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Exploring phase control with square pulsed perturbations

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Abstract. We discuss the phase control technique consisting of an applied square pulsed periodic perturbation. We explore the effect of such perturbations to the different terms of the Duffing oscillator. We find that the effect depends sensitively on how the perturbation is applied, indeed, it is specially effective when it modulates the cubic and the linear term and uneffective when applied to the driving term. Our results highlight the highly nontrivial role of the phase when applying a second periodic perturbation to a chaotic system.

1 Introduction

Chaos is characterized by the sensitive dependence on the initial conditions, which implies also that small but accurately chosen perturbations of a chaotic system can lead to substantial changes in the dynamics, and even to its stabilization in different periodic orbits. This is the rationale behind different methods of chaos control that have been proposed since the pioneering work by Ott, Grebogi and Yorke [1]. In that work, a carefully chosen perturbation is applied on one of the system's parameters leading to the stabilization of the system on one of the attractors' unstable periodic orbits. Such perturbation is computed each time depending on the system's state, and for this reason the OGY method falls in the category of the so-called feedback methods of chaos control.

Another important family of chaos control is given by nonfeedback controls. These are applied to periodically driven nonlinear dynamical systems. In these methods, a second small harmonic perturbation is applied to the system [2] and can lead to the stabilization over different periodic orbits. It was soon appreciated that the phase difference between this second harmonic perturbation and the main driving can lead

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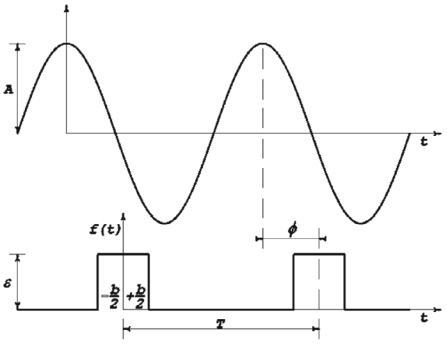

Fig. 1. Controlling perturbation $f(t)$ of period $T = 1/f_c$ representing a rectangular pulse of width b and height ϵ. The duty cycle is defined as $D = b/T$. The controlling frequency f_c is assumed to be equal to the driving frequency f_d. The phase difference is defined by considering the maximum of the sinusoidal signal and the midpoint of the pulsed perturbation.

to different behaviours. This was first observed in the control of chaos in a CO_2 laser [3] and in numerical simulations with the Duffing oscillator [4], a model of paramount importance in science and engineering.

The concept of *phase control of chaos* was then precisely formulated and applied to the control of chaos in a two-well Duffing system and also in its implementation in an electronic circuit [5]. Interestingly, we also found that such method can be applied to control other types of complex dynamics, such as crisis induced intermittency [6], escapes in an open system [8] and pulses in an excitable system [7], both numerically and in experimental implementations. This underscores the versatility and the robustness of the phase control method.

In spite of this progress, we are far from a complete understanding on the mechanism of this control scheme and on how it can be further optimized. For this reason, in a recent work we used the Duffing oscillator to understand how different ways to apply the controlling perturbation have different effectiveness in stabilizing the system's trajectories [9]. A second possibility that we have recently explored is using square perturbations [10], showing that it is possible to stabilize the trajectories for an adequate value of the phase difference, which indicates that this technique could be in principle applied to any periodically driven system independently of the periodic driving considered. Here we compare the role of applying this periodic controlling perturbation to different terms of the Duffing oscillator. We find that optimal control is more effective on the cubic term as compared to the linear one and that it is scarcely effective when applied to the main forcing amplitude. The harmonic content in the square wave forcing provides a simple explanation for this fact.

Our numerical investigations refer to a double-well Duffing oscillator:

$$\dot{x} = y, \quad \dot{y} = -\gamma y + x - x^3 + A\sin(2\pi f_d t) \tag{1}$$

where $\gamma = 0.25$ is the damping constant, $A = 0.41$ is the amplitude of the sinusoidal driving signal with frequency $f_d = 1$.

Following [10] we consider the possibility of applying a pulsed control perturbation $f(t)$, consisting of a square wave of period $T = 1/f_c$ and amplitude ϵ, so that $f(t) = \epsilon$ for $t = [-b/2, b/2]$ and 0 during the rest of the period as shown in Figure 1. In our case we consider the resonant case where the control frequency f_c is equal to the driving frequency f_d. The parameter b is related to the duty cycle D of the square wave through the relation $D = b/T$. A relative phase $\phi \in [0, 2\pi]$ of the pulsed perturbation $f(t + \phi)$ with respect to the driving has been introduced considering that it is the key parameter of our control strategy, see Figure 1.

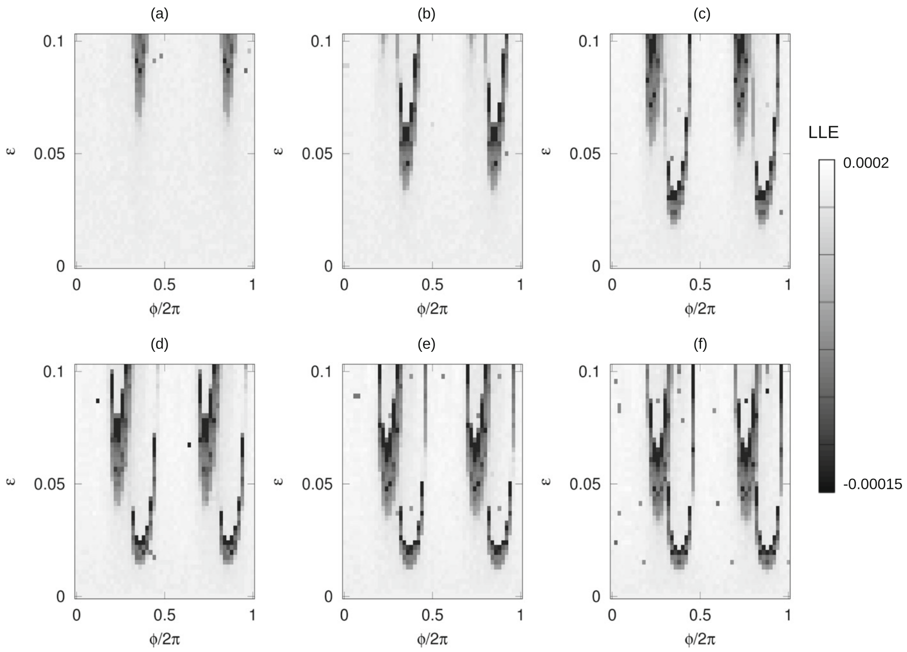

Fig. 2. Largest Lyapunov Exponent computed for a control on x^3, (a) $D = 0.02$, (b) $D = 0.04$, (c) $D = 0.08$, (d) $D = 0.12$, (e) $D = 0.16$, (f) $D = 0.2$.

2 Control on the cubic term

A first possibility is to apply the controlling perturbation to the x^3 term:

$$\dot{x} = y, \quad \dot{y} = -\gamma y + x - (1 + f(t))x^3 + A\sin(2\pi f_d t). \tag{2}$$

To characterize the effect of the controlling perturbation we compute the Largest Lyapunov Exponents (LLEs) in the parameter plane $\epsilon - \phi$ using the Wolf algorithm [11] for different values of the duty cycle D. The trajectories were obtained using a fourth-order Runge-Kutta algorithm with a time step $h = 0.001$. After running the integrator for 1000 cycles, to make sure the convergence towards the attractor, the LLE was calculated using another 1000 forcing cycles of the system.

Numerical results extending the range of the duty cycle D are reported in Figure 2 for six different values of D from $D = 0.02$ to $D = 0.2$ [10]. The minimum value of ϵ required for the control of the system decreases when the duty cycle increases, while the phase remains a crucial parameter to control the system. It is important to observe that the two stability domains separated by a phase difference of π show the appearance of secondary domains as the parameter D increases.

3 Control on the linear term

Following what we did previously in [9], we apply the square pulsed perturbation to the other terms. The second application is to the x term:

$$\dot{x} = y, \quad \dot{y} = -\gamma y + (1 + f(t))x - x^3 + A\sin(2\pi f_d t). \tag{3}$$

We calculate now the stability zones in the $\epsilon - \phi$ plane as in the previous section, using the same algorithm to compute the LLE. The results are reported in Figure 3, where we plot again the stability zones for the duty cycle ranging from $D = 0.02$

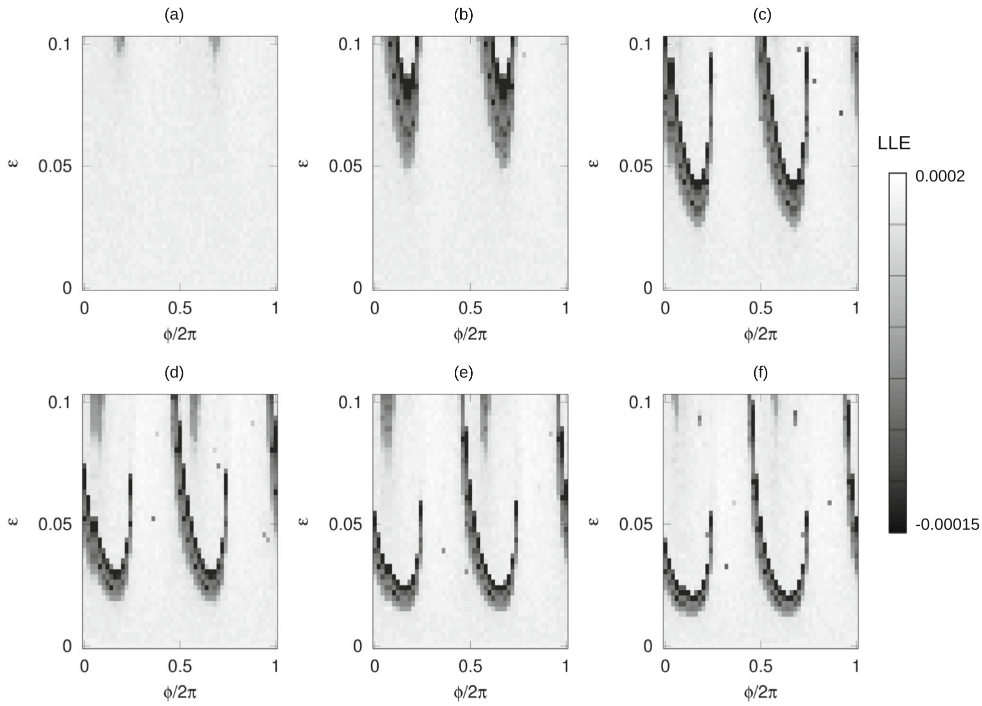

Fig. 3. Largest Lyapunov Exponent for a control on the x term: (a) $D = 0.02$, (b) $D = 0.04$, (c) $D = 0.08$, (d) $D = 0.12$, (e) $D = 0.16$, (f) $D = 0.2$.

to $D = 0.2$. We find again that the minimum value of ϵ required for the control of the system decreases when the duty cycle increases, and again the phase plays a fundamental role. However the control areas are different from the previous case and no secondary structures seem to arise in this case, although a clear π symmetry in the figures is present. All summed up we stress that we stabilize the orbits of the system only with an adequate selection of the phase ϕ.

4 Control on the forcing amplitude term

Eventually we apply the control perturbation to the forcing term:

$$\dot{x} = y, \quad \dot{y} = -\gamma y + x - x^3 + A(1 + f(t)) \sin(2\pi f_d t). \tag{4}$$

The results displayed in Figure 4 clearly indicate that a different mechanism here is at place. The control is less effective than in the two previous schemes, something that was also evident when the phase control was applied as a sinusoidal modulation of the amplitude [9]. The use of a pulsed square wave provides a simple physical reason for this result. In particular, it is clear that the pulsed wave acts as a modulation in the amplitude that only works during the duty cycle, and that out of the duty cycle the system is essentially identical to the unperturbed one. When such modulation is too short in time, its effect is negligible, and only when it is active for a consistent fraction of time ($D > 0.1$) the effects become evident. The modulation effect has a π symmetry.

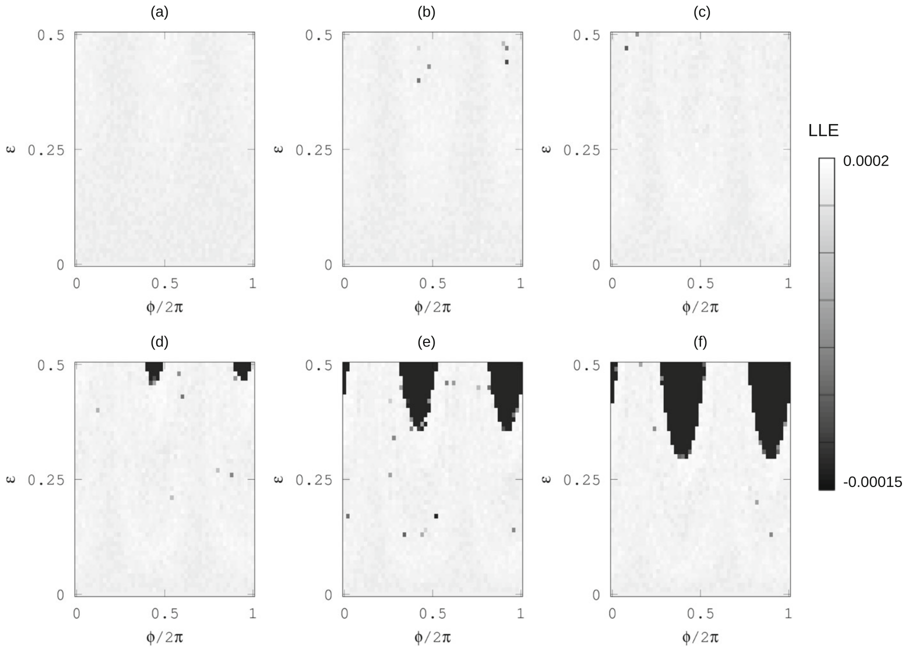

Fig. 4. Largest Lyapunov Exponent computed for a control modulating the amplitude A, (a) $D = 0.02$, (b) $D = 0.04$, (c) $D = 0.08$, (d) $D = 0.12$, (e) $D = 0.16$, (f) $D = 0.2$.

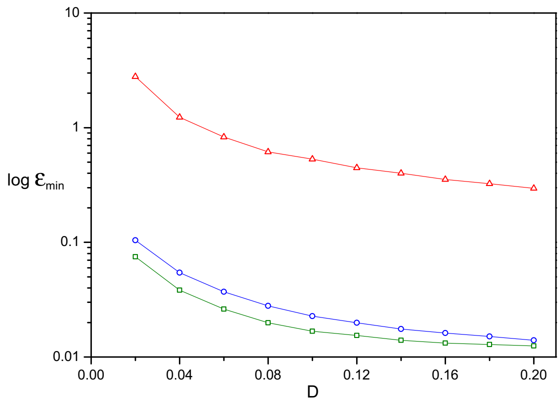

Fig. 5. Minimum ϵ required to control the system for different values of the duty cycle D for control applied on the forcing amplitude A (red, triangles), to the x term (blue, circles) and to the x^3 term (green, squares).

5 Comparison of the results

From Figures 2, 3 and 4, it appears that control on x^3 term is more efficient than the control on the x term. This comparison was performed along different values of D in Figure 5. This result is to be compared with our recent paper [9] where pure sinusoidal phase control was considered. In such a case, phase control was more efficient when applied to the linear term. We speculate that the reason of this difference is related to the role of high harmonics present in the square wave perturbation.

6 Conclusions

The effectiveness of the phase control when applied in a periodic pulsed configuration to the double well Duffing oscillator has been investigated in three distinct configurations. Periodic pulsed perturbations are effective to stabilize the chaotic orbits and the best efficiency is reached when applied to the cubic term of the equation. The opposite result was found when c.w. sinusoidal perturbations are applied to the linear term [9]. Phase control is less effective by more than one order of magnitude when applied to the driving term.

The advantage of using pulsed perturbations is also related to the fact the region of optimal control can be easily detected due to the presence of a saturated regime as the duty cycle is increased. In this regime, if the pulse duration is reduced, we have to compensate with an increase of its amplitude and vice versa. Short pulsed perturbations will be effective only for a narrow range of the phase difference; long perturbations will lose effectiveness as they imply the effect of several sinusoidal harmonic components with opposite phases, which occurs soon as the duty cycle becomes large. Our investigations confirm the versatility of the phase control technique and how the constraints imposed by the energy play a key role when considering potential applications in physics, engineering and biomedical applications [12,13].

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