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Robust projective lag synchronization in drive-response dynamical networks via adaptive control

G. Al-mahbashi[1,a], M.S. Md Noorani[1,b], S.A. Bakar[1,c], and M.M. Al-sawalha[2,d]

[1] School of mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 UKM, Selangor Darul Ehsan, Malaysia
[2] Faculty of Science, Mathematics Department, University of Hail, Hail, Saudi Arabia

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Abstract. This paper investigates the problem of projective lag synchronization behavior in drive-response dynamical networks (DRDNs) with identical and non-identical nodes. An adaptive control method is designed to achieve projective lag synchronization with fully unknown parameters and unknown bounded disturbances. These parameters were estimated by adaptive laws obtained by Lyapunov stability theory. Furthermore, sufficient conditions for synchronization are derived analytically using the Lyapunov stability theory and adaptive control. In addition, the unknown bounded disturbances are also overcome by the proposed control. Finally, analytical results show that the states of the dynamical network with non-delayed coupling can be asymptotically synchronized onto a desired scaling factor under the designed controller. Simulation results show the effectiveness of the proposed method.

1 Introduction

A complex dynamical network consists of coupling nodes. Each node is a nonlinear dynamical system connecting with the others via a topology defined on the network edges. Many real-life systems can be modelled as complex networks including world wide web, food webs, electrical power grids, social networks and ecosystems. Therefore, investigation of dynamical complex networks becomes important with the development of industry and various sciences [1–3]. In particular, one of the interesting and significant phenomena in complex dynamical network is the synchronization of all dynamical nodes. The synchronization of complex dynamical networks means that all the nodes in a complex networks eventually approaches to trajectory of a target

[a] e-mail: mahbashighada@yahoo.com
[b] e-mail: msn@ukm.my
[c] e-mail: sakhinah@ukm.my
[d] e-mail: sawalha_moh@yahoo.com

node. Currently, there are many types of concepts of synchronization, such as complete synchronization [4,5], lag synchronization [6], zero-lag synchronization [7], generalized synchronization [8], anti-synchronization [9], projective synchronization [10], quasi-synchronization [11] and breathing synchronization [12]. In practical cases, time delay appears in the electronic applications of dynamical systems. Therefore, it is very important to develop synchronization methods with time delay.

Recently, some effective methods have been proposed to synchronize chaotic systems. For example, using a pinning control scheme, a general method of lag synchronization without assuming the symmetry and irreducibility of the coupling matrix was achieved by Wanli [13].

Using an impulsive control method, Zhang and Zhao[14] investigated the projective and lag synchronization between general complex networks where the coupling matrix in this model is not assumed to be symmetric, diffusive or irreducible. Based on an active nonlinear control technique, Banerjee et al. [15] and Ghosh et al. [16] discussed the problem of projective synchronization and generalized synchronization between two different time delayed systems respectively.

Adaptive control is an active field in the design of control systems which deals with uncertainties. In recent years, adaptive control has been receiving significant attention in different industrial fields. In aerospace applications, adaptive control has been demonstrated in a number of flight vehicles. In the last decade, NASA conducted a flight test program of a neural net intelligent flight control system on board a modified F-15 test aircraft [17]. In practice, mismatch terms, unknown parameters and external disturbance are unavoidable and these can destroy the system's stability. Therefore, some researchers have developed methods to deal with the uncertainty, mismatch, and disturbance. Based on the sliding mode control method, robust synchronization for a coupled FitzHugh-Nagumo (FHN) neurobiological network with parameter disturbances was investigated [18]. Chen et al. studied robust modified function projective synchronization in networks with unknown parameters and mismatch parameters [19]. Using an adaptive control method, lag synchronization (LS) [20] and function projective synchronization (FPS) [21] were proposed for uncertain CDNs having delay coupling, unknown parameter and bounded external disturbances. In the delayed neural network model, Zhou et al. [22] investigated lag synchronization of coupled chaotic delayed neural networks without noise perturbation by using adaptive feedback control techniques.

Recently, based on hybrid feedback control, a general method of projective lag synchronization (PLS) with non-delay coupling and with delay coupling was investigated. In addition, parameters mismatch, constant and varying time coupling delay were considered in [23,24] respectively. Feng et al. in [25] proposed projective-anticipating, projective and projective lag synchronization of time-delayed chaotic systems on random networks. However, few results have been reported with respect to the projective lag synchronization in DRDNs.

Motivated by the above discussion, in this paper, we propose a general PLS scheme in DRDNs with identical and non-identical nodes. Both the drive and the network nodes have unknown parameters and bounded disturbances. A simple adaptive control method is proposed and all of the unknown parameters are estimated by adaptive laws based on Lyapunov stability theory. Based on updating laws, the controller was designed to overcome the unknown bounded disturbances. For the coupling matrix, we do not assume it to be symmetric or irreducible. As a result, the network is asymptotically synchronized with the proposed method. Moreover, numerical simulations are performed to verify the effectiveness of the theoretical results.

The rest of this paper is organized as follows: the DRDNs model with unknown parameters is introduced in Sect. 2. A general method of PLS in DRDNs with unknown parameters using an adaptive method is discussed in Sect. 3. Section 4 deals with a general method of PLS with unknown parameters and bounded disturbances by adaptive method. Section 5 deals with examples and their simulations. Finally, conclusions are drawn in Sect. 6.

2 Model description

We consider a controlled non-delay complex dynamical network consisting of N linearly and diffusively different nodes with uncertain parameters described as follows:

$$\dot{x_i^r}(t) = g_i(x_i^r(t)) + G_i(x_i^r(t))\theta_i + c\sum_{j=1}^{N} a_{ij}\Gamma x_j^r(t) + \Delta_i(t) + u_i(t), \quad i = 1, 2, \ldots, N \quad (1)$$

where the superscripts r stand for the response networks, $x_i^r = (x_{i1}^r, x_{i2}^r, \ldots, x_{in}^r)^T \in \mathbf{R}^n$ denotes the state vector of the ith node, $g_i : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ and $G_i : \mathbf{R}^n \longrightarrow \mathbf{R}^{n\times m_i}$ are the known continuous nonlinear function matrices determining the dynamic behavior of the node, θ_i is the unknown constant parameter vector, Δ_i contains external disturbance terms. $u_i \in \mathbf{R}^n$ is the control input,c is the coupling strength. Here $\Gamma = diag(\gamma_1, \gamma_2, \ldots, \gamma_n)$ is the inner coupling matrix with $\gamma_i = 1$ for the ith state variable, i.e. matrix Γ determines which nodes in the system are coupled. $A = (a_{ij} \in \mathbf{R}^{N\times N})$ is the coupling configuration matrix representing the topological structure of the networks, where a_{ij} is defined as follows: if there exists a connection between node i and j ($j \neq i$), then $a_{ij} > 0$, otherwise $a_{ij} = 0$, and the diagonal elements of matrix A are defined by

$$a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij}, i = 1, 2, \ldots, N. \quad (2)$$

The reference node is describe as follows:

$$\dot{x}^d(t) = f(x^d(t)) + F(x^d(t))\Phi, \quad (3)$$

where superscripts d stand for the drive system $x^d = (x_1^d, x_2^d, \ldots, x_n^d)^T \in \mathbf{R}^n$ denotes the state vector of the drive system, $f : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ and $F : \mathbf{R}^n \longrightarrow \mathbf{R}^{n\times m_i}$ are the known continuous nonlinear function matrices determining the dynamic behavior of the node, Φ is the unknown constant parameter vector.

The projective lag synchronization error is define as

$$e_i(t) = x_i^r(t) - \alpha x^d(t - \tau), \quad i = 1, \ldots, N \quad (4)$$

where α is a nonzero scaling factor, $\tau > 0$ is a constant representing time delay or lag. Then the objective of this paper is to design a controller $u_i(t)$ such that the reference nodes (1) and dynamical networks (3) are asymptotically synchronized such that

$$\lim_{t\longrightarrow\infty} \|x_i^r(t) - \alpha x^d(t - \tau)\| = 0, \quad i = 1, \ldots, N \quad (5)$$

which means that the network (1) is projective lag synchronized with reference node (3).

Assumption 21. *[20] For any positive constant ε_i the time varying disturbance $\Delta_i(t)$ is bounded i.e $\| \Delta_i(t) \| \leq \varepsilon_i$.*

3 The controller design for projective lag synchronization

In this section, we designed an adaptive control method to realize projective lag synchronization with non-delayed coupling in drive-response dynamical networks consisting of different nodes with unknown parameters. In addition, we assume that there are no terms of external disturbances added on the networks dynamics.

The error dynamics for projective lag synchronization are obtained as

$$\dot{e}_i(t) = g_i(x_i^r(t)) + G_i(x_i^r(t))\theta_i + c\sum_{j=1}^{N} a_{ij}\Gamma e_j(t) + u_i(t) - \alpha\Big(f(x^d(t-\tau)) + F(x^d(t-\tau))\Phi\Big), \quad i = 1, \ldots, N. \tag{6}$$

The initial condition corresponding to the error dynamical system (6) is given as $e(\kappa) = \varphi(\kappa), \kappa \in [\tau, 0], \varphi \in \mathcal{C}([\tau, 0], \mathbf{R}^n)$ denotes the continuous vector valued functions mapping the delay interval $[\tau, 0]$ into $\mathbf{R}^n$.

Theorem 31. *The projective lag synchronization error (6) is asymptotically stable with a given time delay τ and scaling factor α, if the control input and adaptive laws are chosen as*

$$u_i(t) = -q_i e_i(t) - g_i(x_i^r(t)) - G_i(x_i^r(t))\hat{\theta}_i(t) + \alpha\Big(f(x^d(t-\tau)) + F(x^d(t-\tau))\hat{\Phi}(t)\Big), \tag{7}$$

$$\dot{\hat{\theta}}_i(t) = k_1 G_i^T(x_i^r(t))e_i(t), \tag{8}$$

$$\dot{\hat{\Phi}}(t) = -k_2 F_i^T(x_i^d(t-\tau))e_i(t), \tag{9}$$

$$\dot{q}_i(t) = k_3 e_i^T(t)e_i(t), \tag{10}$$

where k_1, k_2 and k_3 are any positive constants and $\hat{\Phi}(t)$ and $\hat{\theta}_i(t)$ are the estimated parameters for the drive (3) and network dynamics (1), respectively.

Proof. *Choose the following Lyapunov function candidate*

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i(t)^T e_i(t) + \frac{1}{2k_1}\sum_{i=1}^{N}\tilde{\theta}_i^T(t)\tilde{\theta}_i(t) + \frac{1}{2k_2}\sum_{i=1}^{N}\tilde{\Phi}_i^T(t)\tilde{\Phi}_i(t) + \frac{1}{2k_3}\sum_{i=1}^{N}\tilde{q}_i^2(t) \tag{11}$$

where
$\tilde{\Phi}_i(t) = \hat{\Phi}_i(t) - \Phi, \tilde{\theta}_i(t) = \hat{\theta}_i(t) - \theta_i, \tilde{q}_i(t) = q_i(t) - q_i^*$, *where q_i^* is positive constant.*

The time derivative of $V(t)$ along the error dynamics (6) is

$$\dot{V} = \sum_{i=1}^{N}\Big[e_i^T(t)\dot{e}_i(t) + \frac{1}{k_1}\dot{\hat{\theta}}_i^T(t)\tilde{\theta}_i(t) + \frac{1}{k_2}\dot{\hat{\Phi}}_i^T(t)\tilde{\Phi}_i(t) + \frac{1}{k_3}\dot{q}_i\tilde{q}_i(t)\Big]\cdot \tag{12}$$

By application of the control input (7) to error dynamics $\dot{e}_i(t)$ we have

$$\dot{V} = \sum_{i=1}^{N}\Big[e_i^T(t)\Big(-q_i e_i(t) - G_i(x_i^r(t))\tilde{\theta}_i(t) + \alpha F(x^d(t-\tau))\tilde{\Phi}(t) + c\sum_{j=1}^{N} a_{ij}\Gamma e_j(t)\Big)\Big] + \sum_{i=1}^{N}\Big[\frac{1}{k_1}\dot{\hat{\theta}}_i^T(t)\tilde{\theta}_i(t) + \frac{1}{k_2}\dot{\hat{\Phi}}_i^T(t)\tilde{\Phi}_i(t) + \frac{1}{k_3}\dot{q}_i\tilde{q}_i(t)\Big]\cdot \tag{13}$$

From the adaptation laws (8)–(10), $\dot{V}$ is led as follows:

$$\dot{V} = -\sum_{i=1}^{N} q^* e_i^T(t) e_i(t) + c \sum_{i=1}^{N} e_i^T(t) \sum_{j=1}^{N} a_{ij} \Gamma e_j(t). \tag{14}$$

Let $e(t) = (e_1^T(t), e_2^T(t), \ldots, e_N^T(t))^T \in \mathbf{R}^{nN}), P = (A \otimes \Gamma)$ where $\otimes$ represents the Kronecker product. Then we have,

$$\begin{aligned}\dot{V} &= ce^T(t)Pe(t) - q^* e(t)^T e(t)\\ &= \frac{1}{2} ce^T(t)\left(P + P^T\right) e(t) - q^* e(t)^T e(t)\\ &\le \left[\lambda_{max}\left(\frac{P+P^T}{2}\right) - q^*\right] e(t)^T e(t)\end{aligned}$$

where $\lambda_{max}\left(\frac{P+P^T}{2}\right)$ is the maximum eigenvalue of the matrix $\frac{P+P^T}{2}$.
Taking the condition

$$q^* = \lambda_{max}\left(\frac{P+P^T}{2}\right) + 1. \tag{15}$$

According to the Lyapunov stability theory and the sufficient condition (15), we can obtain

$$\dot{V} \le -e^T(t)e(t).$$

Thus, the error dynamics $e_i(t)$ is asymptotically stable by the control (7) and the update laws (8)–(10). This completes the proof.

4 The controller design for projective lag synchronization with disturbance

As it is well-known, the disturbance causes an unsteadiness phenomenon which degrades the stability of the controlled system. Therefore projective lag synchronization with non-delayed coupling in DRDNs with fully unknown parameters and disturbances is further investigated in this section.

From (3) and (1), the error dynamics for projective lag synchronization is obtained as

$$\begin{aligned}\dot{e}_i(t) &= g_i(x_i^r(t)) + G_i(x_i^r(t))\theta_i + c\sum_{j=1}^{N} a_{ij}\Gamma e_j(t) + \Delta_i(t)\\ &\quad + u_i(t) - \alpha\Big(f(x^d(t-\tau)) + F(x^d(t-\tau))\Phi\Big), \quad i = 1, \ldots, N.\end{aligned} \tag{16}$$

Theorem 41. *Consider the projective lag synchronization error (16) is asymptotically stable with a given time delay τ and scaling factor α, if the control input and adaptive laws are chosen as*

$$\begin{aligned}u_i(t) &= -q_i e_i(t) - \beta_i(t) sgn(e_i(t)) - g_i(x_i^r(t)) - G_i(x_i^r(t))\hat{\theta}_i(t)\\ &\quad + \alpha\Big(f(x^d(t-\tau)) + F(x^d(t-\tau))\hat{\Phi}(t)\Big), \quad i = 1, \ldots, N.\end{aligned} \tag{17}$$

$$\dot{\hat{\theta}}_i(t) = k_1 G_i^T(x_i^r(t))e_i(t), \tag{18}$$
$$\dot{\hat{\Phi}}(t) = -k_2 F_i^T(x_i^d(t-\tau))e_i(t), \tag{19}$$
$$\dot{q}_i(t) = k_3 e_i^T(t)e_i(t), \tag{20}$$
$$\dot{\beta}_i(t) = k_4 e_i^T(t)sgn(e_i(t)), \tag{21}$$

where k_1, k_2, k_3 *and* k_4 *are positive constants and* $\hat{\Phi}(t)$ *and* $\hat{\theta}_i(t)$ *are the estimated parameters for the reference node (3) and network (1) respectively.*

Proof. Choose the following Lyapunov function candidate

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^T(t)e_i(t) + \frac{1}{2k_1}\sum_{i=1}^{N}\tilde{\theta}_i^T(t)\tilde{\theta}_i(t) + \frac{1}{2k_2}\sum_{i=1}^{N}\tilde{\Phi}_i^T(t)\tilde{\Phi}_i(t)$$
$$+ \frac{1}{2k_3}\sum_{i=1}^{N}\tilde{q}_i^2(t) + \frac{1}{2k_4}\sum_{i=1}^{N}\tilde{\beta}_i^2(t) \tag{22}$$

where
$\tilde{\Phi}_i(t) = \hat{\Phi}_i(t) - \Phi, \tilde{\theta}_i(t) = \hat{\theta}_i(t) - \theta_i, \tilde{q}_i(t) = q_i(t) - q_i^*, \tilde{\beta}_i(t) = \beta_i(t) - \beta_i^*$ where q_i^* and β_i^* are positive constants.

The time derivative of $V(t)$ along the error dynamics (16) is

$$\dot{V} = \sum_{i=1}^{N}\Big[e_i^T(t)\dot{e}_i(t) + \frac{1}{k_1}\dot{\hat{\theta}}_i^T(t)\tilde{\theta}_i(t) + \frac{1}{k_2}\dot{\hat{\Phi}}_i^T(t)\tilde{\Phi}_i(t) + \frac{1}{k_3}\dot{q}_i\tilde{q}_i(t) + \frac{1}{k_4}\dot{\beta}_i\tilde{\beta}_i(t)\Big]\cdot \tag{23}$$

By application of the control input (17) to error dynamics $\dot{e}_i(t)$ we have

$$\dot{V} = \sum_{i=1}^{N}\Big[e_i^T(t)\Big(-q_ie_i(t) - \beta_i(t)sgn(e_i(t)) - G_i(x_i^r(t))\tilde{\theta}_i(t) + \alpha F(x^d(t-\tau))\tilde{\Phi}(t)\Big)\Big]$$
$$+\sum_{i=1}^{N}\Bigg[e_i^T(t)\Big(c\sum_{j=1}^{N}a_{ij}\Gamma e_j(t) + \Delta_i(t)\Big)\Bigg] + \sum_{i=1}^{N}\Big[\frac{1}{k_1}\dot{\hat{\theta}}_i^T(t)\tilde{\theta}_i(t) + \frac{1}{k_2}\dot{\hat{\Phi}}_i^T(t)\tilde{\Phi}_i(t)$$
$$+\frac{1}{k_3}\dot{q}_i\tilde{q}_i(t) + \frac{1}{k_4}\dot{\beta}_i\tilde{\beta}_i(t)\Big]. \tag{24}$$

From the adaptation laws (18)–(21), $\dot{V}$ *is led as follows:*

$$\dot{V} = -\sum_{i=1}^{N} q^* e_i^T(t)e_i(t) - \sum_{i=1}^{N}\beta^* e_i^T(t)sgn(e_i(t)) + \sum_{i=1}^{N} e_i^T(t)\Delta_i(t)$$
$$+ c\sum_{i=1}^{N} e_i^T(t)\sum_{j=1}^{N} a_{ij}\Gamma e_j(t). \tag{25}$$

Let $e(t) = (e_1^T(t), e_2^T(t), \ldots, e_N^T(t))^T \in \mathbf{R}^{nN}), P = (A \otimes \Gamma)$ *where* $\otimes$ *represents the Kronecker product and from Assumption (21). Then we have,*

$$\begin{aligned}
\dot{V} &\leq ce^T(t)Pe(t) - q^* e(t)^T e(t) + \sum_{i=1}^{N} \left[\varepsilon_i - \beta^*\right] \parallel e_i(t) \parallel \\
&\leq \frac{1}{2} ce^T(t)\left(P + P^T\right)e(t) - q^* e(t)^T e(t) + \sum_{i=1}^{N} \left[\varepsilon_i - \beta^*\right] \parallel e_i(t) \parallel \\
&\leq e^T(t)\left(\frac{1}{2}c\left(P + P^T\right) - q^*\right)e(t) + \sum_{i=1}^{N} \left[\varepsilon_i - \beta^*\right] \parallel e_i(t) \parallel \\
&\leq \left[\lambda_{max}\left(\frac{P + P^T}{2}\right) - q^*\right] e(t)^T e(t) + \sum_{i=1}^{N} \left[\varepsilon_i - \beta^*\right] \parallel e_i(t) \parallel \\
&\leq \left(\lambda - q^*\right) e^T(t)e(t) + \sum_{i=1}^{N} \left[\varepsilon_i - \beta^*\right] \parallel e_i(t) \parallel
\end{aligned}$$

where $\lambda = \lambda_{max}\left(\frac{1}{2}c\left(P + P^T\right)\right)$ *is the maximum eigenvalue of the matrix* $\frac{1}{2}c\left(P + P^T\right)$. *Taking the conditions*

$$\lambda - q^* < 0 \tag{26}$$

$$\varepsilon_i - \beta^* < 0. \tag{27}$$

According to the Lyapunov stability theory and the sufficient conditions (26), (27), we can obtain

$$\dot{V} \leq 0.$$

Which means the response networks (3) projective lag synchronizes the drive system (1) asymptotically by the control (17) and the update laws (18)–(21). This completes the proof.

Remark 42. [20] The inclusion of the $sgn(e_i(t))$ function in (17) provides robustness against unknown disturbances. In order to alleviate the unsteadiness, the boundary layer approach is used by replacing the $sgn(e_i(t))$ function with the following saturation function

$$sat\left(\frac{e_i(t)}{\delta}\right) = \begin{cases} sgn(e_i(t)), & if \parallel e_i(t) \parallel > \delta, \\ \frac{e_i(t)}{\delta}, & if \parallel e_i(t) \parallel \leq \delta, \end{cases} \tag{28}$$

where δ is a small positive constant and the function (28) can approach the the $sgn(.)$ function, as enough small δ is chosen.

5 Illustrative example

In this section, a DRDN with three identical and different nodes systems is used. Each node with and without disturbance has unknown parameters to show the effectiveness of the proposed schemes obtained in the previous sections. We use the Lorenz system as drive system, which is described as follows:

$$\begin{pmatrix} \dot{x}_1^d \\ \dot{x}_2^d \\ \dot{x}_3^d \end{pmatrix} = \begin{pmatrix} 0 \\ -x_1^d x_3^d - x_2^d \\ x_1^d x_2^d \end{pmatrix} + \begin{pmatrix} x_2^d - x_1^d & 0 & 0 \\ 0 & x_1^d & 0 \\ 0 & 0 & -x_3^d \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix}. \tag{29}$$

Where the unknown parameter vector is $\Phi = [\Phi_1 \quad \Phi_2 \quad \Phi_3] = [10 \quad 28 \quad \frac{8}{3}]$.

The inner coupling matrix $\Gamma = I_{3\times 3}$ and the coupling configuration matrix $A = (a_{ij})$ is chosen to be

$$A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

5.1 Synchronization with identical nodes

In this subsection, we focus on studying PLS in complex dynamical networks with identical nodes, unknown parameters and disturbance.

5.1.1 Synchronization with unknown parameters

Taking a chaotic Chen system as the ith networks nodes with unknown parameters to realize PLS in DRDNs and verify the effectiveness of the proposed scheme which can be described as follows:

$$\dot{x}_1^r(t) = g(x_1^r(t)) + c(x_2^r(t) + x_3^r(t) - 2x_1^r(t)) + u_1(t) \tag{30}$$

$$\dot{x}_2^r(t) = g(x_2^r(t)) + c(x_1^r(t) - x_2^r(t)) + u_2(t) \tag{31}$$

$$\dot{x}_3^r(t) = g(x_3^r(t)) + c(x_2^r(t) - x_3^r(t)) + u_3(t). \tag{32}$$

Where $u_i(t)$ for $(i = 1, 2, 3)$ can be designed by Eq. (7) in the Theorem 41 and

$$g(x_i^r(t)) = \begin{pmatrix} \dot{x}_{i1}^r \\ \dot{x}_{i2}^r \\ \dot{x}_{i3}^r \end{pmatrix} = \begin{pmatrix} 0 \\ -x_{i1}^r x_{i3}^r \\ x_{i1}^r x_{i2}^r \end{pmatrix} + \begin{pmatrix} x_{i2}^r - x_{i1}^r & 0 & 0 \\ -x_{i1}^r & x_{i1}^r + x_{i2}^r & 0 \\ 0 & 0 & -x_{i3}^r \end{pmatrix} \begin{pmatrix} \theta_{i1} \\ \theta_{i2} \\ \theta_{i3} \end{pmatrix}. \tag{33}$$

Here the unknown parameters vector is $\theta_1 = [35 \quad 28 \quad 3]^T$.

In these numerical simulations, we assume that $c = 0.2, \alpha = 2$ and $\tau = 1$. The gain of adaptive laws (8)–(10) are $k_1 = 5, k_2 = 3, k_3 = 1.4$ and $q_i = 0$. We take the initial states as $x^d[-1, 0] = [3 \quad 1 \quad -2]^T$ and $x_i^r(0)$ are chosen in $[-4, 4]$ randomly.

The numerical results are presented in Figs. 1–2. The time evolution of the synchronization errors is depicted in Fig. 1, which displays $e \longrightarrow 0$ with $t \longrightarrow \infty$. The estimated parameters of the reference node and network nodes are depicted in Fig. 2(a) and Fig. 2(b) which converge to their real values. It also shows the PLS are achieved after small time interval.

5.1.2 Synchronization with unknown parameters and disturbance

The previous chaotic Chen system is chosen as three nodes of complex dynamical networks with unknown parameters and disturbances to verify the effectiveness of the proposed scheme which can be described as follows:

$$\dot{x}_1^r(t) = g(x_1^r(t)) + c(x_2^r(t) + x_3^r(t) - 2x_1^r(t)) + \Delta_1(t) + u_1(t) \tag{34}$$

$$\dot{x}_2^r(t) = g(x_2^r(t)) + c(x_1^r(t) - x_2^r(t)) + \Delta_2(t) + u_2(t) \tag{35}$$

$$\dot{x}_3^r(t) = g(x_3^r(t)) + c(x_2^r(t) - x_3^r(t)) + \Delta_3(t) + u_3(t). \tag{36}$$

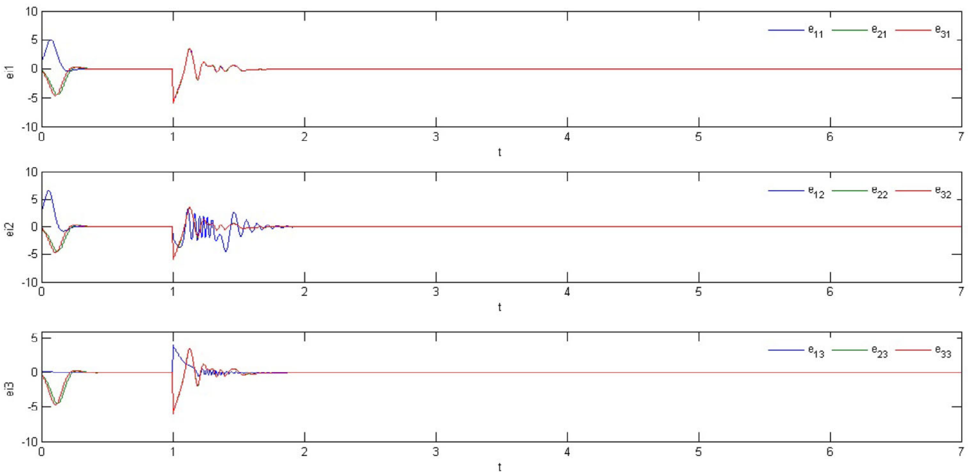

Fig. 1. The synchronization error $e_i(t) = x_i^r(t) - \alpha x^d(t - \tau)$.

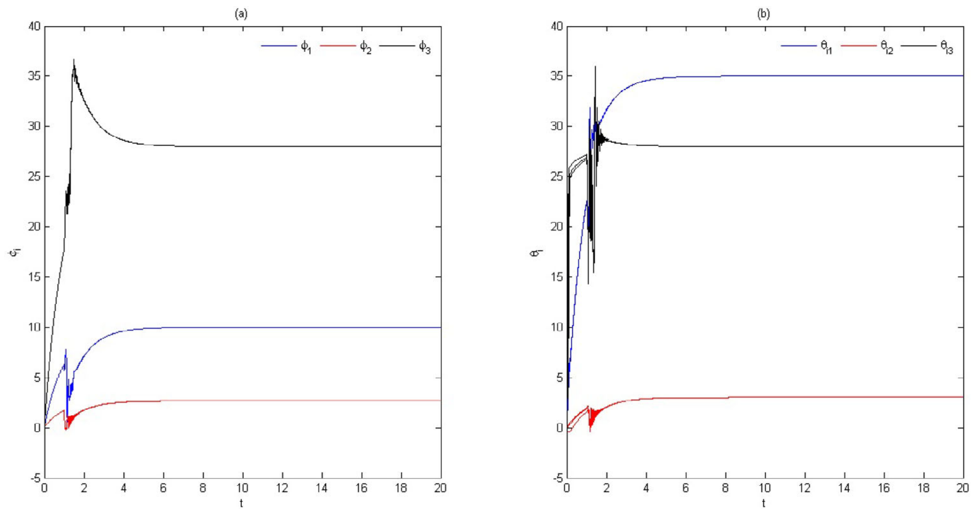

Fig. 2. The estimated parameters (a) $\hat{\phi}_i$ and (b)$\hat{\theta}_i$ of identical nodes.

Where $\Delta_i = [0.3cos(t)sin(t) \quad 0.1sin(t) \quad 0.5cos(t)]$. For these numerical simulations, we assume that $c = 0.2, \alpha = 2$ and $\tau = 1$. The gain of adaptive laws (8)–(10) are $k_1 = 6, k_2 = 5, k_3 = 2, k_4 = 0.2q_i = \beta_i = 0$. we take the initial states as $x^d[-1, 0] = [3 \quad 3 \quad -2]^T$ and $x_i^r(0)$ which are chosen in $[-3, 3]$ randomly. As noticed in remark (4), we replace the $sgn(e_i(t))$ function in (17) with (28) with $\delta = 0.00002$ to reduce the unsteadiness phenomenon.

The numerical results are presented in Figs. 3, 4. Figure 3 displays the time evolution of the synchronization errors, which shows $e \longrightarrow 0$ with $t \longrightarrow \infty$. The estimated parameters of the reference node and network nodes are depicted in Fig. 4(a) and Fig. 4(b) respectively, which converge to their real values. These results verify the proposed control (17) with adaptive laws (18)–(21) makes the network (1) projective lag synchronized, even if both the reference node and the network have unknown parameters and disturbances.

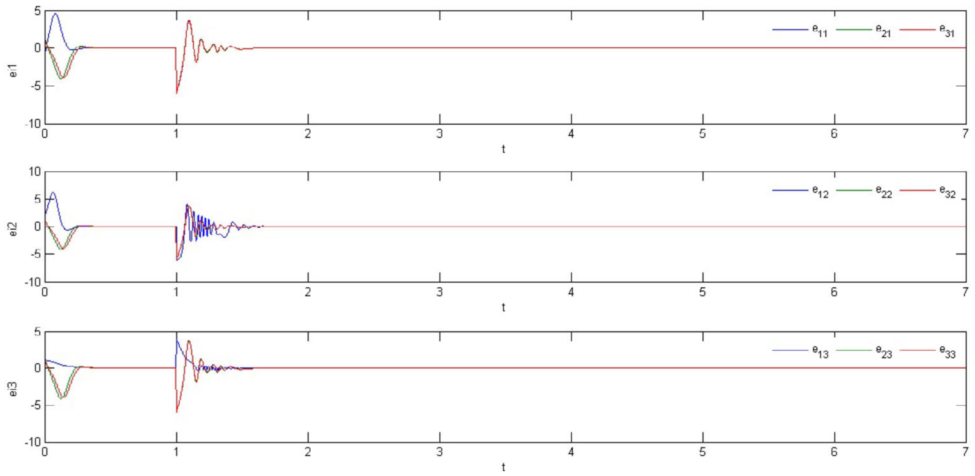

Fig. 3. The synchronization error of identical nodes with fully unknown parameters and disturbance.

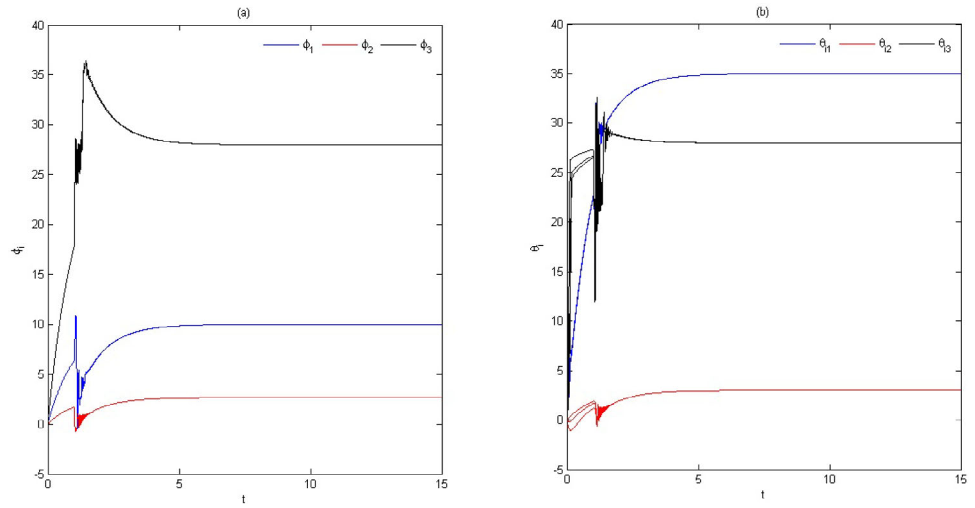

Fig. 4. The estimated parameters (a) $\hat{\phi}_i$ and (b) $\hat{\theta}_i$ of identical nodes with disturbance.

5.2 Synchronization with different nodes

In this subsection, we focus on studying PLS in complex dynamical networks with different nodes, unknown parameters and disturbance.

5.2.1 Synchronization with unknown parameters

Taking the response networks with three different nodes and unknown parameters consist of the Chen system, the Lu system and the Rössler system, respectively which are as follows:

$$\dot{x}_1^r(t) = g_1(x_1^r(t)) + c(x_2^r(t) + x_3^r(t) - 2x_1^r(t)) + u_1(t) \tag{37}$$

$$\dot{x}_2^r(t) = g_2(x_2^r(t)) + c(x_1^r(t) - x_2^r(t)) + u_2(t) \tag{38}$$

$$\dot{x}_3^r(t) = g_3(x_3^r(t)) + c(x_2^r(t) - x_3^r(t)) + u_3(t). \tag{39}$$

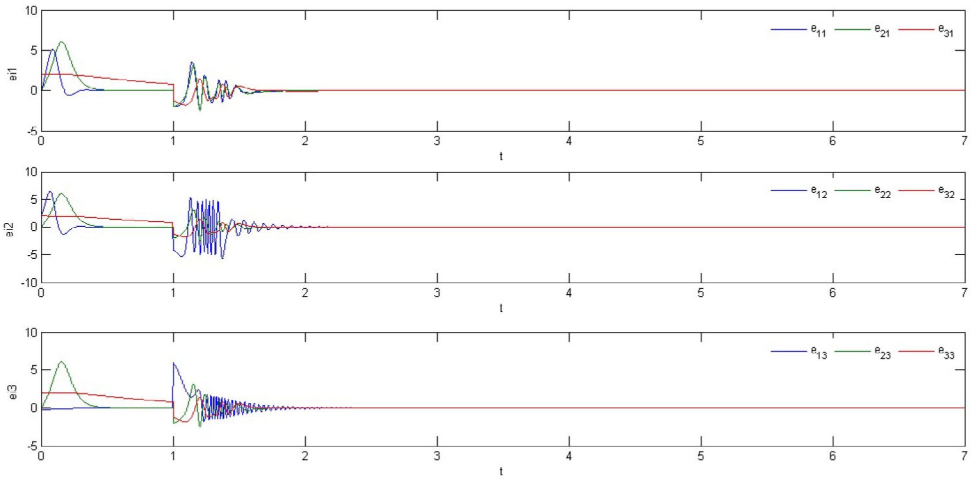

Fig. 5. The synchronization error of different nodes with fully unknown parameters.

Where $u_i(t)$ for $(i = 1, 2, 3)$ can be design by Eq. (7) in the Theorem 41 and

$$g_1(x_1^r(t)) = \begin{pmatrix} \dot{x}_{11}^r \\ \dot{x}_{12}^r \\ \dot{x}_{13}^r \end{pmatrix} = \begin{pmatrix} 0 \\ -x_{11}^r x_{13}^r \\ x_{11}^r x_{12}^r \end{pmatrix} + \begin{pmatrix} x_{12}^r - x_{11}^r & 0 & 0 \\ -x_{11}^r & x_{11}^r + x_{12}^r & 0 \\ 0 & 0 & -x_{13}^r \end{pmatrix} \begin{pmatrix} \theta_{11} \\ \theta_{12} \\ \theta_{13} \end{pmatrix} \quad (40)$$

$$g_2(x_2^r(t)) = \begin{pmatrix} \dot{x}_{21}^r \\ \dot{x}_{22}^r \\ \dot{x}_{23}^r \end{pmatrix} = \begin{pmatrix} 0 \\ -x_{21}^r x_{23}^r \\ x_{21}^r x_{22}^r \end{pmatrix} + \begin{pmatrix} x_{22}^r - x_{21}^r & 0 & 0 \\ 0 & x_{22}^r & 0 \\ 0 & 0 & -x_{23}^r \end{pmatrix} \begin{pmatrix} \theta_{21} \\ \theta_{22} \\ \theta_{23} \end{pmatrix} \quad (41)$$

$$g_3(x_3^r(t)) = \begin{pmatrix} \dot{x}_{31}^r \\ \dot{x}_{32}^r \\ \dot{x}_{33}^r \end{pmatrix} = \begin{pmatrix} -x_{32}^r - x_{33}^r \\ x_{31}^r \\ x_{31}^r x_{33}^r + 0.2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ x_{32}^r & 0 \\ 0 & -x_{33}^r \end{pmatrix} \begin{pmatrix} \theta_{31} \\ \theta_{32} \end{pmatrix}. \quad (42)$$

The unknown parameters vectors are $\theta_1 = [35 \quad 28 \quad 3]^T, \theta_2 = [36 \quad 20 \quad 3]^T$, $\theta_3 = [0.2 \quad 5.7]^T$. In the numerical simulations, we assume that $c = 0.2, \alpha = 2, \tau = 1$. The gain of adaptive laws (8)–(10) are $k_1 = 6, k_2 = 4, k_3 = 0.3$. We take the initial states as $x^d[-1, 0] = [3 \quad 6 \quad -1]^T$ and $x_i^r(0)$ which are chosen in $[-3, 3]$ randomly. The numerical results are presented in Figs. 5, 6. The time evolution of the synchronization errors is depicted in Fig. 5, which displays $e \longrightarrow 0$ with $t \longrightarrow \infty$. The estimated parameters of the reference node and network nodes are depicted in Fig. 6(a) and Fig. 6(b) respectively, which converge to their real values. These results verify the proposed control (7) with adaptive laws (8)–(10) makes the network dynamics (1) when $\Delta_i = 0$ projective lag synchronized, even though the drive system and the network have unknown parameters.

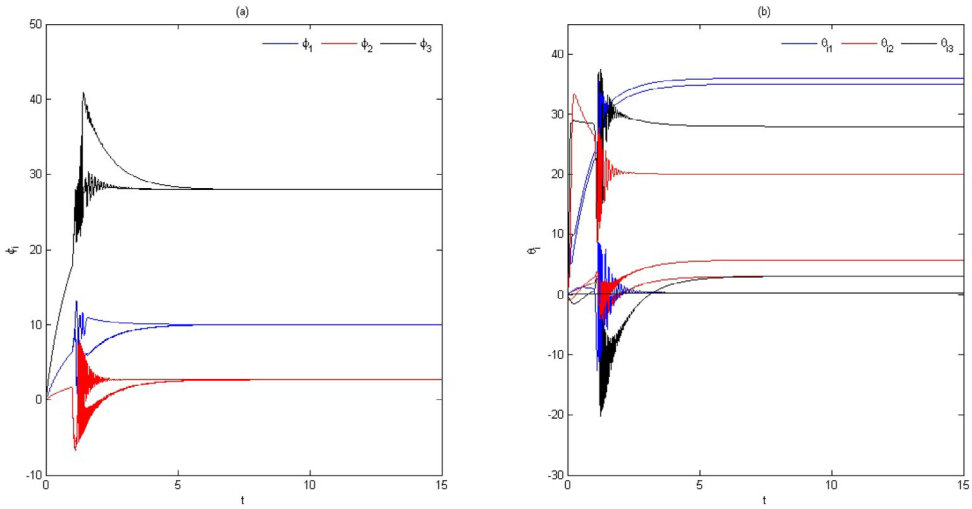

Fig. 6. The estimated parameters (a) $\hat{\phi}_i$ and (b) $\hat{\theta}_i$ of different nodes.

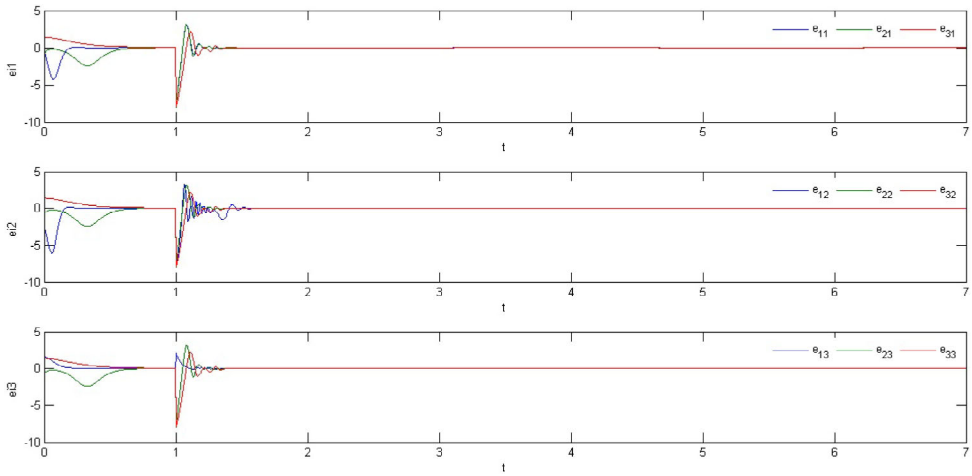

Fig. 7. The synchronization error of different nodes with fully unknown parameters and disturbance.

5.3 Synchronization with different nodes and disturbance

The response networks with the three previous different nodes (40–42), unknown parameters and disturbance can be described as follows:

$$\dot{x}_1^r(t) = g_1(x_1^r(t)) + c(x_2^r(t) + x_3^r(t) - 2x_1^r(t)) + \Delta_1(t) + u_1(t) \tag{43}$$

$$\dot{x}_2^r(t) = g_2(x_2^r(t)) + c(x_1^r(t) - x_2^r(t)) + \Delta_2(t) + u_2(t) \tag{44}$$

$$\dot{x}_3^r(t) = g_3(x_3^r(t)) + c(x_2^r(t) - x_3^r(t)) + \Delta_3(t) + u_3(t). \tag{45}$$

We assume that $\Delta_i = [0.3cos(t)sin(t) \quad 0.1sin(t) \quad 0.5cos(t)], c = 0.2, \alpha = 2, \tau = 1$. The gain of adaptive laws (18)–(21) are $k_1 = 5, k_2 = 6, k_3 = 4, k_4 = 0.2q_i = \beta_i = 0$. We take the initial states as $x^d{-}1, 0 = [4 \quad 4 \quad -1]^T$ and $x_i^r(0)$ which are chosen in

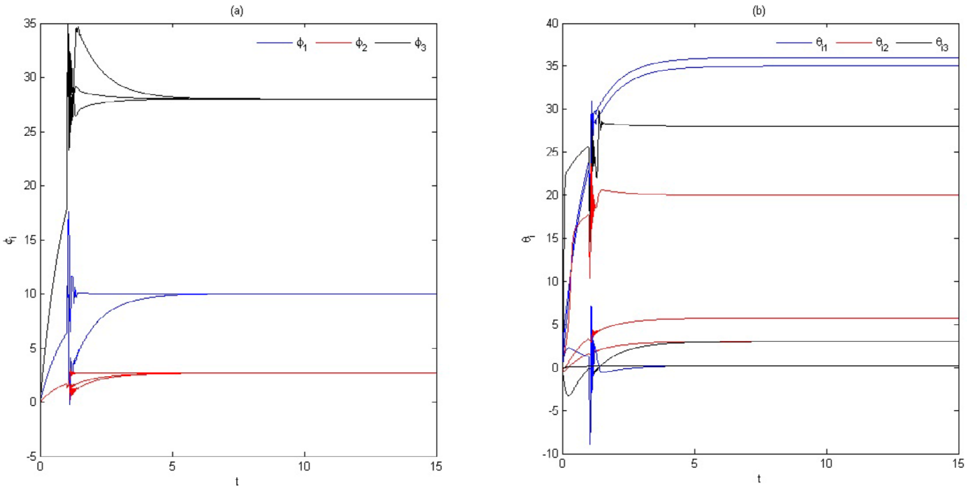

Fig. 8. The estimated parameters (a)$\hat{\phi}_i$ and (b) $\hat{\theta}_i$ of different nodes with disturbance.

$[-3, \quad 3]$ randomly. As noticed in remark (42), we replace the $sgn(e_i(t))$ function in (17)with (28) with $\delta = 0.00002$ to reduce the unsteadiness phenomenon.

The numerical results are presented in Figs. 7, 8. Figure 7 displays the time evolution of the synchronization errors, which shows $e \longrightarrow 0$ with $t \longrightarrow \infty$. The estimated parameters of the reference node and network nodes are depicted in Fig. 8(a) and Fig. 8(b) respectively, which converge to their real values. These results verify the proposed control (17) with adaptive laws (18)–(21) makes the network (1) projective lag synchronized with reference node (3), even though the drive and the network systems have unknown parameters and disturbance.

6 Conclusion

In this paper, a general projective lag synchronization (PLS) scheme was proposed in DRDNs with identical and different nodes. Both the reference nodes and network nodes have fully unknown parameters and disturbances. Adaptive control and update laws were designed to achieve the PLS. The unknown parameters were estimated using the adaptive laws obtained based on the Lyapunov stability theory. In addition, sufficient conditions for synchronization are derived analytically using the Lyapunov stability theory and adaptive techniques. The simulation results are presented to show the effectiveness of this approach.

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