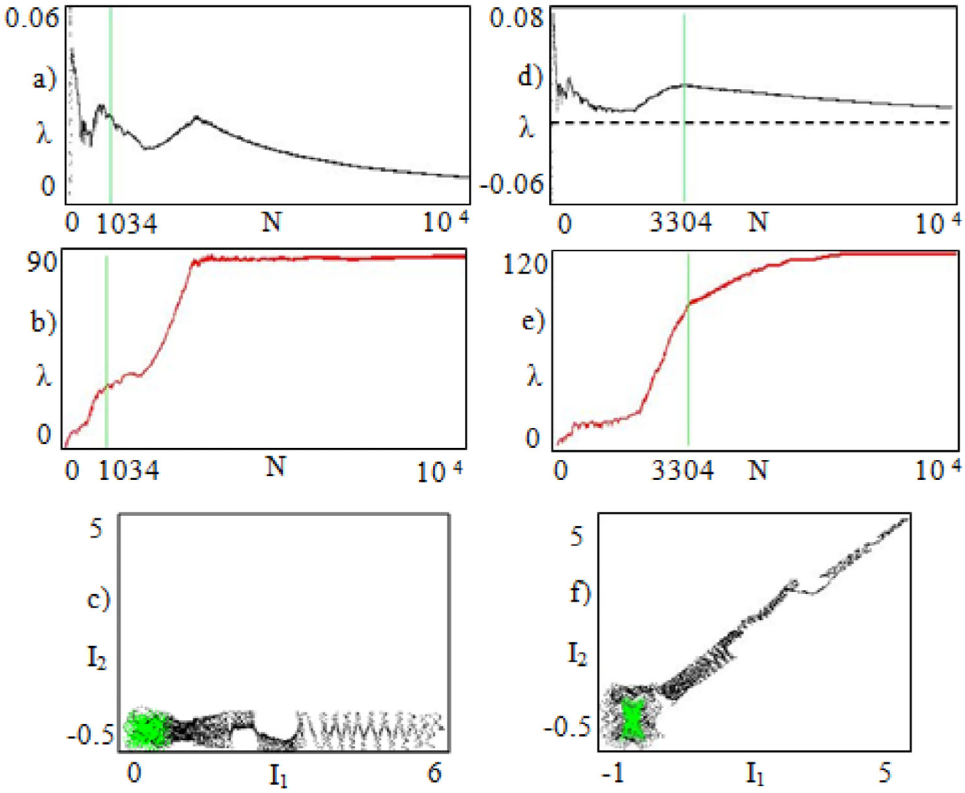

Fig. 4. Dependencies of finite time Lyapunov exponent (a,d) and the Lyapunov sum (b,e) on the realization length N and corresponding phase portraits (c,f) for the system (2) with $\alpha = 0.999$, $\varepsilon = 0.6$. The orbits start from the points marked 1 (left column) and 2 (right column) on Figure 6a.

exponent for Figure 5a but only 1000 for Figure 5b. We can see that Lyapunov exponents calculated on short time are larger for any point of action plane which confirm the existence of chaotic transition process.

Figure 6 shows the dependence of whole (i.e., the number of iterations required for initial point to come closer than 10^{-4} to attractor) and chaotic transition process duration on the initial conditions (the darker color means the greater number of iterations). One can see that the transition process is the longest for orbits started from the resonance regions. The left and right figures seem to be quite similar, which means that chaotic transient process contributes significantly to the whole transient process.

However, the features revealed above take place only when the dissipation is extremely small. Even for $\varepsilon = 0.99$ the Lyapunov exponent decreases uniformly (Fig. 7) and demonstrates no chaotic transition process. Also the sum of the Lyapunov exponents quickly reaches a constant value without intermediate maxima.

4 Search for the chaotic saddle

Results discussed above show that some unstable set with chaotic behavior exists in the system (2). We try to obtain it by using the stagger-and-step method, introduced in [31] to find the chaotic saddle.

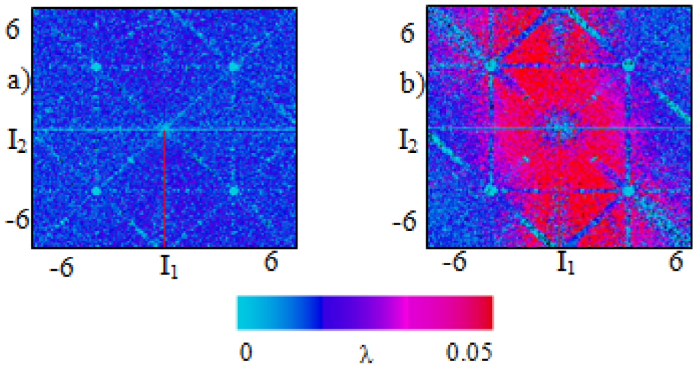

Fig. 5. Maps of Lyapunov exponents for the system (2) with $\alpha = 0.999$, $\varepsilon = 0.6$; different lenghts of realization used for calculation: (a) $N = 10000$; (b) $N = 1000$. The color of point depends on Lyapunov exponent value as shown on palette.

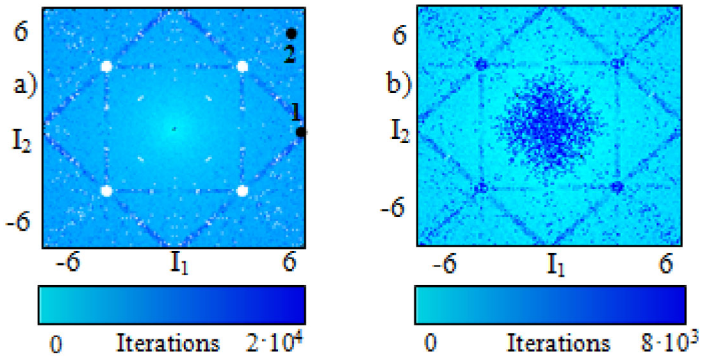

Fig. 6. Maps of the duration of the whole (a) and chaotic (b) transition process for the system (2) with $\alpha = 0.999$, $\varepsilon = 0.6$ (numbered points correspond to the initial conditions for Figs. 3,4,7). The color of point depends on Lyapunov exponent value as shown on palettes.

This method consists of two stages. On first we should find the initial point which does not leave some region during some large time T^*. We obtain it by trying consequently random points in some region without attractors. On the second stage we consider the next iteration of this point as the new initial point. If the orbit starting from the new point leaves the region for times larger that T^* we repeat the second stage, in another case we try the orbits starting from the small vicinity r_n of this point until the one which leaves the region for more that T^* is found. This procedure can be shortly described by

$$x_{n+1} = \begin{cases} F(x_n), T(x_n) > T^*(step) \\ F(x_n + r_n), T(x_n) \leq \ T^*(stagger) \end{cases} \tag{3}$$

where $F(x_n)$ – the initial dynamic system, r_n – perturbation. We use $T^* = 3000$ iterations and $|r_n| < 10^{-7}$ while numerical simulation.

Trajectories calculated by Stagger-and-Step method for the system (2) are shown in Figure 8. It is rather blurred because the method used is statistical and requires extremely long time to cover the whole saddle set especially if it has a complex structure. But it seems to be similar to the resonance lines of Arnold's web, that is, the trajectory of points on the way to the attractor located along the web.

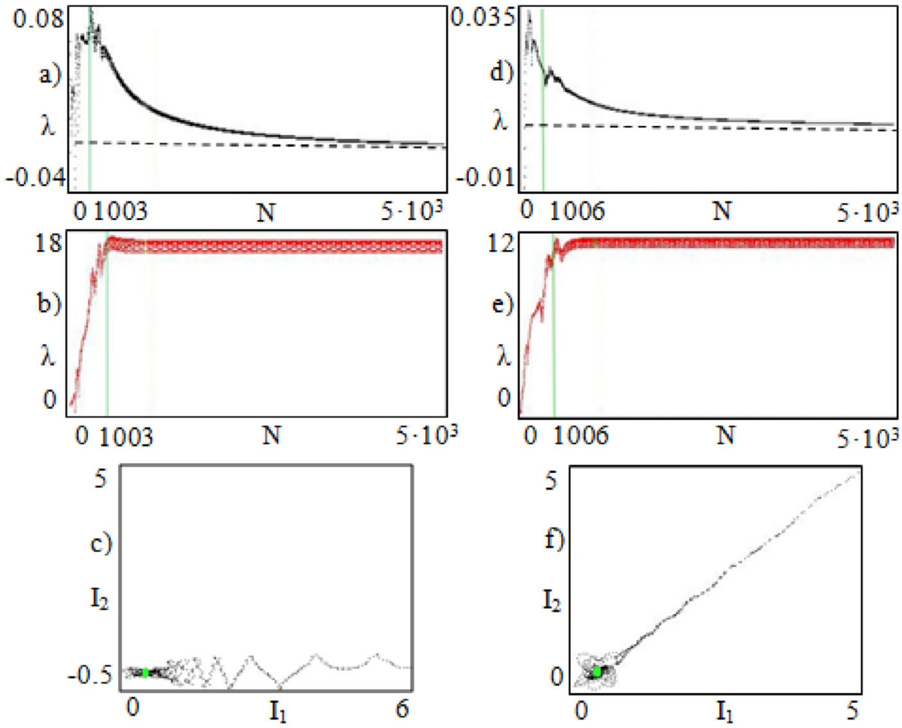

Fig. 7. Dependencies of finite time Lyapunov exponent (a,d) and the Lyapunov sum (b,e) on the realization length N and corresponding phase portraits (c,f) for the system (2) with $\alpha = 0.99$, $\varepsilon = 0.6$. The orbits start from the points marked 1 (left column) and 2 (right column) on Figure 6a.

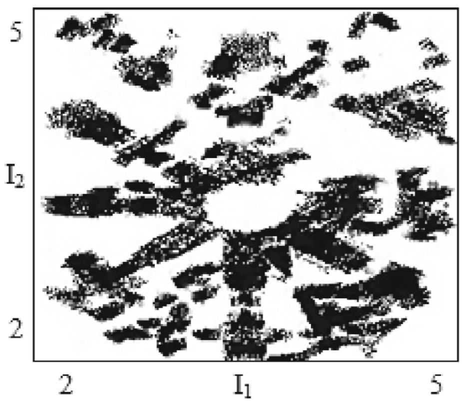

Fig. 8. The chaotic saddle of the system (2) obtained by Stagger-and-Step method at $\alpha = 0.9999$, $\varepsilon = 0.6$.

For larger values of dissipation (already at the $\alpha = 0.999$) the described procedure works worse because it becomes increasingly difficult to implement the first stage of the method as orbit often comes to attractors with a very small basins of attraction. So we can suppose that the set with chaotic dynamics disappear.

5 Conclusion

We revealed that chaotic transient processes occur in the system of two coupled twist maps with extermely weak dissipation although the attractors are regular. The chaotic transient process is associated with a chaotic saddle situated near the resonance web of the Hamiltonian system. This set destructs with increase of dissipation.

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