Eur. Phys. J. Special Topics **226**, 443–453 (2017)

DOI: 10.1140/epjst/e2016-60178-1

THE EUROPEAN PHYSICAL JOURNAL SPECIAL TOPICS

Regular Article

Fairy circles and their non-local stochastic instability

Miguel Angel Fuentes[1,2,a] and Manuel O. Cáceres[3]

[1] Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, New Mexico, 87501, USA
IIF-Sadaf, Bulnes 642, 1428, Buenos Aires, Argentina
[2] Facultad de Ingeniería y Tecnología, Universidad San Sebastián, Lota 2465, Santiago 7510157, Chile
[3] Centro Atómico Bariloche (CNEA), Instituto Balseiro (Uni. Nac. Cuyo), and CONICET, Av. E. Bustillo 9500, CP 8400, Bariloche, Argentina

Received 10 June 2016 / Received in final form 26 September 2016
Published online 6 March 2017

Abstract. We study *analytically* a non local stochastic partial differential equation describing a fundamental mechanism for patterns formation, as the one responsible for the so called fairy circles appearing in two different bio-physical scenarios; one on the African continent and another in Australia. Using a stochastic multiscale perturbation expansion, and a minimum coupling approximation we are able to describe the life-times associated to the stochastic evolution from an unstable uniform state to a patterned one. In this way we discuss how two different biological mechanisms can be collapsed in one analytical framework.

1 Introduction

The recent discovery of fairy-circle patterns in the remote outback of Australia [1] shows the same geometrical characteristics of the well know patterns in Africa [2] and open up the discussion about the fundamental causes of these mysterious patterns.

So-called Fairy circles are circular patches of land barren of any vegetation encircled by a ring of grass. These patterns vary between 4 to 13 meters [3]. Until recently, the phenomenon was only observed in the arid grasslands of the Namib desert, South Africa. In 2014, ecologists were alerted to similar rings of vegetation outside of Africa [1], in a remote region of the Pilbara, Western Australia.

Biological phenomena that drive these surprising patterns are driven by different mechanisms, mainly related with biomass-water feedback [4,5]. While Namibia's porous soils rainwater diffuses and plants will experience an intense competition for water forming finally a patch of bare soil, in Australia the hard clay form a crust-like terrain, forcing the water to flow to nearest plants. This flow reduces the seedling establishment, and finally produces the particular patterns shown in the fairy circles.

[a] e-mail: fuentesm@santafe.edu

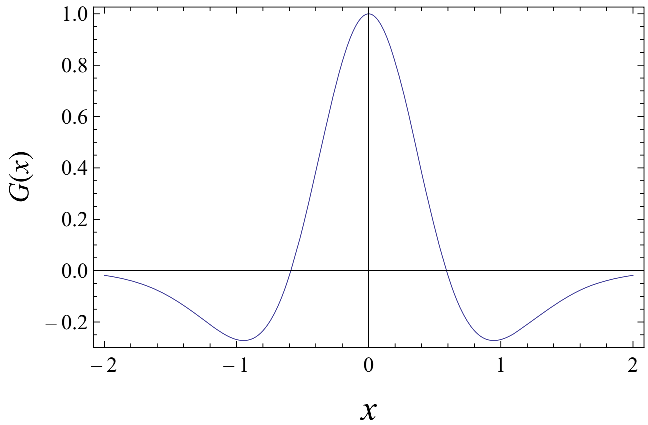

Fig. 1. Typical migration kernel $G\,(x)$ showing two regimes. The positive effect near the origin and a negative effect due to external competition at certain range. Here we chose $B = 2$, $A = 3$ in equation (12).

In order to understand these structures, several models have been applied, most of them related with Turing's mechanisms [6], showing pattern morphologies changing from labyrinths to spots [7–9].

Even though the ecological system appears to be completely different [1], common characteristics of the geometrical structures appearing in Africa and Australia could refer to certain universal principle present in the bio-physical mechanisms behind the phenomena. In this work we explore the possibility of modeling this universal principle throughout a minimal fundamental model in which classical birth/decay mechanism is complemented by non-local migration and stochastic effects [10–12]. Then using the framework of stochastic partial integro-differential equations, we will discuss how a instability can appear unifying these two systems under a same equation.

In the next section we will introduce the mathematical model, then we will discuss the appearance of the pattern. Using a minimum coupling approximation and a multiple scale expansion, we will obtain analytical results of relevant quantities like the first passage time distribution, mean first passage time, etc.

2 The mathematical model

The model shown in the next equations, takes into account a local exponential growth of the population, and a competition term, also local. This part of the model defines a classical Lotka-Volterra-like dynamics. We denote it with a general non-linear function $f[u(x,t)]$, been $u(x,t)$ the density of the biomass.

Subsequently, we introduce the non-local interactions, where a normalized kernel function $G\,(x)$ models the two mechanisms for the creation of these particular patterns: uptake-diffusion feedback and infiltration feedback [1,2]. The fundamental aspect of this non-locality is that at short range, there is a positive feedback mechanism; while at long distances there is a negative one, that finally acts against seedling establishment (see Fig. 1 and Fig. 2). The use of this type of non-local mechanisms is typically used in ecology, after the pioneer research of Volterra in its classical work [13].

The model ends with a Gaussian delta-correlated white noise that trigger the patterns stochastically (note that in the critical state the fluctuation are very important for the appearance of the final structures). The noise field, in space and

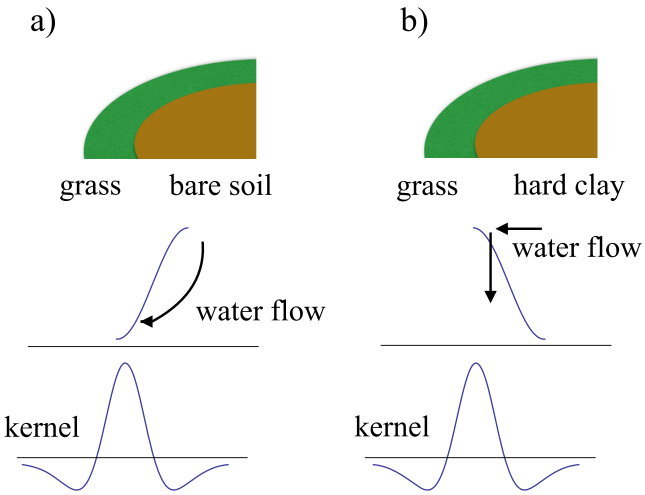

Fig. 2. (a) Namibia's porous soils rainwater diffuses, forming a patch of bare soil at the location where competition for water is most intense. (b) Australia's hard clay, the overland-water flow acts against seedling establishment. In both figures, the black arrows sketch the water (nutrients) flow in the two scenarios.

time, corresponds to $\xi(x,t)$ in the dynamical equation. This is a plausible ansatz when the unspecified random contributions are more important at low density, see Appendix 3 in [14]. We characterize the strength of the noise with a small parameter ϵ.

Thus, to begin with our mathematical analysis we will write down the general one dimensional model with the explicit properties mentioned before, for the kernel and the stochastic field

$$\frac{\partial u(x,t)}{\partial t} = f[u(x,t)] + \sqrt{\epsilon}\xi(x,t) + g[u(x,t)] \int_{-\infty}^{\infty} u\left(x-x',t\right) G(x')dx', \tag{1}$$

$$1 = \int_{-\infty}^{\infty} G(x)dx \tag{2}$$

$$\langle \xi(x,t) \rangle = 0, \tag{3}$$

$$\langle \xi(x,t)\xi(x',t') \rangle = \delta(x-x')\delta(t-t'). \tag{4}$$

Here $g[u(x,t)]$ can be also a general non-linear function, see for example [15].

The homogeneous stationary state comes from the deterministic analysis

$$0 = f\left[u_0\right] + g(u_0)u_0.$$

Doing a linear analysis around the nontrivial stable point u_0, we can use the perturbation

$$u(x,t) \simeq u_0 + v(x,t), \tag{5}$$

and arrive to the following equation.

$$\frac{\partial v(x,t)}{\partial t} = f'(u_0)v(x,t) + u_0 g'(u_0)v(x,t) + g(u_0) \int_{-\infty}^{\infty} v\left(x-x',t\right) G(x')dx'. \tag{6}$$

Introducing a Fourier wave for the perturbation around u_0, i.e.,: $v(x,t) \propto \exp(\varphi t + ikx)$. The equation for the growth/decay rate of the Fourier modes (the so-called dispersion relation) is given by

$$\varphi = f'(u_0) + g'(u_0)u_0 + g(u_0)G(k), \tag{7}$$

with

$$G(k) = \int_{-\infty}^{+\infty} e^{ikx} G(x)dx. \tag{8}$$

As we mention above, it is our intention to introduce the *minimal fundamental model* that shows, with elegance and simplicity, a general migration mechanism with a Lotka-Volterra dynamics. Then, for the general non-linear functions we use

$$f[u(x,t)] = u(x,t) - bu(x,t)^2, \tag{9}$$

$$g[u(x,t)] = D. \tag{10}$$

With these features the dispersion relation becomes

$$\varphi = DG(k) - 2D - 1. \tag{11}$$

2.1 The model for the migration kernel

For the case treated in this manuscript, i.e., an ecological model with two distinct regimes at different length scales, the kernel must contains on the one hand a positive region which takes into account the opportunities and lack of strong competition near the origin, and on the other a negative effect which model the competition effect outside a certain range [16], as it is schematically shown in Figure 2. This nonlocal migration is well described by the kernel shown in Figure 1, which analytically can be modelled by the function

$$G(x) = \mathcal{N}(Be^{-Ax^2} - e^{-x^2}). \tag{12}$$

Here, $\mathcal{N}$ is a normalization factor

$$\mathcal{N}^{-1} = \sqrt{\pi}\left(\frac{B}{\sqrt{A}} - 1\right). \tag{13}$$

Notice that $G(x)$ is not a probability. If this function is positive, its contribution is asymptotically equivalent to the Laplacian operator.

We will use this particular function in this work, but the general results does not depend on the specific functional form of $G(x)$, but on the positive and negative regions as shown in Figure 1. The Fourier transform of this kernel is the following

$$G(k) = \mathcal{N}\sqrt{\pi}\left(\frac{B\exp\left(-k^2/4A\right)}{\sqrt{A}} - e^{-k^2/4}\right), \tag{14}$$

in Figure 3 we show this function, $G(k)$, with parameters $B = 2$, $A = 3$.

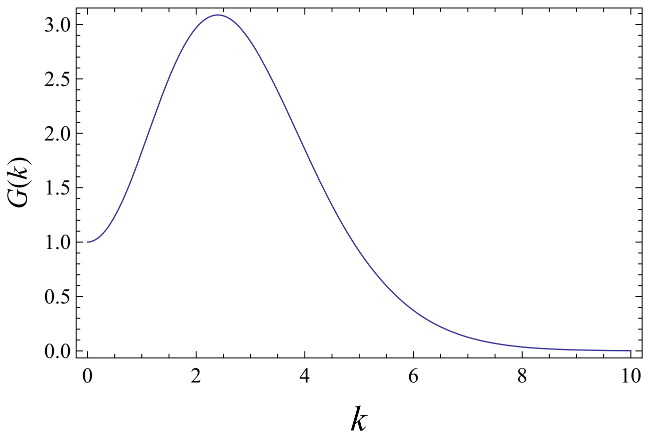

Fig. 3. Fourier transform of the nonlocal kernel $G\,(x)$ for $B = 2$, $A = 3$.

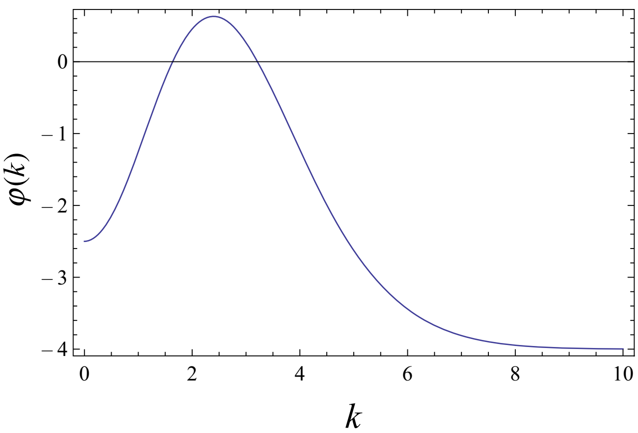

Fig. 4. Dispersion relation φ, see equation (11) for $D = 1.5$ and $G(k)$ from Figure 3, as a function of Fourier number k. The instability of the homogeneous mode occurs when $\varphi\,(k) > 0$.

With all these characteristics, and noticing that $G(k)$ has a peak with a positive value, the dispersion relation φ will show an instability if

$$G(k) > 2 + \frac{1}{D}, \tag{15}$$

as it is shown in Figure 4 ($\varphi > 0$).

As can be seen, a model which contains the shown fundamental interactions can produce a selection in the mode for the occurrence of structures of the system. These interactions are: the local growth and competition, and the nonlocal migration taking into account the surrounding environment, see Figures 1 and 2. Finally fluctuations are included, which are very important for the trigger mechanism from the unstable homogeneous stationary state. In the next section we will present in detail an *analytical procedure* allowing to obtain the minimal set of equations that decrease drastically the dimension of the mathematical problem presented in the full stochastic integro–differential equation of the model.

3 The stochastic pattern formation

Due to the instability, around the fully populated state u_0, a nonhomogeneous pattern may occur if it is triggered by fluctuations. To study this phenomenon we introduce, now explicitly, a finite system with periodic boundary conditions on $x \in [-1, 1]$. Thus we will use – from now on – a discrete Fourier transform ($k_n = n\pi, n = 0, \pm 1, \pm 2, \cdots$) characterized by the expansion in modes

$$u(x,t) = \sum_{n=-\infty}^{\infty} A_n(t) \exp(in\pi x), \tag{16}$$

$$\xi(x,t) = \sum_{n=-\infty}^{\infty} \xi_n(t) \exp(in\pi x), \tag{17}$$

$$G(x) = \sum_{n=-\infty}^{\infty} G_n \exp(in\pi x), \tag{18}$$

$$2\delta_{n,l} = \int_{-1}^{1} e^{i\pi(n-l)x} dx, \tag{19}$$

$$G_n = \frac{1}{2} \int_{-1}^{1} G(x) e^{-in\pi x} dx, \tag{20}$$

and similar inverse relations for $A_n(t)$, $\xi_n(t)$. We note here the relation that exists among the discrete and continuous Fourier transform:

$$2G_n = G(k). \tag{21}$$

Introducing (16)–(18) in the field equation (1) and using (19) we arrive to an infinite set of stochastic coupled Fourier modes. It is worth mentioning that this set of equations is completely equivalent to the nonlocal stochastic partial differential equation (1).

$$\dot{A}_l = A_l(t)(1 + 2G_l D) - b \sum_{n=-\infty}^{\infty} A_n(t) A_{l-n}(t) + \xi_l(t), \tag{22}$$

$$G_l = G_{-l}, \tag{23}$$

$$\langle \xi_l \rangle = 0, \tag{24}$$

$$\langle \xi_l(t) \xi_n(t') \rangle = \delta_{n+l,0}\ \delta(t - t'). \tag{25}$$

Obviously, due to fluctuations, only in mean value we can consider that the modes A_l are symmetric.

In order to study the emerge of the pattern formation from this infinite set of coupled modes, we can introduce a Minimum Coupling Approximation (MCA) considering only the coupling between A_0, $A_{\pm e}$, been this last mode the one that shows a positive dispersion relation.

Then keeping this approximation in mind we arrive to the set of MCA equations

$$\frac{dA_0}{dt} = A_0(1 + 2G_0 D) - bA_0^2 - 2bA_e^2 + \sqrt{\epsilon}\xi_0(t), \tag{26}$$

$$\frac{dA_e}{dt} = A_e(1 + 2G_e D - 2bA_0) + \sqrt{\epsilon}\xi_e(t), \tag{27}$$

in this approximation $A_e = A_{-e}$. The MCA is a good approximation during the initial state of the pattern formation and it is enough to predict the random escape time [17].

The stochastic modes introduced in the MCA can be analyzed introducing a multiple scale expansion. Due to the fact that originally the initial state is the fully populated state u_0, we consider that $A_n(0) = 0, \forall n \neq 0$. Then we can propose an expansion for the homogeneous and the explosive modes in the form

$$A_0(t) = H_0 + H_1\sqrt{\epsilon} + H_2\epsilon + \cdots, \quad H_0 = \text{const.}, \tag{28}$$

$$H_j(0) = 0, \forall j \geq 1,$$

$$A_e(t) = 0 + E_1\sqrt{\epsilon} + E_2\epsilon + \cdots, \tag{29}$$

$$E_j(0) = 0, \forall j \geq 1,$$

and then we introduce these expansions in (26) and (27). Therefore to $\mathcal{O}(\epsilon^0)$ we get

$$0 = H_0(1+D) - bH_0^2 \;\Rightarrow H_0 = (1+D)/b,$$

and to $\mathcal{O}(\epsilon^{1/2})$ we get

$$\frac{dH_1}{dt} = H_1(1+D) - bH_0H_1 + \xi_0, \tag{30}$$

$$\Rightarrow H_1(t) = \int_0^t \xi_0(s)ds, \tag{31}$$

$$\frac{dE_1}{dt} = E_1(1 + 2G_eD - 2(1+D)) + \xi_e, \tag{32}$$

$$\Rightarrow E_1(t) = \int_0^t e^{\varphi(t-s)}\xi_e(s)ds, \tag{33}$$

here $\varphi = 2G_eD - 1 - 2D$ is the dispersion relation in the discrete Fourier representation.

The perturbation $H_1(t)$ is a Wiener process with mean value zero and it is statistically independent of the process $E_1(t)$, this last process can be re-written in the form

$$H_1(t) = e^{\varphi t}h(t), \tag{34}$$

with

$$\frac{dh}{dt} = e^{-\varphi t}\ \xi_e(t). \tag{35}$$

Note that the process $h(t)$ is Gaussian with mean zero and dispersion

$$\langle h(t)^2\rangle = \frac{1 - e^{-2\varphi t}}{2\varphi}. \tag{36}$$

Therefore for times $2\varphi t \gg 1$ the process $h(t)$ saturates and it is described by a Gaussian random variable Ω, which is characterized by the pdf

$$P(\Omega) = \frac{e^{-\Omega^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}, \tag{37}$$

$$\sigma^2 = \frac{1}{2\varphi}, \tag{38}$$

$$\varphi = 2G_eD - 1 - 2D.$$

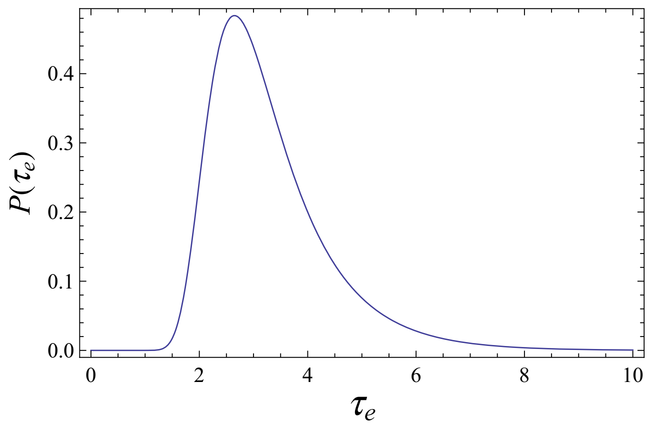

Fig. 5. First passage time distribution, equation (43), as a function of the dimensionless time $\tau_e = \varphi t_e$ for $K = 10$ (the universal group parameter of the model).

With all this information we can finally calculate the lifetime around the unstable state u_0, and therefore to study the random times for pattern formation we were talking about. In order to do this we need to find the first passage time distribution, for the process $A_e(t)$, from the value 0 to some macroscopic (threshold) value A_e^*. We will do these calculations in the next section.

4 The first passage time distribution

The solution of $H_1(t)$, see (34)–(35), can be used as the mapping to define a random escape times associated to the unstable mode $A_e(t)$ to pass from the initial condition to a given threshold value, i.e.: $0 \to A_e^*$.

If $t \gg 1/2\varphi$ we can approximate $h(t)$ by the random variable Ω, then we get a map for the random escape times t_e as a function of the random variable Ω

$$H_1^* \equiv H_1(t_e) = \exp\left(\varphi t_e\right)\ \Omega, \tag{39}$$

$$\Rightarrow t_e = \frac{1}{2\varphi}\ln\left(\frac{H_1^*}{\Omega}\right)^2. \tag{40}$$

Now using the scaling (29)–(30) we write

$$t_e = \frac{1}{2\varphi}\ln\left(\frac{A_e^*}{\sqrt{\epsilon}\Omega}\right)^2, \tag{41}$$

which can be used to calculate the lifetime as the mean value $\langle t_e \rangle$.

In addition, from this map, the first passage timedistribution can also be calculated as

$$P\left(t_e\right) = \int_{\mathcal{D}} \delta\left(t_e - \frac{1}{2\varphi}\ln\left(\frac{A_e^*}{\sqrt{\epsilon}\Omega}\right)^2\right) P\left(\Omega\right) d\Omega, \tag{42}$$

here the domain $\mathcal{D}$ have to be chosen in order to assure that $t_e \geq 0$.

Finally using a non dimensional unit of time, we get

$$P\left(\tau_e\right) = \frac{2K}{erf\left(K\right)\sqrt{\pi}} \exp\left(-\tau_e - K^2 \exp\left(-2\tau_e\right)\right), \tag{43}$$

with

$$K = A_e^* \sqrt{\frac{\varphi}{\epsilon}}, \tag{44}$$

$$\tau_e = \varphi t_e = \left(2G_e D - 1 - 2D\right) t_e,$$

here $2G_e = G\left(k_e\right)$ is given in (21) with $k_e = n_e \pi$ and n_e indicates the most unstable modes (see Fig. 4). Note that this approach allows us to find the group of parameters: $A_e^* \sqrt{\frac{\varphi}{\epsilon}}$. Therefore all the stochastic dynamics can be described in terms of this unique dimensionless universal parameter K. The distribution (43) fully characterizes the description of the relaxation from the unstable state and the stochastic occurrence of the pattern formation (Fig. 5).

Interestingly, we have arrived from first principles to the Gumbel distribution (43), after E.J. Gumbel who described it in his original work published in 1935 [18]. It was used to model the distribution of the maximum of a number of samples in various distributions. It is used heuristically to represent the distribution of the maximum level crossing. It is useful, for example, in predicting the chance that an extreme earthquake, flood or other natural disaster will occur in nature. The applicability of the Gumbel distribution to represent the distribution of maxima relates to extreme value theory, as we have done in the present paper characterizing a threshold value of level crossing from the dynamics of the most unstable Fourier's mode of the harmonic analysis of the concentration field $u(x,t)$.

We should note that if the instability is non linear this universal parameter, K, does not apply, for example, as in the stochastic relaxation from the state that loses its stability in a saddle-node bifurcation [19].

Using (43) the mean first passage time can be calculated as

$$\langle \tau_e \rangle = \int_0^\infty P\left(\tau_e\right) d\tau_e \simeq \ln\left(K\right) + \frac{E + \ln 4}{2erf\left(K\right)}, \tag{45}$$

where

$$K \gg 1, \tag{46}$$

$$E = \text{Euler constant}. \tag{47}$$

This characteristic time is given in terms of the universal group K, see Figure 6, it represents the mean time for the occurrence of the pattern formation.

4.1 Discussion

In this work we described an analytical, fundamental and minimal model to study ecological systems where different mechanisms (activation or inhibition) occur at two length scales, and where fluctuations trigger the stochastic pattern formation.

We would like to note that two mainly different, and highly non–trivial, models can also be tackled using the present approach. First: when the generalized migration is associated to an interaction term. The second possibility arises when the effects of fluctuations have a multiplicative character. Therefore, in general, we can write the following stochastic evolution equation for the concentration field $u(x,t)$

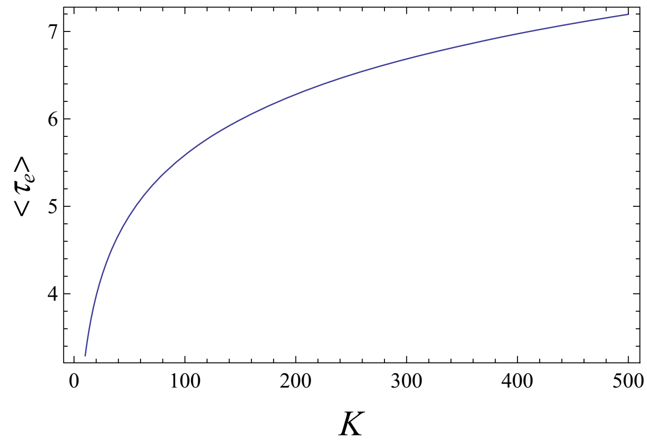

Fig. 6. Mean first passage time (in dimensionless units, see Eq. (45)) as a function of the universal parameter K (see Eq. (44) for its definition).

$$\frac{\partial u(x,t)}{\partial t} = f[u(x,t)] + h\,[u(x,t)]\,\sqrt{\epsilon}\xi(x,t) + \; g\,[u(x,t)] \int u\,(x - x', t)\, G(x')dx'. \tag{48}$$

Here, we have a full nonlinear non-local problem with a multiplicative noise field. Interestingly if this dynamic presents a change of stability as the result of the non-local migration, then the theory of the first passage time that we have presented-using the MCA and the multiple scale transformation-can also be applied to study the lifetime of the unstable sate, and therefore precluding a complex pattern formation scenario.

MAF and MOC thank grant CONICET, PIP 112-201501-00216 CO. Argentina. MAF thanks FONDECYT 1140278.

References

1. S. Getzin, et al., Discovery of fairy circles in Australia supports self-organization theory, Proc. Natl. Acad. Sci. **113**, 3551 (2016)
2. N. Juergens, The biological underpinnings of Namib Desert fairy circles, Science **339**, 1618 (2013)
3. W.R. Tschinkel, The life cycle and life span of Namibian fairy circles, PloS one **7**, e38056 (2012)
4. F. Carteni, A. Marasco, G. Bonanomi, S. Mazzoleni, M. Rietkerk, F. Giannino, Negative plant soil feedback explaining ring formation in clonal plants, J. Theor. Biol. **313**, 153 (2012)
5. M.D. Cramer, N.N. Barger, Are Namibian fairy circles the consequence of self-organizing spatial vegetation patterning? PloS one **8**, e70876 (2013)
6. A.M. Turing, The chemical basis of morphogenesis, Philos. T. Roy. Soc. B **237**, 37 (1952)
7. V. Deblauwe, N. Barbier, P. Couteron, O. Lejeune, J. Bogaert, The global bio-geography of semi-arid periodic vegetation patterns, Glob. Ecol. Biogeogr. **17**, 715 (2008)
8. J. von Hardenberg, E. Meron, M. Shachak, Y. Zarmi, Diversity of vegetation patterns and desertification, Phys. Rev. Lett. **87**, 198101 (2001)
9. M. Rietkerk, J. van de Koppel, Regular pattern formation in real ecosystems, Trends. Ecol. Evol. **23**, 169 (2008)

10. M.A. Fuentes, M.N. Kuperman, V.M. Kenkre, Nonlocal interaction effects on pattern formation in population dynamics, Phys. Rev. Lett. **91**, 158104 (2003)
11. M.O. Cáceres, *Elementos de Estadistica de no Equilibrio y sus Aplicaciones al Transporte en Medios Desordenados* (Reverte, Barcelona, 2003)
12. N.G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981)
13. V. Volterra, *Theory of Functional and Integro Differential Equations* (Dover, N.Y., 2005)
14. M.O. Cáceres, Passage Time Statistics in Exponential Distributed Time-Delay Models: Noisy Asymptotic Dynamics, J. Stat. Phys. **156**, 94 (2014)
15. M.A. Fuentes, M.N. Kuperman, V.M. Kenkre, Analytical considerations in the study of spatial patterns arising from nonlocal interaction effects, J. Phys. Chem. B **108**, 10505 (2004)
16. J.D. Murray, *Mathematical Biology I: An Introduction*, Vol. 17 of Interdisciplinary Applied Mathematics (Springer, New York, 2002)
17. M.O. Cáceres, M.A. Fuentes, First-passage time for pattern formation nonlocal partial differential equations, Phys. Rev. E **92**, 042122 (2015)
18. E.J. Gumbel, Les valeurs extrêmes des distributions statistiques, Annales de l'institut Henri Poincaré **5**, 115 (1935)
19. P. Colet, F. de Pasquale, M.O. Cáceres, M. San Miguel, Theory for relaxation at a subcritical pitchfork bifurcation, Phys. Rev. A **41**, 1901 (1990)