

From dynamical scaling to local scale-invariance: a tutorial

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Abstract. Dynamical scaling arises naturally in various many-body systems far from equilibrium. After a short historical overview, the elements of possible extensions of dynamical scaling to a local scale-invariance will be introduced. Schrödinger-invariance, the most simple example of local scale-invariance, will be introduced as a dynamical symmetry in the Edwards-Wilkinson universality class of interface growth. The Lie algebra construction, its representations and the Bargman superselection rules will be combined with non-equilibrium Janssen-de Dominicis field-theory to produce explicit predictions for responses and correlators, which can be compared to the results of explicit model studies.

At the next level, the study of non-stationary states requires to go over, from Schrödinger-invariance, to ageing-invariance. The ageing algebra admits new representations, which acts as dynamical symmetries on more general equations, and imply that each non-equilibrium scaling operator is characterised by two distinct, independent scaling dimensions. Tests of ageing-invariance are described, in the Glauber-Ising and spherical models of a phase-ordering ferromagnet and the Arcetri model of interface growth.

1 Dynamical symmetries out of equilibrium

Symmetries have played an important rôle in physics since a long time [1,2], and new types of symmetry and new applications are continuously being discovered. The best-known example of a time-space symmetry is the special-relativistic Poincaré-invariance, of either classical mechanics or classical electrodynamics [3]. While the principle of relativity acts essentially as a scaffold on which more specific physical theories can be constructed, larger time-space symmetries can be realised if the physical system under study is specified in more detail: for example, Maxwell's equations

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of a free electromagnetic field in the vacuum, in $d = 1 + 3$ time-space dimensions,¹ admit a conformal symmetry [4, 5]. Continuous phase transitions, at thermodynamic equilibrium, constitute a very widespread set of examples where the strong interactions of a large number of degrees of freedom may create first a scale-invariance [8–11], at least at certain specific ‘critical points’ in parameter space,² which in many ‘favourable’ cases can be extended further to conformal invariance [16, 17]. In particular, there is a proof of conformal invariance in the Ising model universality class, in any dimension [18]. Schematically, scale-invariance defines the critical exponents and through the renormalisation group establishes mainly qualitative properties, such as their universality (and produces the scaling relations between the exponents) but does not fix their values. Conformal invariance rather makes quantitative predictions in fixing the form of the scaling functions and, at least in $d = 2$ dimensions, produces the admissible values of the exponents from the unitary representations of the Virasoro algebra. At a phase transition, conformal invariance is a property of the effective theory, which describes the long-distance properties of a critical system. On the other hand, conformal invariance also arises as a ‘fundamental’ symmetry from the reparametrisation-invariance in string theory, see [19, 20] and references therein.

In condensed-matter and non-equilibrium statistical physics, one is often led to study time-dependent critical phenomena, of which Brownian motion is one of the best-known examples [21, 22]. Their time-space dynamical symmetries have since a very long time been known to mathematicians [23–25] as a dynamical symmetry, originally either of the motion of free particles or of free diffusion of an ensemble of particles, and the corresponding Lie group is nowadays usually called the *Schrödinger group*. This Lie group, and its associated Lie algebra, caught the attention of physicists much later [26–29]. Here, we shall be interested in a class of applications of extensions of dynamical scaling to the collective non-equilibrium behaviour, as it arises (i) in the phase-ordering kinetics of simple magnets [30] quenched into the coexistence phase below the critical temperature $T_c > 0$, from a disordered initial state, or (ii) in the kinetics of interface growth [31–33]. The description of these examples of non-equilibrium critical phenomena owes a lot to earlier studies on the *physical ageing* in glassy and non-glassy systems [34–36]. Remarkably, experiments on the mechanical relaxation in many polymer systems, to be followed later by analogous studies in many different kinds of glassy and non-glassy systems, established that physical ageing has indeed many *reproducible* and *universal* aspects [37]. This allows one to present a formal definition of ageing in complex physical systems [35]: *a system undergoes physical ageing*³ *if its relaxational behaviour has the following properties:*

1. *slow relaxation of the observables (not described in terms of a single exponential)*
2. *breaking of time-translation-invariance*
3. *dynamical scaling*⁴

¹ Remarkably, the conformal invariance of the free Maxwell field no longer holds true in $d \neq 4$ time-space dimensions, although the theory certainly is scale-invariant: for $d = 3$, the theory can be embedded into an unitary conformally invariant field-theory, while for $d \geq 5$, only non-unitary extensions exist [6, 7].

² For historical overviews on critical phenomena see [12, 13], and also [14, 15].

³ We use european spelling throughout.

⁴ Physically, this means that there is a single time-dependent length scale $L(t)$, but *a priori* nothing is yet said on its precise form. Indeed, for systems with frustrations and/or disorder, one expects for large times logarithmic growth $L(t) \sim \ln^{1/\psi} t$ (eventually after a very long cross-over regime), whereas for simple systems without disorder and frustrations, an algebraic law $L(t) \sim t^{1/z}$ is expected. We shall restrict throughout to the latter case, also referred to as *simple ageing*.

Table 1. Comparison between the kinetics of phase-ordering and of interface growth.

Phase-ordering	Interface growth
thermodynamic equilibrium state	growth continues forever
magnetisation $m(t, \mathbf{r})$	height profile $h(t, \mathbf{r})$
phase transition at $T = T_c$	same generic behaviour for $T > 0$
(ageing for $T \leq T_c$, no ageing for $T > T_c$)	(deterministic for $T = 0$, probabilistic for $T > 0$)
variance $\langle (m(t, \mathbf{r}) - \langle m(t) \rangle)^2 \rangle \sim t^{-\beta/(z\nu)}$	roughness $\langle (h(t, \mathbf{r}) - \langle h(t) \rangle)^2 \rangle \sim t^\beta$
relaxation, after quench to $T \leq T_c$	relaxation, from initial substrate
autocorrelator $C(t, s) = \langle m(t)m(s) \rangle_c$	autocorrelator $C(t, s) = \langle h(t)h(s) \rangle_c$

A central quantity for the description of such non-equilibrium systems is either the time-space-dependent magnetisation $m(t, \mathbf{r})$ in the case of phase-ordering or the interface height $h(t, \mathbf{r})$ for interface growth. Both are instances of a time-space-dependent *order-parameter* $\varphi = \varphi(t, \mathbf{r})$. Adopting a continuum description, this order-parameter is assumed to obey a stochastic Langevin equation

$$2\mathcal{M}\partial_t\varphi = \Delta_{\mathbf{r}}\varphi - \frac{\delta\mathcal{V}[\varphi]}{\delta\varphi} + \left(\frac{T}{\mathcal{M}}\right)^{1/2}\eta \quad (1)$$

where \mathcal{M} is a kinetic coefficient, $\Delta_{\mathbf{r}}$ the spatial laplacian and the potential $\mathcal{V}[\varphi]$ fixes the detailed behaviour of the model. The solution φ is a random variable, since η is a centred Gaussian noise of unit variance and T plays the rôle of a temperature and also, the initial state is assumed to obey a centred Gaussian distribution with variance $\langle \varphi(0, \mathbf{r})\varphi(0, \mathbf{r}') \rangle = \Delta_0\delta(\mathbf{r} - \mathbf{r}')$. The special case $T = 0$ includes the physics of phase-ordering kinetics, while the special case $\Delta_0 = 0$ includes interface growth. In both cases, one has $\langle \varphi(t, \mathbf{r}) \rangle = 0$, where $\langle \cdot \rangle$ denotes the average over the thermal or initial distributions. See Table 1 for a schematic comparison.⁵

A large part of this work will concentrate on the paradigmatic special case $\mathcal{V} = 0$. As we shall see, it already contains many important features of more general systems which can thereby explained in a simple way. In our paradigm, the case $T = 0$ is often called the *free Gaussian model* of phase-ordering kinetics, whereas the case $\Delta_0 = 0$ is known as the *Edwards-Wilkinson model* of interface growth [38]. In the latter situation, the interface width, on a hypercubic lattice $\mathcal{L} \subset \mathbb{Z}^d$ with $|\mathcal{L}| = L^d$ sites, usually shows, for large times, *Family-Vicsek scaling* [39]

$$w^2(t; L) := \frac{1}{L^d} \sum_{\mathbf{r} \in \mathcal{L}} \langle (h(t, \mathbf{r}) - \bar{h}(t))^2 \rangle = L^{2\beta z} f_w(tL^{-z}) \sim \begin{cases} t^{2\beta}; & \text{if } tL^{-z} \ll 1 \\ L^{2\alpha}; & \text{if } tL^{-z} \gg 1 \end{cases}$$

where $\bar{h}(t)$ is the spatially averaged height, β is the *growth exponent* and α is the *roughness exponent*, see Figure 1. The dynamical exponent $z = \alpha/\beta > 0$. When $tL^{-z} \gg 1$, one speaks of the *saturation regime* and when $tL^{-z} \ll 1$, one speaks of the *growth regime*. All studies of ageing in interface growth are in the growth regime, on which we shall focus from now on. In contrast to equilibrium critical phenomena, non-equilibrium scaling, as in phase-ordering or interface growth, can be achieved *without* having to fine-tune one or several thermodynamic parameters of the macroscopic system.

⁵ For magnets, a quench to $T = T_c$ produces critical dynamics: depending on the initial state either *at* or *out of* equilibrium.

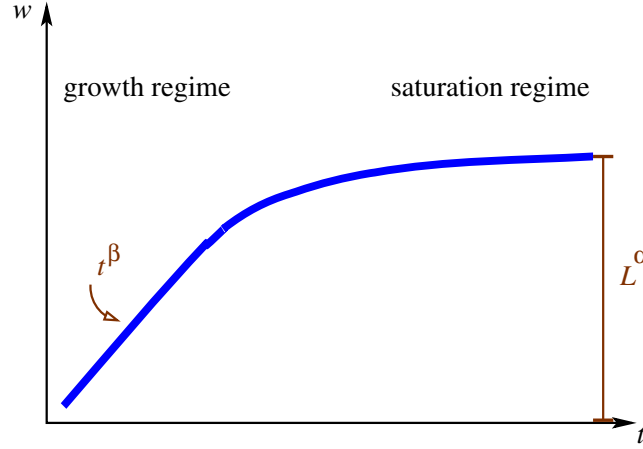


Fig. 1. Schematic evolution of the interface width on a substrate with linear dimension L . The growth regime (where ageing occurs) and the saturation regime, both with the associated scaling, are indicated.

The ageing behaviour of the solutions of the Langevin equation (1) is conveniently studied through the two-time correlators C and responses R , defined as

$$C(t, s; \mathbf{r}) = \langle \varphi(t, \mathbf{r} + \mathbf{r}_0) \varphi(s, \mathbf{r}_0) \rangle - \langle \varphi(t, \mathbf{r} + \mathbf{r}_0) \rangle \langle \varphi(s, \mathbf{r}_0) \rangle \quad (2a)$$

$$R(t, s; \mathbf{r}) = \left. \frac{\delta \langle \varphi(t, \mathbf{r} + \mathbf{r}_0) \rangle}{\delta j(s, \mathbf{r}_0)} \right|_{j=0} = \langle \varphi(t, \mathbf{r} + \mathbf{r}_0) \tilde{\varphi}(s, \mathbf{r}_0) \rangle \quad (2b)$$

where $j = j(t, \mathbf{r})$ is an external perturbation conjugate to the order-parameter φ , to be added to equation (1), and $\tilde{\varphi}$ is the associated response field, in the context of Janssen-de Dominicis field theory, see [36]. In many cases, for instance when $\mathcal{V} = 0$, spatial translation-invariance holds true, as anticipated in (2). In addition, one has the following dynamical scaling (also assuming rotation-invariance for $d > 1$ dimensions)

$$C(t, s; \mathbf{r}) = s^{-b} F_C \left(\frac{t}{s}; \frac{|\mathbf{r}|^z}{t-s} \right), \quad R(t, s; \mathbf{r}) = s^{-1-a} F_R \left(\frac{t}{s}; \frac{|\mathbf{r}|^z}{t-s} \right) \quad (3)$$

which defines the ageing exponents a, b . The scaling forms (3), often referred to as *simple ageing*, implicitly assume the existence of a single time-dependent length scale $L = L(t)$, and with an algebraic long-time behaviour $L(t) \sim t^{1/z}$, which defines the *dynamical exponent* z . Often, one focuses on the autocorrelator and the autoresponse

$$C(t, s) = C(t, s; \mathbf{0}) = s^{-b} f_C \left(\frac{t}{s} \right), \quad f_C(y) = F_C(y; 0) \stackrel{y \gg 1}{\sim} y^{-\lambda_C/z} \quad (4a)$$

$$R(t, s) = R(t, s; \mathbf{0}) = s^{-1-a} f_R \left(\frac{t}{s} \right), \quad f_R(y) = F_R(y; 0) \stackrel{y \gg 1}{\sim} y^{-\lambda_R/z} \quad (4b)$$

and defines the *autocorrelation exponent* λ_C and the *autoresponse exponent* λ_R . The exponent b is simply related to stationary exponents: one has $b = 0$ in phase-ordering, $b = 2\beta/(z\nu)$ for critical dynamics and $b = -2\beta$ for interface growth. The value of a can be fixed if a fluctuation-dissipation theorem (FDR), i.e. a relationship between C and

R , holds true.⁶ In the known cases where such a relationship exists, this also implies $\lambda_C = \lambda_R$, but for a fully disordered initial state, the autocorrelation and autoresponse exponents are independent of all equilibrium exponents, see [35,36].⁷

Does there exist any extension of dynamical scaling, which would constrain the form of the scaling functions in (4)? Can one use conformal invariance, at equilibrium critical points, as a guide to find such extensions?

In order to present the basic elements of a possible answer,⁸ we shall concentrate for quite a while on the paradigmatic case $\mathcal{V}[\varphi] = 0$ in equation (1). This gives the *Edwards-Wilkinson equation* [38]. It is well-known that its dynamical exponent $z = 2$. For $z = 2$, an extension of dynamical scaling is given by the *Schrödinger group* of time-space transformations

$$t \mapsto t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \mathbf{r} \mapsto \mathbf{r}' = \frac{\mathcal{R}\mathbf{r} + \mathbf{v}t + \mathbf{a}}{\gamma t + \delta}; \alpha\delta - \beta\gamma = 1 \quad (5)$$

with a rotation matrix $\mathcal{R} \in SO(d)$, $\mathbf{v}, \mathbf{a} \in \mathbb{R}^d$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. The transformation in time is indeed a (projective) conformal transformation, the transformations in space are rotations, Galilei-transformations and translations, as parametrised by \mathcal{R} , \mathbf{v} and \mathbf{a} . The Schrödinger group is *not* semi-simple, its representations are therefore projective: co-variant solutions of Schrödinger-invariant equations transform also through the presence of a ‘companion function’.⁹ Consequently, since the Edwards-Wilkinson equation, equation (1) with $\mathcal{V}[\varphi] = 0$, describes the coupling of the system with a bath, this coupling is incompatible with any dynamical symmetries, beyond translation- and rotation-invariance, and dynamical scaling. Therefore, symmetries of (1) will be studied in two steps, as follows:

1. find the dynamical symmetries of the noiseless simple diffusion equations, with $T = 0$ and $\Delta_0 = 0$ [23,24,26]. In particular, derive the Bargman superselection rules [47] which follow from the combination of spatial translation-invariance and Galilei-invariance;
2. using the non-equilibrium Janssen-de Dominicis theory, derive reduction formulae, in order to express any correlator or response of the full, noisy theory in terms of averages computed only in terms of the noise-less, deterministic theory [48].

Therefore, Schrödinger-invariance of a Langevin equation (1) is a *hidden symmetry* in the sense that formally it is only a symmetry of its *deterministic part*, obtained from (1) by setting $T = 0$ and $\Delta_0 = 0$.

The quest for *local scale-invariance* (LSI) is to find non-trivial extensions of dynamical scaling which would allow (i) to predict the form of the scaling functions of responses and correlators, once the scaling dimensions are known and (ii) to fix,

⁶ At equilibrium, one has Kubo’s well-known result: $TR(t-s; \mathbf{r}) = \partial_s C(t-s; \mathbf{r})$. Non-equilibrium stationary systems with a known FDR include critical directed percolation [40, 41] or the Kardar-Parisi-Zhang universality class in one dimension [42,43]. Their FDRs are distinct from Kubo’s form.

⁷ Since response functions, defined in (2b), are difficult to measure directly in a numerical simulation, it is common practise to use time-integrated dynamical susceptibilities instead, as introduced first in glassy systems [34]. However, there are several pitfalls for the correct interpretation of the scaling of dynamical susceptibilities, especially so for phase-ordering [44,45], see [35] for full details. For interfaces, the analogues of dynamical susceptibilities are computed from a damage-spreading simulation [46].

⁸ Presented for the first time in [49,50].

⁹ This is well-known from the Galilei-transformation of the wave function $\psi(t, \mathbf{r})$ of a free particle in non-relativistic quantum mechanics.

Table 2. Lie algebra generators \mathcal{X} of the Schrödinger algebra $\mathfrak{sch}(1)$.

\mathcal{X}	Generator	Interpretation
X_{-1}	$-\partial_t$	time-translation
$Y_{-1/2}$	$-\partial_r$	space-translation
M_0	$-\mathcal{M}$	phase shift
X_0	$-t\partial_t - \frac{1}{2}r\partial_r - \frac{x}{2}$	dilatation
$Y_{1/2}$	$-t\partial_r - \mathcal{M}r$	Galilei-transformation
X_1	$-t^2\partial_t - tr\partial_r - xt - \frac{1}{2}\mathcal{M}r^2$	'special' Schrödinger transformation

or at least to constrain, the values of these scaling dimensions. At present, some progress has been achieved on the first objective, while the second is still out of reach. Schrödinger-invariance is the most simple example of LSI. It arises as dynamical symmetry of the Edwards-Wilkinson equation and will be discussed at length in Section 2. Section 3 considers what may happen for truly non-equilibrium systems where time-translation-invariance no longer holds. Then we must instead study the *ageing algebra* $\mathfrak{age}(d)$, a true subalgebra of the Schrödinger algebra $\mathfrak{sch}(d)$. New features arise in the representations of $\mathfrak{age}(d)$ and we shall present some of the physical consequences. The 1D Glauber-Ising model, the spherical model of a ferromagnet and the Arcetri model of interface growth are examples of ageing-invariant systems.

We close with a short overview of some further tests and a brief outlook on current and possible future work. This tutorial is not intended as a review: we did not attempt completeness of neither the themes treated, nor the references quoted.

2 Schrödinger-invariance and the Edwards-Wilkinson equation

Here the elements of the dynamical symmetry of the Edwards-Wilkinson equation will be presented one after the other, step by step. We shall show how to derive the form of the Schrödinger-covariant correlators and responses.

2.1 Schrödinger algebra

It is convenient to consider the infinitesimal form of the Schrödinger transformations (5). The corresponding infinitesimal generators are, for technical simplicity in $d = 1$ dimensions,

$$X_n = -t^{n+1}\partial_t - \frac{n+1}{2}t^n r\partial_r - \frac{x}{2}(n+1)t^n - \frac{n(n+1)}{4}\mathcal{M}t^{n-1}r^2 \quad (6a)$$

$$Y_m = -t^{m+1/2}\partial_r - \left(m + \frac{1}{2}\right)\mathcal{M}t^{m-1/2}r \quad (6b)$$

$$M_n = -t^n\mathcal{M} \quad (6c)$$

where $n \in \mathbb{Z}$ and $m \in \mathbb{Z} + \frac{1}{2}$. The Lie algebra of the finite-dimensional Schrödinger group in (5) is the *Schrödinger algebra* $\mathfrak{sch}(1) = \text{Lie } \text{Sch}(1) = \langle X_{\pm 1,0}, Y_{\pm 1/2}, M_0 \rangle$, see Table 2 for their interpretation. Then the non-vanishing commutators are

$$\begin{aligned} [X_n, X_{n'}] &= (n - n')X_{n+n'}, & [X_n, Y_m] &= \left(\frac{n}{2} - m\right)Y_{n+m} \\ [X_n, M_{n'}] &= -n'M_{n+n'}, & [Y_m, Y_{m'}] &= (m - m')M_{m+m'} \end{aligned} \quad (7)$$

This shows the inclusion $\mathfrak{sch}(1) \subset \mathfrak{sv}(1)$ of the six-dimensional Schrödinger algebra into the infinite-dimensional *Schrödinger-Virasoro algebra* $\mathfrak{sv}(1) = \langle X_n, Y_{n+1/2}, M_n \rangle_{n \in \mathbb{Z}}$ [49–51]. Integrating these infinitesimal transformations gives the Schrödinger-Virasoro Lie group $t \mapsto t'$, $\mathbf{r} \mapsto \mathbf{r}'$ and $\varphi \mapsto \varphi'$ [52–54]. From the X_n , one has,

$$t = \beta(t'), \quad \mathbf{r} = \mathbf{r}' \dot{\beta}(t')^{1/2}, \quad \varphi(t, \mathbf{r}) = \dot{\beta}(t')^{-x/2} \exp \left[-\frac{\mathcal{M} \mathbf{r}'^2 \dot{\beta}(t')}{4 \beta(t')} \right] \varphi'(t', \mathbf{r}') \quad (8)$$

with $\dot{\beta}(t) = d\beta(t)/dt$ and $\beta(t)$ is an arbitrary, but non-decreasing, differentiable function of time. Herein, x and \mathcal{M} , respectively, are the scaling dimension and the mass of φ . From the Y_m , one has

$$t = t', \quad \mathbf{r} = \mathbf{r}' - \alpha(t'), \quad \varphi(t, \mathbf{r}) = \exp \left[\mathcal{M} \left(\frac{1}{2} \dot{\alpha}(t') \cdot \alpha(t') - \mathbf{r}' \cdot \alpha(t') \right) \right] \varphi'(t', \mathbf{r}') \quad (9)$$

with a differentiable vector function $\alpha(t)$ of time. From the M_n , one obtains a time-dependent phase-shift. In contrast to the conformal group, this infinite-dimensional extension is possible for all dimensions $d \geq 1$, with an obvious extension of the commutators (7) [35, 53].

2.2 Schrödinger-invariance of the diffusion equation

The deterministic part of the Edwards-Wilkinson equation reads $\mathcal{S}\varphi = 0$, with the Schrödinger operator $\mathcal{S} = 2\mathcal{M}\partial_t - \Delta_{\mathbf{r}}$. The Schrödinger operator commutes with several elements of $\mathfrak{sch}(1)$:

$$[\mathcal{S}, X_{-1}] = [\mathcal{S}, Y_{\pm 1/2}] = [\mathcal{S}, M_0] = 0 \quad (10)$$

such that the corresponding transformations are symmetries of the operator \mathcal{S} , which takes here a rôle analogously to the hamiltonian with respect to symmetries in quantum mechanics. However, there are two non-vanishing commutators

$$[\mathcal{S}, X_0] = -\mathcal{S}, \quad [\mathcal{S}, X_1] = -2t\mathcal{S} + (2x - 1). \quad (11)$$

If one considers a solution φ_0 of the equation $\mathcal{S}\varphi_0 = 0$, then $\mathcal{X}\varphi_0$ is the transformed solution, with $\mathcal{X} = X_{0,1} \in \mathfrak{sch}(1)$. Now, (11) implies that also $\mathcal{S}(\mathcal{X}\varphi_0) = 0$, but only if the scaling dimension $x = \frac{1}{2}$. We consider ‘*symmetries*’ in a generalised sense: a *dynamical symmetry* is the Lie algebra \mathfrak{g} of transformations $\varphi_0 \mapsto \mathcal{X}\varphi_0$, with $\mathcal{X} \in \mathfrak{g}$, which leave the solution space of $\mathcal{S}\varphi_0 = 0$ invariant.

Generalising to d dimensions, one has: *the free diffusion equation $\mathcal{S}\varphi = 0$ is Schrödinger-invariant, i.e. its space of solutions is invariant under the action of $\mathfrak{sch}(d)$, if the scaling dimension $x = x_\varphi = d/2$ [23, 24, 26].*

2.3 Ward identities

Consider n -point correlation functions

$$C^{(n)} = C^{(n)}(t_1, \dots, t_n; \mathbf{r}_1, \dots, \mathbf{r}_n) = \langle \varphi_1(t_1, \mathbf{r}_1) \dots \varphi_n(t_n, \mathbf{r}_n) \rangle$$

built from scaling operators φ_i . Such a $C^{(n)}$ is *sch-covariant*, if it vanishes under the action of its infinitesimal generators $\mathcal{X}^{[n]}C^{(n)} = 0$ with $\mathcal{X}^{[n]} = \sum_{i=1}^n \mathcal{X}_i$ and \mathcal{X}_i can be any of the generators $\mathcal{X} \in \mathfrak{sch}(d)$ acting on the i^{th} scaling operator φ_i .¹⁰ These *Ward identities* permit to restrict the form of the $C^{(n)}$.

¹⁰ Analogous to correlators of quasi-primary scaling operators in conformal field-theory [16].

2.4 Bargman superselection rule

As an example, we consider the consequences of spatial translation- and Galilei-invariance. The n -body operators are, with $D_i = \partial_{r_i}$ and for $d = 1$

$$Y_{-1/2}^{[n]} = \sum_{i=1}^n [-D_i], \quad Y_{1/2}^{[n]} = \sum_{i=1}^n [-t_i D_i - \mathcal{M}_i r_i]. \quad (12)$$

The Ward-identities $Y_{-1/2}^{[n]} C^{(n)} = Y_{1/2}^{[n]} C^{(n)} = 0$ lead to the differential equations

$$\sum_{i=1}^n \frac{\partial}{\partial r_i} C^{(n)}(t_1, \dots, t_n; r_1, \dots, r_n) = 0, \quad (13a)$$

$$\sum_{i=1}^n \left[t_i \frac{\partial}{\partial r_i} + \mathcal{M}_i r_i \right] C^{(n)}(t_1, \dots, t_n; r_1, \dots, r_n) = 0. \quad (13b)$$

Equation (13a) implies that $C^{(n)} = C^{(n)}(t_1, \dots, t_n; u_1, \dots, u_{n-1})$, with $u_i = r_i - r_n$, which we abbreviate as $C^{(n)}(\{t\}; \{u\})$. Then equation (13b) becomes

$$\sum_{i=1}^{n-1} \left[-(t_i - t_n) \frac{\partial}{\partial u_i} - \mathcal{M}_i u_i \right] C^{(n)}(\{t\}; \{u\}) + r_n (\mathcal{M}_1 + \dots + \mathcal{M}_n) C^{(n)}(\{t\}; \{u\}) = 0.$$

Because of spatial translation-invariance, an explicit dependence on r_n is inadmissible. Hence, the last term must vanish, leading to the *Bargman superselection rule* [47]

$$(\mathcal{M}_1 + \dots + \mathcal{M}_n) C^{(n)}(\{t\}; \{u\}) = 0. \quad (14)$$

2.5 Non-equilibrium field-theory

The Langevin equation (1) can be recast as the equation of motion following from a dynamical functional. Formally, the essential steps are as follows, see [36] for details. Consider the average of an observable \mathcal{A}

$$\langle \mathcal{A} \rangle = \int \mathcal{D}\eta \mathcal{P}[\eta] \int \mathcal{D}\varphi \mathcal{A}[\varphi] \delta \left((2\mathcal{M}\partial_t - \Delta_{\mathbf{r}}) \varphi - \mathcal{V}'[\varphi] - j\varphi - \sqrt{\frac{T}{\mathcal{M}}} \eta \right). \quad (15)$$

Here, $\mathcal{P}[\eta]$ is the distribution of the noise η , assumed Gaussian. This noise represents the average over the initial conditions for phase-ordering and is ‘thermal’ for interface growth. One also uses an integral representation of the Dirac distribution $\delta(x) = (2\pi)^{-1} \int_{\mathbb{R}} d\tilde{\varphi} \exp(i\tilde{\varphi}x)$. Inserting this into the generating functional, the Gaussian integrals over the noises can be carried out and one finally arrives at

$$\langle \mathcal{A} \rangle = \int \mathcal{D}\tilde{\varphi} \mathcal{D}\varphi \mathcal{A}[\varphi] \exp(-\mathcal{J}[\varphi, \tilde{\varphi}]) \quad (16)$$

where the dynamical functional $\mathcal{J}[\varphi, \tilde{\varphi}] = \mathcal{J}_0[\varphi, \tilde{\varphi}] + \mathcal{J}_b[\tilde{\varphi}]$ is naturally decomposed into a ‘deterministic part’ \mathcal{J}_0 and a ‘noise part’ \mathcal{J}_b . These take the form

$$\mathcal{J}_0[\varphi, \tilde{\varphi}] = \int dt d\mathbf{r} \tilde{\varphi} ((2\mathcal{M}\partial_t - \Delta_{\mathbf{r}} - j) \varphi - \mathcal{V}'[\varphi]), \quad \mathcal{J}_b[\tilde{\varphi}] = -T \int dt d\mathbf{r} \tilde{\varphi}^2 - \frac{\Delta_0}{2} \int d\mathbf{r} \tilde{\varphi}_0^2 \quad (17)$$

with $\tilde{\varphi}_0 = \tilde{\varphi}|_{t=0}$. The dynamical functional, or action, $\mathcal{J}[\varphi, \tilde{\varphi}]$ depends on both the order-parameter scaling operator φ and the associated response operator $\tilde{\varphi}$.

Using the Euler-Lagrange equations of motion, derived from the action (17), scaling and response operators are schematically characterised as follows [51]

scaling operator φ :	scaling dimension x	mass $\mathcal{M} > 0$
response operator $\tilde{\varphi}$:	scaling dimension \tilde{x}	mass $\tilde{\mathcal{M}} = -\mathcal{M} < 0$

2.6 Bargman superselection rule, again

If one defines the combined $(n + m)$ -point functions

$$\begin{aligned} C^{(n,m)} &= C^{(n,m)}(t_1, \dots, t_{n+m}; \mathbf{r}_1, \dots, \mathbf{r}_{n+m}) \\ &= \langle \varphi_1(t_1, \mathbf{r}_1) \dots \varphi_n(t_n, \mathbf{r}_n) \tilde{\varphi}_{n+1}(t_{n+1}, \mathbf{r}_{n+1}) \dots \tilde{\varphi}_{n+m}(t_{n+m}, \mathbf{r}_{n+m}) \rangle \end{aligned}$$

the Bargman superselection rule (14) can be reformulated as follows: *the co-variant $(n + m)$ -point function $C^{(n,m)} = 0$ unless $n = m$.* This has immediate consequences:

1. all co-variant correlators $C^{(n,0)}$ must vanish.
2. only response functions $R^{(n)} = C^{(n,n)}$ can be non-vanishing. The most simple example is the two-time auto-response $R(t, s) = C^{(1,1)}(t, s; \mathbf{0}, \mathbf{0}) = \langle \varphi(t, \mathbf{0}) \tilde{\varphi}(s; \mathbf{0}) \rangle = \delta \langle \varphi(t; \mathbf{0}) / \delta j(s; \mathbf{0}) \rangle|_{j=0}$.

2.7 Schrödinger-covariant response functions

The co-variant two-time response function $R(t, s; \mathbf{r}_1 - \mathbf{r}_2) = \langle \varphi(t; \mathbf{r}_1) \tilde{\varphi}(s; \mathbf{r}_2) \rangle$, built from scalar scaling and response operators, obeys the conditions

$$(\partial_t + \partial_s) R = 0 \quad (18a)$$

$$\left(t\partial_t + s\partial_s + \frac{r_1}{2}\partial_{r_1} + \frac{r_2}{2}\partial_{r_2} + \frac{x}{2} + \frac{\tilde{x}}{2} \right) R = 0 \quad (18b)$$

$$\left(t^2\partial_t + s^2\partial_s + tr_1\partial_{r_1} + sr_2\partial_{r_2} + xt + \tilde{x}s + \frac{\mathcal{M}}{2}r_1^2 + \frac{\tilde{\mathcal{M}}}{2}r_2^2 \right) R = 0 \quad (18c)$$

$$(\partial_{r_1} + \partial_{r_2}) R = 0 \quad (18d)$$

$$\left(t\partial_{r_1} + s\partial_{r_2} + \mathcal{M}r_1 + \tilde{\mathcal{M}}r_2 \right) R = 0 \quad (18e)$$

$$(\mathcal{M} + \tilde{\mathcal{M}}) R = 0 \quad (18f)$$

which follow from the Ward identities for $X_{-1}, X_0, X_1, Y_{-1/2}, Y_{1/2}, M_0$, respectively, see Table 2. Spatial rotations were not included explicitly, since for a two-point function built from scalars, any two spatial points can be brought onto a pre-defined line, so that the problem reduces to the case $d = 1$. Their solution follows standard lines, essentially analogous to conformal invariance [16]. From time- and space-translation invariance (18a, 18d), it follows $R = R(\tau, r)$, with $\tau = t - s$ and $r = r_1 - r_2$. As discussed above, Galilei-invariance (18e) produces the Bargman superselection rule $\mathcal{M} + \tilde{\mathcal{M}} = 0$, in agreement with (18f). Then (18b) and (18e) lead to the equations

$$\left(\tau\partial_\tau + \frac{1}{2}r\partial_r + \frac{1}{2}(x + \tilde{x}) \right) R = 0, \quad (\tau\partial_r + \mathcal{M}r) R = 0 \quad (19)$$

whereas the condition (18c) can be simplified, by repeated application of (18b, 18e) to the condition

$$\tau r (x - \tilde{x}) R = 0 \quad (20)$$

which hence produces the constraint $x = \tilde{x}$. The final form can be found from the scaling ansatz $R = \tau^{-(x+\tilde{x})/2} f(r^2/\tau)$, in d spatial dimensions [35, 51]

$$R(t, s; \mathbf{r}) = \langle \varphi(t, \mathbf{r}) \tilde{\varphi}(s, \mathbf{0}) \rangle = \delta_{x, \tilde{x}} \delta(\mathcal{M} + \tilde{\mathcal{M}}) r_0 (t - s)^{-x} \exp \left[-\frac{\mathcal{M}}{2} \frac{\mathbf{r}^2}{t - s} \right] \quad (21)$$

where r_0 is a normalisation constant. The constraint $x = \tilde{x}$ is analogous to conformal invariance [16]. However, the constraint in the masses and the heat-kernel form of the response are specific properties of Schrödinger-covariance.

The Schrödinger-covariant three-point response function is found similarly [51]:

$$\begin{aligned} \langle \varphi_1(t_1, \mathbf{r}_1) \varphi_2(t_2, \mathbf{r}_2) \tilde{\varphi}_3(t_3, \mathbf{r}_3) \rangle &= \delta(\mathcal{M}_1 + \mathcal{M}_2 + \tilde{\mathcal{M}}_3) \exp \left[-\frac{\mathcal{M}_1}{2} \frac{\mathbf{r}_{13}^2}{t_{13}} - \frac{\mathcal{M}_2}{2} \frac{\mathbf{r}_{23}^2}{t_{23}} \right] \\ &\times t_{13}^{-x_{13,2}/2} t_{23}^{-x_{23,1}/2} t_{12}^{-x_{12,3}/2} \Psi_{12,3} \left(\frac{(\mathbf{r}_{13} t_{23} - \mathbf{r}_{23} t_{13})^2}{t_{12} t_{13} t_{23}} \right) \end{aligned} \quad (22)$$

where $t_{ab} = t_a - t_b$, $\mathbf{r}_{ab} = \mathbf{r}_a - \mathbf{r}_b$, $x_{ab,c} = x_a + x_b - x_c$ (replace $x_3 \mapsto \tilde{x}_3$) and $\Psi_{12,3}$ is an arbitrary differentiable function. A similar expression exists for $\langle \varphi_1 \tilde{\varphi}_2 \tilde{\varphi}_3 \rangle$ [35].

2.8 Noisy responses and correlators

The results derived so far are consequences of the dynamical Schrödinger symmetry of the noise-less free diffusion equation. We now show how responses and correlators for noisy diffusion equations can be found.

First, we use the decomposition $\mathcal{J}[\varphi, \tilde{\varphi}] = \mathcal{J}_0[\varphi, \tilde{\varphi}] + \mathcal{J}_b[\tilde{\varphi}]$ of the dynamic action and define a *deterministic average*

$$\langle \mathcal{A} \rangle_0 := \int \mathcal{D}\varphi \mathcal{D}\tilde{\varphi} \mathcal{A}[\varphi, \tilde{\varphi}] e^{-\mathcal{J}_0[\varphi, \tilde{\varphi}]} \quad (23)$$

If the deterministic action \mathcal{J}_0 is Schrödinger-invariant, the deterministic average will obey the Ward identities of the Schrödinger algebra.

Second, the full average is rewritten as follows

$$\begin{aligned} \langle \mathcal{A} \rangle &= \int \mathcal{D}\varphi \mathcal{D}\tilde{\varphi} \mathcal{A}[\varphi, \tilde{\varphi}] e^{-\mathcal{J}[\varphi, \tilde{\varphi}]} \\ &= \int \mathcal{D}\varphi \mathcal{D}\tilde{\varphi} \left(\mathcal{A}[\varphi, \tilde{\varphi}] e^{-\mathcal{J}_b[\tilde{\varphi}]} \right) e^{-\mathcal{J}_0[\varphi, \tilde{\varphi}]} \\ &= \langle \mathcal{A} e^{-\mathcal{J}_b[\tilde{\varphi}]} \rangle_0. \end{aligned} \quad (24)$$

Expanding the noise part of the action and applying the Bargman superselection rule will produce simple reduction formulæ for responses and correlators [48]. We shall illustrate the idea through three examples:

1. The noisy two-time response function is (for brevity, suppress spatial arguments)

$$R(t, s) = \langle \varphi(t) \tilde{\varphi}(s) e^{-\mathcal{J}_b[\tilde{\varphi}]} \rangle_0 = \sum_{k=0}^{\infty} \frac{1}{k!} \langle \varphi(t) \tilde{\varphi}(s) (-\mathcal{J}_b[\tilde{\varphi}])^k \rangle_0 = \langle \varphi(t) \tilde{\varphi}(s) \rangle_0 = R_0(t, s). \quad (25)$$

Here, $R_0(t, s)$ is the noise-less response which was found from Schrödinger-invariance and given explicitly in (21). On the other hand, $R(t, s)$ is the response which is found in an explicit model calculation or in an experiment. Because of the Bargman superselection rules, these two responses are identical. In other words, *the covariant two-time response function does not depend explicitly on the noise*. Certainly, this result does depend on that a decomposition of the dynamical action into a ‘deterministic’ and a ‘noise’ part is possible.¹¹

Therefore, the two-point and three-point responses are given by equations (21, 22) for the noisy Langevin equation (1), if only its deterministic part is Schrödinger-invariant.

2. The noisy two-time correlator is

$$C(t, s) = \langle \varphi(t)\varphi(s)e^{-\mathcal{J}_b[\tilde{\varphi}]} \rangle_0 = \sum_{k=0}^{\infty} \frac{1}{k!} \langle \varphi(t)\varphi(s) (-\mathcal{J}_b[\tilde{\varphi}])^k \rangle_0. \quad (26)$$

Here, two response operators $\tilde{\varphi}$ are needed in order to retain a non-vanishing result and consequently, the detailed structure of \mathcal{J}_b becomes essential. We shall use here the simple form (17). For phase-ordering, where $T = 0$, this gives

$$C(t, s; \mathbf{r}) = \frac{\Delta_0}{2} \int_{\mathbb{R}^d} d\mathbf{R} \langle \varphi(t, \mathbf{r} + \mathbf{r}_0)\varphi(s, \mathbf{r}_0)\tilde{\varphi}^2(0, \mathbf{R}) \rangle_0 \quad (27)$$

while for interface growth, with $\Delta_0 = 0$, we have

$$C(t, s; \mathbf{r}) = T \int_{\mathbb{R}_+ \times \mathbb{R}^d} dud\mathbf{R} \langle \varphi(t, \mathbf{r} + \mathbf{r}_0)\varphi(s, \mathbf{r}_0)\tilde{\varphi}^2(u, \mathbf{R}) \rangle_0 \quad (28)$$

so that in both cases, the correlator is found from an integral of a co-variant three-point response function, in turn given by (22).¹²

3. The single-time correlator $C(t, \mathbf{r}) := C(t, t; \mathbf{r}) = \langle \varphi(t, \mathbf{r})\varphi(t, \mathbf{0}) \rangle$ cannot be read off from (26–28) by simply setting $t = s$. Rather, we must return to the three-point response (22) and perform the limit $t_1 - t_2 \rightarrow 0$ more carefully. We set $\varphi_1 = \varphi_2 = \varphi$, $\tilde{\varphi}_3 = \tilde{\varphi}^2$ such that the scaling dimensions $x_1 = x_2 = x$ and $\tilde{x}_3 = 2\tilde{x}$. The Bargman superselection rule $\mathcal{M}_1 + \mathcal{M}_2 + \tilde{\mathcal{M}}_3 = 2\mathcal{M} - 2\mathcal{M} = 0$ is obeyed. Also let $t_1 = t_2 + \varepsilon$ and $t_3 = u$. For the scaling function, we make the ansatz $\Psi_{12,3}(A) = \Psi_0 A^{-\omega}$ and look for consistency in the $\varepsilon \rightarrow 0$ limit. This gives for $\varepsilon \rightarrow 0$ (with $t > u$ implied)

$$\begin{aligned} \langle \varphi(t + \varepsilon, \mathbf{r}_1)\varphi(t, \mathbf{r}_2)\tilde{\varphi}^2(u, \mathbf{r}_3) \rangle_0 &= \Psi_0 \varepsilon^{\omega - (x - \tilde{x})} \\ &\times (t - u)^{-2\tilde{x}} (\mathbf{r}_1 - \mathbf{r}_2)^{-2\omega} \exp \left[-\frac{\mathcal{M}}{2(t - u)} [(\mathbf{r}_1 - \mathbf{r}_3)^2 + (\mathbf{r}_2 - \mathbf{r}_3)^2] \right]. \end{aligned}$$

The dependence on ε only disappears if $\omega = x - \tilde{x}$. Since $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, we can now insert this in the explicit expression for the single-time correlator. For example, for interface growth we have from (28)

$$C(t, \mathbf{r}) = \frac{T\Psi_0}{(|\mathbf{r}|^2)^{x - \tilde{x}}} \int_0^t du u^{-2\tilde{x}} \int_{\mathbb{R}^d} d\mathbf{R} \exp \left(-\frac{\mathcal{M}}{2u} [(\mathbf{r} - \mathbf{R})^2 + \mathbf{R}^2] \right). \quad (29)$$

¹¹ The precise structure of \mathcal{J}_b need not be exactly the one of (17), but at least it should contain more $\tilde{\varphi}$'s than φ 's for equation (25) to hold.

¹² Notice that pure responses $\langle \tilde{\varphi}_1 \dots \tilde{\varphi}_n \rangle = 0$, as it should be because of causality, see [36].

We see that the scaling function depends not only on the scaling dimension x of the scaling operator φ , but also on the scaling dimension \tilde{x} of the associated response operator $\tilde{\varphi}$.

For phase-ordering, replace T by $\Delta_0/2$, set $u \mapsto t$ and drop the integration over u .

2.9 Tests of Schrödinger-invariance in the Edwards-Wilkinson model

The *Edwards-Wilkinson equation* [38] is the special case of (1) with $\mathcal{V}[\varphi] = 0$. This is exactly solvable and one readily obtains the exact expressions for the height response and correlators, in the frame where $\langle h(t, \mathbf{r}) \rangle = 0$ [55–57]

$$R(t, s; \mathbf{r}) = \left. \frac{\delta \langle h(t, \mathbf{r}) \rangle}{\delta j(s, \mathbf{0})} \right|_{j=0} = r_0 (t-s)^{-d/2} \exp \left[-\frac{\mathcal{M}}{2} \frac{\mathbf{r}^2}{t-s} \right] \quad (30a)$$

$$C(t, s; \mathbf{r}) = \langle h(t, \mathbf{r}) h(s, \mathbf{0}) \rangle = \frac{c_0 T}{|\mathbf{r}|^{d-2}} \left[\Gamma \left(\frac{d}{2} - 1, \frac{\mathcal{M}}{2} \frac{\mathbf{r}^2}{t+s} \right) - \Gamma \left(\frac{d}{2} - 1, \frac{\mathcal{M}}{2} \frac{\mathbf{r}^2}{t-s} \right) \right] \quad (30b)$$

$$C(t, \mathbf{r}) = \bar{c}_0 T |\mathbf{r}|^{-d} \Gamma \left(\frac{d}{2} - 1, \frac{\mathcal{M}}{4} \frac{\mathbf{r}^2}{t} \right) \quad (30c)$$

where $\Gamma(a, x)$ is an incomplete Gamma function [58] and r_0, c_0, \bar{c}_0 are known normalisation constants. In the $T \rightarrow 0$ limit, the correlators indeed vanish, as predicted by the Schrödinger-invariance of the noise-less diffusion equation.

Are the exact expressions (30) compatible with the predictions (21, 28, 29) of Schrödinger-invariance?

1. The exact response (30a) is independent of the ‘temperature’ T , as expected from the Bargman superselection rule and (25). The precise forms of (30a) and (21) completely agree, so that we can identify $1 + a = x = \tilde{x} = d/2$.
2. Turning to the single-time correlator (29), we symmetrise the \mathbf{R} -integration through the shift $\mathbf{R} \mapsto \mathbf{R} + \frac{1}{2}\mathbf{r}$. Expanding the terms in the exponential, we find

$$\begin{aligned} C(t, \mathbf{r}) &= \frac{T\Psi_0}{(|\mathbf{r}|^2)^{x-\tilde{x}}} \int_0^t du u^{-2\tilde{x}} \int_{\mathbb{R}^d} d\mathbf{R} \exp \left[-\frac{\mathcal{M}}{2u} \left[\left(\frac{\mathbf{r}}{2} - \mathbf{R} \right)^2 + \left(\frac{\mathbf{r}}{2} + \mathbf{R} \right)^2 \right] \right] \\ &= \frac{T\Psi_0}{(|\mathbf{r}|^2)^{x-\tilde{x}}} \int_0^t du u^{-2\tilde{x}} \int_{\mathbb{R}^d} d\mathbf{R} \exp \left[-\frac{\mathcal{M}}{4u} \mathbf{r}^2 \right] \exp \left[-\frac{\mathcal{M}}{u} \mathbf{R}^2 \right] \\ &= \frac{T\Psi_0}{(|\mathbf{r}|^2)^{x-\tilde{x}}} \left(\frac{\pi}{\mathcal{M}} \right)^{d/2} \int_0^t du u^{d/2-2\tilde{x}} \exp \left[-\frac{\mathcal{M}}{4} \frac{\mathbf{r}^2}{u} \right] \\ &= T\bar{c}_0 |\mathbf{r}|^{d-2x-2\tilde{x}} \Gamma \left(2\tilde{x} - \frac{d}{2} - 1, \frac{\mathcal{M}}{4} \frac{\mathbf{r}^2}{t} \right) \end{aligned} \quad (31)$$

which agrees with (30c), with the same identifications as above for the response.

3. Finally, the identity of the exact the two-time correlator (30b) with the prediction (28), using the three-point response (22), is shown analogously [55, 56].

Therefore, with the same consistent identification of the scaling dimensions for all three measurable quantities, we find full consistency with the predictions of Schrödinger-invariance for the Edwards-Wilkinson equation. That was the main purpose of

this section: *to go through all steps of the formulation and of the derivation of simple consequences of Schrödinger-invariance, such that tests of Schrödinger-invariance in a simple model can be followed closely.*

These tests of Schrödinger-invariance do not depend on being able to produce an exact solution of the model under study. Indeed, a lattice realisation of the Edwards-Wilkinson universality class is given by the *Family model* [59]: it describes the heights $h_i(t)$ on the sites $i \in \mathcal{L}$ of the (hypercubic) lattice $\mathcal{L} \subset \mathbb{Z}^d$. At each time step, a site i is randomly selected for a deposition attempt. Before depositing a new particle, all sites j in the vicinity of i (usually, one takes the nearest neighbours) are considered and one looks for the site j_{\min} of minimal height: $h_{j_{\min}}(t) \leq h_j(t)$. Then the particle is deposited at the site j_{\min} , i.e. $h_{j_{\min}}(t+1) = h_{j_{\min}}(t) + 1$, and all other $h_i(t)$ are unchanged at this time step. The procedure is repeated for the next time step. A coarse-graining procedure shows that this reproduces the Edwards-Wilkinson equation [60]. A careful simulational study of the Family model, for both $d=1$ and $d=2$, reproduces precisely the exact time-space behaviour of the two-time correlator (30b) [55,56] and thereby confirms the Schrödinger-invariance of the Family model.

2.10 Test of Schrödinger-invariance of the free Gaussian field

For phase-ordering, the predictions (21, 29) of Schrödinger-invariance can be adapted, with the result

$$R(t, s; \mathbf{r}) = \delta_{x, \tilde{x}} \delta(\mathcal{M} + \tilde{\mathcal{M}}) r_0 (t-s)^{-x} \exp\left[-\frac{\mathcal{M}}{2} \frac{\mathbf{r}^2}{t-s}\right] \quad (32a)$$

$$C(t; \mathbf{r}) = \bar{c}_0 |\mathbf{r}|^{-2(x-\tilde{x})} t^{d/2-2\tilde{x}} \exp\left[-\frac{\mathcal{M}}{4} \frac{\mathbf{r}^2}{t}\right]. \quad (32b)$$

If one identifies $x = \tilde{x} = d/4$, these predictions are indeed reproduced from the exact solution of the free Gaussian field [61,62].

3 Ageing-invariance and the spherical and Arcetri models

The predictions (21, 26, 29) are not the final word of local scale-invariance. We now illustrate what can happen with a more general form of the deterministic action \mathcal{J}_0 or the Schrödinger operator \mathcal{S} . Of course, ageing-invariance is not the last word either.

3.1 Ageing algebra

The Schrödinger algebra contains time-translations $X_{-1} = -\partial_t$ and hence can only describe the behaviour of systems at their stationary state. The description of generic systems far from a stationary state requires that at least this generator is dropped. We therefore define the *ageing algebra* $\mathbf{age}(1) = \langle X_{0,1}, Y_{\pm 1/2}, M_0 \rangle$ [35,63]. Moreover,

it turns out that the generators X_n now admit a more general form, namely

$$X_n = -t^{n+1}\partial_t - \frac{n+1}{2}t^n r \partial_r - \frac{x}{2}(n+1)t^n - \xi n t^n - \frac{n(n+1)}{4}\mathcal{M}t^{n-1}r^2 \quad (33a)$$

$$Y_m = -t^{m+1/2}\partial_r - \left(m + \frac{1}{2}\right)\mathcal{M}t^{m-1/2}r \quad (33b)$$

$$M_n = -t^n \mathcal{M} \quad (33c)$$

whereas the generators Y_m and M_n are not modified with respect to the Schrödinger algebra, equation (6). The new feature is that now a *scaling operator* φ is characterised by two independent scaling dimensions $x = x_\varphi$ and $\xi = \xi_\varphi$. We chose the convention that the generator X_0 is unmodified with respect to (6a).¹³ We point out that ξ cannot be re-absorbed into x through a change of variables.

One readily verifies that the commutators (7) also hold for $\mathfrak{age}(1)$. As for the Schrödinger algebra, there is an infinite-dimensional extension, which is called the *ageing-Virasoro algebra* $\mathfrak{av}(1) = \langle X_k, Y_{n+1/2}, M_n \rangle_{k \in \mathbb{N}, n \in \mathbb{Z}}$. It is a true subalgebra of $\mathfrak{sv}(1)$. As for Schrödinger-invariance, the extension to any dimension $d > 1$ is obvious. The rôle of the second scaling dimension ξ becomes more clear when considering a finite transformation $t = \beta(t')$ and $\mathbf{r} = \mathbf{r}'\dot{\beta}(t')^{1/2}$ and

$$\varphi(t, \mathbf{r}) = \dot{\beta}(t')^{-x/2} \left(t' \frac{d \ln \beta(t')}{dt'} \right)^{-\xi} \exp \left[-\frac{\mathcal{M} \mathbf{r}'^2 \dot{\beta}(t')}{4 \beta(t')} \right] \varphi'(t', \mathbf{r}') \quad (34)$$

where $\beta(t)$ is again an arbitrary, but non-decreasing, differentiable function of time, which also obeys the condition $\beta(0) = 0$. Equation (34) replaces equation (8) of Schrödinger-invariance, whereas the transformation (9) remains unchanged.

3.2 Ageing-invariance of a generalised diffusion equation

A first appreciation of the physical relevance of the new representation (33a) comes from the form of the \mathfrak{age} -invariant Schrödinger operator. This operator now takes the form [64–66]

$$\mathcal{S} = 2\mathcal{M}\partial_t - \Delta_{\mathbf{r}} + 2\mathcal{M} \left(x + \xi - \frac{d}{2} \right) t^{-1}. \quad (35)$$

It differs from the Schrödinger operator of free diffusion by the explicitly time-dependent potential term. The non-vanishing commutators of $\mathfrak{age}(d)$ with \mathcal{S} are

$$[\mathcal{S}, X_0] = -\mathcal{S}, [\mathcal{S}, X_1] = -2t\mathcal{S}. \quad (36)$$

Generalising from section 2, we now have: *the space of solutions of the generalised diffusion equation $\mathcal{S}\varphi = 0$, with \mathcal{S} given by (35), is $\mathfrak{age}(d)$ -invariant [64–66].* Note that here no condition, neither on x nor on ξ , has to be imposed.

Diffusion equations $(\partial_t - \Delta_{\mathbf{r}} - V)\varphi = 0$, with time- or space-dependent potentials $V = V(t, \mathbf{r})$, have been studied intensively, and since a long time. For example, the dynamical symmetry algebra for an inverse-square potential $V \sim |\mathbf{r}|^{-2}$ is isomorphic to $\mathfrak{sch}(d)$, a fact already known to Jacobi, along with the case of a free particle [25]. In turn, this is related to the *Fick-Jacobs equation*

$$\partial_t \varphi(t, r) = \nu \frac{\partial}{\partial r} \left[A(r) \frac{\partial}{\partial r} \frac{\varphi(t, r)}{A(r)} \right]$$

¹³ If we add the generator $X_{-1} = -\partial_t$ to the algebra, the commutator $[X_1, X_{-1}] = 2X_0$ would imply $\xi = 0$.

which describes diffusion in a rotation-symmetric channel, of cross-sectional area $A(r)$ [67], with application to diffusion in biological channels or zeolites, e.g. [68]. If $A(r) = A_0 r^{2\mu}$, then one can map the problem onto an inverse-square potential $V(r) = V_0 \mu(\mu - 1)r^{-2}$ [69,70]. Niederer [64] gave a classification of the dynamical symmetry of the diffusion equation with *any* time-space-dependent potential $V = V(t, \mathbf{r})$. Generalised representations of the ageing algebra for Schrödinger operators with an arbitrary time-dependent potential have been found recently [71,72].

We shall see below that a potential $V \sim t^{-1}$ arises naturally in certain models of interface growth or phase-ordering.

3.3 Non-equilibrium field-theory and Bargman selection rules

The dynamical functional $\mathcal{J}[\varphi, \tilde{\varphi}] = \mathcal{J}_0[\varphi, \tilde{\varphi}] + \mathcal{J}_b[\tilde{\varphi}]$ now takes the form

$$\begin{aligned} \mathcal{J}_0[\varphi, \tilde{\varphi}] &= \int dt d\mathbf{r} \tilde{\varphi} \left(\left(2\mathcal{M}\partial_t - \Delta_{\mathbf{r}} - \frac{2\mathcal{M}}{t} \left(x + \xi - \frac{d}{2} \right) - j \right) \varphi - \mathcal{V}'[\varphi] \right) \\ \mathcal{J}_b[\tilde{\varphi}] &= -T \int dt d\mathbf{r} \tilde{\varphi}^2 - \frac{\Delta_0}{2} \int d\mathbf{r} \tilde{\varphi}_0^2. \end{aligned} \quad (37)$$

According to the new representation of $\mathbf{age}(d)$, we have the characterisation [63]

scaling operator φ :	scaling dimensions x, ξ	mass $\mathcal{M} > 0$
response operator $\tilde{\varphi}$:	scaling dimensions $\tilde{x}, \tilde{\xi}$	mass $\tilde{\mathcal{M}} = -\mathcal{M} < 0$

The emergence of the second, independent scaling dimension ξ in non-stationary systems is a new feature, not present in dynamical symmetries of the stationary state, such as conformal or Schrödinger invariance.

The Bargman superselection rules apply as for Schrödinger-invariance. In particular, the average $C^{(n,m)} = 0$ unless $n = m$. The dynamical symmetries of the deterministic part will therefore fix response functions.

3.4 Ageing-covariant response functions

In order to find ageing-covariant response function, one might again write down the Ward identities. However, it is more simple to rewrite the transformation (34), generated by the X_n , as follows: $t = \beta(t')$, $\mathbf{r} = \mathbf{r}'\beta(t')^{1/2}$ and

$$t^{-\xi} \varphi(t, \mathbf{r}) = \dot{\beta}(t')^{-(x+2\xi)/2} \exp \left[-\frac{\mathcal{M}\mathbf{r}'^2 \dot{\beta}(t')}{4 \beta(t')} \right] t'^{-\xi} \varphi'(t', \mathbf{r}'). \quad (38)$$

Hence, if one sets $\varphi(t, \mathbf{r}) = t^\xi \Phi(t, \mathbf{r})$, the scaling operator Φ is Schrödinger-covariant, with the scaling dimension $x + 2\xi$ [63]. The transformations from $Y_{n+1/2}$ and M_n are unchanged.¹⁴

The \mathbf{age} -covariant two-point response can be read from (21):

$$\begin{aligned} R(t, s; \mathbf{r}) &= \langle t^\xi \Phi(t, \mathbf{r}) s^{\tilde{\xi}} \tilde{\Phi}(s, \mathbf{0}) \rangle \\ &= \delta_{x+2\xi, \tilde{x}+2\tilde{\xi}} \delta(\mathcal{M} + \tilde{\mathcal{M}}) r_0 s^{-(x+\tilde{x})/2} \left(\frac{t}{s} \right)^\xi \left(\frac{t}{s} - 1 \right)^{-(x+2\xi)} \exp \left[-\frac{\mathcal{M}}{2} \frac{\mathbf{r}^2}{t-s} \right]. \end{aligned} \quad (39)$$

A similar generalisation is read from (22) for the three-point response.

¹⁴ One may extend these transformations to the entire Schrödinger (-Virasoro) group, but then time-translations are generated by $X_{-1} = -\partial_t + \xi t^{-1}$ and also change the scaling operator φ . Such modifications of X_{-1} also apply to more general potentials [71].

3.5 Noisy responses and correlators

The Bargman superselection rules are the same as those for Schrödinger-invariance. Therefore, equations (25, 27, 28) can be taken over for ageing-invariance as well. In particular, the two-time response does not explicitly depend on the noise. Comparing with the scaling form (3, 4) and the explicit expression (39), ageing-invariance predicts

$$\begin{aligned} R(t, s; \mathbf{r}) &= R(t, s) \exp \left[-\frac{\mathcal{M}}{2} \frac{\mathbf{r}^2}{t-s} \right] \\ R(t, s) &= r_0 s^{-1-a} \left(\frac{t}{s} \right)^{1+a'-\lambda_R/2} \left(\frac{t}{s} - 1 \right)^{-1-a'} \end{aligned} \quad (40)$$

where the three independent exponents a , a' , λ_R are related to the three independent scaling dimensions x, \tilde{x}, ξ by

$$1 + a = \frac{1}{2}(x + \tilde{x}), \quad 1 + a' - \lambda_R/2 = \xi, \quad 1 + a' = x + 2\xi. \quad (41)$$

Notice that because of the constraint $x + 2\xi = \tilde{x} + 2\tilde{\xi}$, the difference $a' - a = \xi + \tilde{\xi}$ measures the contribution of the second scaling dimensions.

Noisy correlators can be derived, analogously to Schrödinger-invariance, from the three-point response functions, see [35] for details.

3.6 Spherical model of a ferromagnet and Arcetri model of an interface

The *spherical model* [73–75] is a widely studied, classical model for magnetic ordering, which has a critical temperature $T_c > 0$ for dimensions $d > 2$. Its collective properties are distinct from mean-field behaviour for $d < 4$, yet it remains exactly solvable in all dimensions. Its dynamical variables are no longer discrete Ising spins $s_i = \pm 1$, both rather continuous ‘spin’ variables $s(t, \mathbf{r}) \in \mathbb{R}$ which obey the *spherical constraint* $\int d\mathbf{r} \langle s^2(t, \mathbf{r}) \rangle = 1$. Its dynamics is given by the Langevin equation [76]

$$\partial_t s(t, \mathbf{r}) = \Delta_{\mathbf{r}} s(t, \mathbf{r}) + \mathfrak{z}(t) s(t, \mathbf{r}) + (2T)^{1/2} \eta(t, \mathbf{r}) \quad (42)$$

where η is a standard white noise of unit variance and $\mathfrak{z}(t)$ is a Lagrange multiplier to ensure the spherical constraint, at all times. The solution is conveniently specified in terms of the function $g(t) = \exp \left[-2 \int_0^t d\tau \mathfrak{z}(\tau) \right]$. Because of the spherical constraint, it obeys a Volterra integral equation [77]

$$g(t) = f(t) + 2T \int_0^t d\tau f(t-\tau)g(\tau) \quad (43)$$

where $f(t) = (e^{-4t} I_0(4t))^d$, in the case of a totally disordered initial state (I_0 is a modified Bessel function [58]). If $T \leq T_c$, the long-time behaviour $g(t) \sim t^F$ is found [77]. For example, phase-ordering occurs at $T = 0$, when $F = -d/2$ (for $d > 2$, this remains true for all temperatures $T < T_c$). Critical dynamics occurs if $T = T_c$. If $d > 4$, one finds $F = 0$ and one is back to a free Gaussian field. If $2 < d < 2$, one has $F = d/2 - 2$. Taking the logarithm and then the derivative, this implies

$$\mathfrak{z}(t) \stackrel{t \rightarrow \infty}{\simeq} -\frac{F}{2} \frac{1}{t} + o(t^{-1}) \quad (44)$$

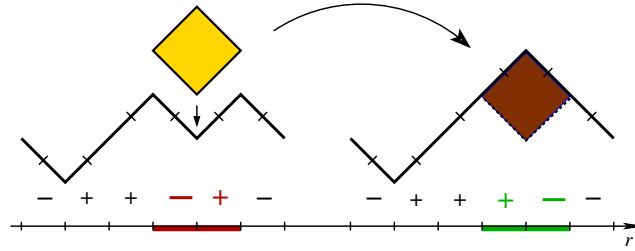


Fig. 2. Interface with the RSOS constraint $h_{i+1/2} - h_{i-1/2} = \pm 1$. A particle (left, golden) can be adsorbed (right, brown) if the RSOS condition is still satisfied after the adsorption. The evolution of the corresponding local slopes $u_i = h_{i+1/2} - h_{i-1/2}$ is also indicated.

for large times. Hence the deterministic part of the Langevin equation (42), for $T \leq T_c$, is an example of the ageing-invariant Schrödinger operator (35). Small-time corrections to (44) will merely generate corrections to the leading scaling behaviour.

The *Arcetri model* [78] is an analogue of the spherical model, adapted to interface growth. Many lattice models of interface growth are specified in terms of RSOS models [79], see Figure 2, such the local slopes $u_i = h_{i+1/2} - h_{i-1/2} = \pm 1$. We relax this condition to a ‘spherical constraint’ $\int d\mathbf{r} u^2(t, \mathbf{r}) = d$ and write down the defining Langevin equation

$$\partial_t u(t, \mathbf{r}) = \Delta_{\mathbf{r}} u(t, \mathbf{r}) + \mathfrak{z}(t) u(t, \mathbf{r}) + (2T)^{1/2} \partial_r \eta(t, \mathbf{r}) \quad (45)$$

in terms of the slopes $u = \partial_r h$. The Lagrange multiplier is analysed as in the spherical model. Again, we find the Volterra equation (43), but now with $f(t) = (e^{-4t} I_0(4t))^{d-1} e^{-4t} I_1(4t)/(4t)$, for a flat initial substrate. There is a critical point with $T_c > 0$ for all $d > 0$. For long times, we have $g(t) \sim t^F$: for $T < T_c$, $F = -1 - d/2$ and for $T = T_c$, $F = d/2 - 1$ if $d < 2$ but $F = 0$ for $d > 2$. Therefore, for $d > 2$, the Arcetri model at $T = T_c$ reduces to the Edwards-Wilkinson model. On the other hand, for $d < 2$, the stationary exponents $z = 2$ and $\beta = (2 - d)/4$ are the same as for the Edwards-Wilkinson model, but the non-stationary exponents $\lambda_C = \lambda_R = 3d/2 - 1$ are distinct from $\lambda_C^{\text{EW}} = \lambda_R^{\text{EW}} = d$. This is an elementary example to illustrate the independence of λ_C, λ_R from the stationary exponents, predicted long ago from field-theory [36, 61, 80]. For $T \leq T_c$, equation (44) applies again and the Arcetri model is an example of ageing-invariant interface growth.

3.7 Tests of ageing-invariance

Finally, we compare the prediction (40) of ageing invariance with the exact solutions of the time-space response $R(t, s; \mathbf{r})$ of the spherical and Arcetri models. We find perfect agreement and extract the four scaling dimensions, as well as the three phenomenological exponents a, a', λ_R . This is listed in Table 3, where we add as well the corresponding result of the magnetic response of the 1D Ising model with Glauber dynamics, at $T = 0$ [81, 82] and also those of the Gaussian theory of phase-ordering [83], also known as Ohta-Jasnow-Kawasaki (OJK) approximation. Several comments are in order:

1. For both the critical spherical and Arcetri models and above their upper critical dimension $d > d^*$, simple Schrödinger-invariance is enough to reproduce the autoresponse. On the other hand, if fluctuation effects do become important, ageing-invariance with its second independent scaling dimension is necessary.

Table 3. Scaling dimensions and exponents of the ageing of the autoresponse $R(t, s)$ in some exactly solved models, with $z = 2$.

Model	Conditions	x	ξ	\tilde{x}	$\tilde{\xi}$	a	a'	λ_R
spherical	$T < T_c$	0	$d/4$	d	$-d/4$	$d/2 - 1$	$d/2 - 1$	$d/2$
	$T = T_c$ $d < 4$	$d - 2$	$1 - d/4$	2	$-1 + d/4$	$d/2 - 1$	$d/2 - 1$	$3d/2 - 2$
	$T = T_c$ $d > 4$	$d/2$	0	$d/2$	0	$d/2 - 1$	$d/2 - 1$	d
Arcetri	$T < T_c$	-1	$(d + 2)/4$	$d + 1$	$-(d + 2)/4$	$d/2 - 1$	$d/2 - 1$	$d/2 - 1$
	$T = T_c$ $d < 2$	$d - 1$	$(2 - d)/4$	1	$(d - 2)/4$	$d/2 - 1$	$d/2 - 1$	$3d/2 - 1$
	$T = T_c$ $d > 2$	$d/2$	0	$d/2$	0	$d/2 - 1$	$d/2 - 1$	d
Ising	$T = 0$ $d = 1$	$1/2$	0	$3/2$	$-1/2$	0	$-1/2$	1
Gaussian theory (OJK)		$d/2$	0	$1/2$	$(d - 1)/4$	$(d - 1)/2$	$(d - 2)/2$	$d/2$

- Since the Langevin equations of the spherical and Arcetri models are linear, the potentials $V(t)$ in the equations of motion of φ and $\tilde{\varphi}$ differ by a sign. Hence $\xi + \tilde{\xi} = 0$ in these two models, which implies $a = a'$.
- The examples of the 1D Glauber-Ising model at $T = 0$, and of the OJK-Gaussian theory, shows that although the equation of motion of the field φ is not linear, it should still transform under the representation (33) of the ageing algebra, but with $\xi + \tilde{\xi} \neq 0$. Then, indeed, $a \neq a'$.
- The analysis of extended scaling symmetries of any more general model must begin with the identification of a ‘deterministic part’ in the Langevin equation and a construction of its symmetry algebra. The predictions (21, 40) are valid for certain Langevin equations only and cannot always be applied to any other model in a straightforward manner.

We refer to the literature for detailed accounts of tests of LSI through the correlators [35]. For example, in the 2D and 3D Ising models, undergoing phase-ordering after a quench to $T \ll T_c$, the exponential spatial dependence of $R(t, s; \mathbf{r})$ in (21) has been confirmed in detail [84]. Two-time correlators in Ising and Potts models undergoing phase-ordering have also been studied in great detail and the predictions of Schrödinger-invariance have been largely confirmed [85, 86].¹⁵ A general result of ageing-invariance is the scaling relation $\lambda_C = \lambda_R$ [48], which had been derived before by other means [30], and has been confirmed numerically many times. One important feature in these tests is that in analysing lattice simulations, one cannot simply use uncorrelated initial data, as in the actions (17, 37), but some phenomenological information about the shape of the equal-time correlator at the onset of the scaling regime must be provided [35]. Our new results on the equal-time correlators, outlined in section 2, might improve the situation.

In most models, the dynamical exponent $z \neq 2$ and neither Schrödinger- nor ageing-invariance can be applied directly. However, if one restricts attention to the auto-response $R(t, s; \mathbf{0})$, then the value of the dynamical exponent only enters through the combination λ_R/z . Therefore, the form of the autoresponse can be predicted successfully from ageing-invariance, including many instance of non-equilibrium critical dynamics. For example, numerical simulations of the critical Glauber-Ising model suggest $a' - a = -0.17(2)$ for $d = 2$ and $a' - a = -0.022(5)$ for $d = 3$ [35, 63]. Studies of this kind must look very precisely into the region $t/s \gtrsim 1$ of the scaling variable, which

¹⁵ For phase-ordering in the 2D Ising model, there is a bound $\lambda_C \leq 5/4$ [87]. Numerical data for λ_C in phase-ordering 2D Ising and various Potts models usually fall slightly below this. Interestingly, the value $\lambda_C = 1.25(2)$ is also found in the ageing of a 2D collapsing homopolymer model, and in agreement with the bounds $\nu_F d - 1 \leq \lambda_C \leq 2(\nu_F d - 1)$, where ν_F is the Flory exponent [88, 89].

requires high-precision data on huge lattices [90]. An open problem is the elaboration of dynamical renormalisation-group schemes which take into account that $a' - a \neq 0$ is possible, in contrast to what happens at the stationary state.

Finishing with a last outlook onto interface growth, the most-studied paradigm is the universality class of the Kardar-Parisi-Zhang (KPZ) equation [91]

$$\partial_t h(t, \mathbf{r}) = \nu \Delta_{\mathbf{r}} h(t, \mathbf{r}) + \frac{\mu}{2} \left(\frac{\partial h(t, \mathbf{r})}{\partial \mathbf{r}} \right)^2 + (2\nu T)^{1/2} \eta(t, \mathbf{r}) \quad (46)$$

which is obtained as the continuum limit of the RSOS model [79] sketched in Figure 2. In the growth regime, it does undergo ageing, quite analogous to the Edwards-Wilkinson and Arcetri models. However, its autoresponse function cannot be described in terms of the representation (33) of the ageing algebra. From a phenomenological point of view, a better approximation appears to be ‘logarithmic representations’, which essentially assume that the scaling operator φ acquires a ‘logarithmic partner’ ψ to form a doublet. Formally, one may treat this by considering the two scaling dimensions x, ξ as matrices. This leads to the form $R(t, s) = s^{-1-a} f_R(t/s)$, where

$$f_R(y) = y^{-\lambda_R/z} \left(1 - \frac{1}{y} \right)^{-1-a'} \left[h_0 - g_0 \ln \left(1 - \frac{1}{y} \right) - f_0 \ln^2 \left(1 - \frac{1}{y} \right) \right] \quad (47)$$

and where the exponent a' and the amplitudes h_0, g_0, f_0 must be fitted to the data [92]. At present, four universality classes are known where the prediction (40) of ageing invariance is no longer enough, but where (47) describes the data well in the entire region where dynamical scaling is found: KPZ for $d = 1$ [46] and very recently also for $d = 2$ [93, 94], critical directed percolation for $d = 1$ [92] and the critical 2D Glauber-Ising model [95]. It appears that the doublet structure only remains in the second scaling dimension $\tilde{\xi}$ of the response operator. Taking the logarithmic terms in (47) into account leads to improved precision in estimates of the exponent λ_R , which is important in order to establish whether the equality $\lambda_C = \lambda_R$ might hold in these models, a question under active discussion [96, 97].

Any substantial further progress will likely require a dynamical symmetry capable to predict the full time-space response for a dynamical exponent $z \neq 2$ or $z \neq 1$, which remains a difficult open problem.

This article is dedicated to Wolfhard Janke at the occasion of his 60th birthday.

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