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Network of mobile systems: mutual influence of oscillators and agents

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Abstract This work presents a network of mobile systems whose nodes are constituted by a moving agent with an internal state (an oscillator), which influences each other. The coupling topology of the agents and internal oscillators changes over time according to the interaction range (also called vision range or vision sizes (Majhi et al. Phys Rev E 99: 012308, 2019)) of their corresponding counterparts. The goal is to investigate the dynamics of the oscillators and the agents in the considered systems. Our results show that the synchronization between agents and that between oscillators depends on the coupling parameter of the oscillators, the velocity of the agents and the interaction range of both agents and oscillators. We have found that the vision range of the oscillators has a great influence on the dynamics of the agents. Among this dynamics, we can mention phase synchronization and clusters formation in the mobile system and complete synchronization as well as clusters formation on the oscillators. The stability of the synchronization in the oscillators is investigated using the Master Stability Function (MSF) developed by Pecora and Carroll (Phys Rev Lett 80: 2109, 1998).

1 Introduction

Mathematical modeling, theory of dynamical systems and numerical simulations are the main tools used to analyze the complex behavior of dynamical systems, and the interest in networks systems is focused on the research on complex systems. According to Edward O. Wilson, "the greatest challenge today, not only in cell biology and ecology, but in all science, is the accurate and complete description of complex systems" [1]. It is well known that different dynamical systems synchronize due to their mutual interaction when the trajectories of their elements approach each other. The fact that it is possible to induce a synchronized state between deterministic chaotic oscillators makes the synchronization a phenomenon of significant interest in many fields of science and technology such as communications, electronics, optics, chemistry, biology and so on [2–6].

Several works exist on the emergence of the phenomenon of synchronization in networks of complex systems, but most of these works are based on interaction networks with a stationary topology. In recent years, the analysis of complex networks in evolution, (those whose topology changes over time) has attracted attention [7–9]. In many cases, they are networks in which nodes (agents) can move and interact with the neighboring nodes. In these networks, the coupling is established according to the proximity of the agents. In the literature, they are called moving neighborhood networks [9,10]. In some fields the interaction varies with time, among these fields we can mention communications [11], social networks [12], the spread of epidemics [13]. The study of the evolutive networks has not yet been well explored, although it may be quite relevant for several applications, such as coordination of the movements of robots, vehicles, animals and even people [14,15]. The mobility that characterizes these types of networks can be of the utmost importance in understanding the phenomenon of synchronization.

The effectiveness of a particular synchronization approach is generally evaluated in terms of its ability to achieve the established objective of proximity between the involved systems. Initial work on mobile systems reveals that the synchronization phenomenon in these systems is influenced by several parameters [9,16–18]. For example, for a set of mobile agents in

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a two dimensional space, it is known that the density (number of individuals per unit area) of the mobile agents is a key parameter on which the phenomenon of synchronization strongly depends. This knowledge actually establishes a relationship between the behavior of the system and the properties of the dynamical network [16]. The time scales between synchronization of local clusters and topological changes due to movement of the agents have a significant influence on the global behavior of the systems (See reference [9]). Indeed, Fujiwara et al. [9] show that, when the change in the time scale related to topology is higher than that related to local synchronization in clusters, it takes more time for the system to achieve synchronization than in the fast-switching approximation prediction. More recent studies on the emergence of synchronization in mobile systems [16,19] use the chaotic dynamics of the Rössler system to mold the nodes of the network. These nodes move by performing a random walk in a two dimensional space and their movement is regulated after each iteration. Majhi et al. show that the mobility parameters induce synchronization in the network system and these parameters are mainly the vision range of the mobile agents and their speed.

According to these works on mobile systems, it is clear that the emphasis is on the impact of agent mobility on oscillators' dynamics. It is now important to know how the internal dynamics can influence the positions of the agents in the space. In this work, we propose a mobile system in which positions and oscillators influence each other. In this system, the interaction structure at the level of the agents varies according to the parameter D_0 which defines the vision range of the oscillators and in turn the oscillators are influenced by the vision range of the agents, d_0.

The work is organized as follows: in Sect. 2 we present the model of our study. It is constituted of a system of mobile agents and an associated system of oscillators. Section 3 presents the numerical results obtained from the model defined in Sect. 2. In this section we study the synchronization of the whole network: first the effects of the space parameters (mobile systems), then the influence of the parameters of the oscillators and the mutual influence of these parameters on the whole network. We end our work with a conclusion in Sect. 4.

2 Model

Here we present a mobile system where the elements are represented by N nodes with two parts, agents and oscillators. Agents move in space following the description in Eq. 1, while at the same time they have an internal dynamics. This internal dynamics will be represented by that of an oscillator (in this case a Rössler oscillator, see Eq. 5), and we refer to them as *oscillators*. Thereby, the numbers of agents and oscillators are identical. Usually, the model used to describe the dynamics of the agents in a finite two dimensional space is given by [9,19]:

$$\begin{cases} \eta_i\left(t+\Delta t\right) = \eta_i\left(t\right) + u_i \cos\left(\theta_i\left(t\right)\right) \\ \xi_i\left(t+\Delta t\right) = \xi_i\left(t\right) + u_i \sin\left(\theta_i\left(t\right)\right) \end{cases} \tag{1}$$

Where $\eta_i\left(t\right)$ and $\xi_i\left(t\right)$ with $(i = 1, 2, ..., N)$ are the position coordinates of the agents, N is the number of agents. Each agent moves with the speed u_i (with the modulus u) which is generally the same for all agents and the direction of each agent is determined by the value of the angle $\theta_i\left(t\right)$ in a 2D space (size $P \times P$) with periodic boundary conditions, and P is the side of the square lattice. The movement of the agents is random, in the finite two dimensional space defined previously. The angles $\theta_i\left(t\right)$ are generated randomly and updated after each step Δt. In previous works the coupling between oscillators is determined by the proximity of the agents in space [9,19]. The neighbors which participate in the interaction are determined by the interaction range of the oscillators, as defined below. In the present manuscript, we propose a model in which the oscillators can influence the interaction between the agents and vice versa. It is important to notice that there is no direct interaction between agents and oscillators, Each element will only interact at a certain range with other elements of the same kind, that is agent i will interact with agent j provided oscillator i interacts with oscillator j, and vice versa, as will be defined below. This type of coupling is generally used in the swarmalator systems, where each oscillator has an internal oscillatory phase. An oscillator's movement and change of internal phase both depend on the positions and internal phases of all other oscillators. Because of this entanglement of spatial forces and phase coupling the oscillators are called swarmalators [6,20,21]. Such a coupling has already been used to couple the dynamics of oscillators in fast communications [9].

We define the dynamics of the agents in the network as follows:

$$\begin{cases} \eta_i\left(t+\Delta t\right) = \eta_i\left(t\right) - u_i \sum_{j}^{N} G_{ij}\left(t\right) \cos\left(\theta_i\left(t\right) - \theta_j\left(t\right)\right) \\ \xi_i\left(t+\Delta t\right) = \xi_i\left(t\right) - u_i \sum_{j}^{N} G_{ij}\left(t\right) \sin\left(\theta_i\left(t\right) - \theta_j\left(t\right)\right) \end{cases} \tag{2}$$

$$G_{ij}\left(t\right) = \begin{cases} 1 & if \;\; D_{ij}\left(t\right) \le D_0 \\ 0 & otherwise \end{cases} \tag{3}$$

D_{ij} measures the distance between states of the oscillators i and j and it is defined below (Eq. 4).

As defined previously, the couple (η_i, ξ_i) is the position coordinates of the i^{th} agents. Updating this position coordinates after each iteration, a new direction is imposed on each agent by randomly choosing a new value of $\theta_i\left(t\right)$ between $\left[0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right]$. $G_{ij}\left(t\right)$ is the agents' connectivity matrix corresponding to the elements i and j. D_0 determines the maximum distance defined in the $3N$ dimensional space of the oscillators' variables, that separates two oscillators which are interacting. It delimits the interaction range of a considered oscillator. D_{ij} measures the distance between states of the oscillators i and j and it is expressed as:

$$D_{ij}(t) = \sqrt{(x_{ij}(t))^2 + (y_{ij}(t))^2 + (z_{ij}(t))^2} \quad (4)$$

with $x_{ij} = (x_i(t) - x_j(t))$, $y_{ij} = (y_i(t) - y_j(t))$ and $z_{ij} = (z_i(t) - z_j(t))$. The $(x_i(t), y_i(t), z_i(t))_{i=1,2,\ldots,N}$ are the state variables of the oscillators. In this manuscript, we are dealing with the Rössler system used in ref [19], defined by:

$$\begin{cases} \dot{x}_i = -y_i - z_i \\ \dot{y}_i = x_i + a y_i - k \sum_j^N g_{ij}(t)(y_i - y_j) \\ \dot{z}_i = b + (x_i - c) z_i \end{cases} \quad (5)$$

with $a = 0.2$, $b = 0.2$ and $c = 5.7$

$$g_{ij}(t) = \begin{cases} 1 & if \;\; d_{ij}(t) \leq d_0 \\ 0 & otherwise \end{cases} \quad (6)$$

$g_{ij}(t)$ is the element connectivity matrix between the oscillators i and j. d_0 is the maximum radius of the vision range of the agents in the space (η, ξ), which we call agents vision range control. $d_{ij}(t)$ is the distance between the agents i and j in this space. The expression for $d_{ij}(t)$ is given by:

$$d_{ij}(t) = \sqrt{(\eta_i(t) - \eta_j(t))^2 + (\xi_i(t) - \xi_j(t))^2} \quad (7)$$

k is the coupling coefficient between the oscillators.

3 Numerical results

We consider a a network of 100 elements (each element is constituted of one agent and one oscillator). The surface or the space where the agents can move is defined by $P \times P$ with $P = 200$. The initial conditions of the state variables of the oscillators (x_i, y_i, z_i) are uniformly distributed in the range ([0.1 , 1.0], [0.0 , 1.0], [0.5 , 1.0]). The initial conditions of the positions of the agents (η_i, ξ_i) are also randomly and uniformly generated in the range ([0.0 , 15.0], [0 , 10.0]).

3.1 Space parameters' influence on the network's dynamics

The goal of this subsection is to investigate the space parameters' impact on the whole network: we study the influence on the dynamics of the network when we vary the parameter that controls the vision range of the agents and their speeds. Using some statistical methods such as the order parameter [27] (developed in Appendix 1) to characterize the phase synchronization, we present in Fig. 1a the evolution towards phase synchronization of the network as a function of d_0, the parameter that controls the vision range of the agents. This figure shows in red the order parameter of the oscillators r and in blue the order parameter of the agents r_p. These results show the transition toward synchronization in the oscillators as well as the agents for the system parameters $u = 0.5$, $k = 1.0$, $D_0 = 2$. For these parameters, we observe that d_0 has a great impact on the dynamics of the whole network. This parameter leads the agents and the oscillators toward a synchronization state. Figure 1b represents the same order parameter as a function of the speed u for the system parameters $d_0 = 3$, $k = 1.5$, $D_0 = 1$. For these values, we observe an abrupt transition towards synchronization in the oscillators and in the agents. This transition is similar to an explosive synchronization in both oscillators and agents. It should be noted that the dynamics of the agents can only have phase synchronization (since two individuals cannot occupy the same position at the same time in a space.) In real systems such as cars, drones, robots. etc., complete synchronization is a state to be avoided as it may lead to disasters. Using the Master Stability Function developed by Pecora and Carroll [24] and explained in the Appendix 1 for our model, we show in Fig. 1c, d the Largest Lyapunov Exponent (LLE) of the oscillators as a function of d_0 and u respectively. For both cases, the control parameter shows that d_0 and u lead the network to a synchronous state and we obtain a class II Master stability function and the network always synchronizes for a large enough coupling strength, as shown by Boccaletti et al. [25], which in our case happens for the critical value of the control parameters ($d_{0c} = 2.5$ and $u_c = 0.23$). After this critical value in each case, the Largest Lyapunov Exponent (LLE) becomes negative and then, all oscillators in the network synchronize.

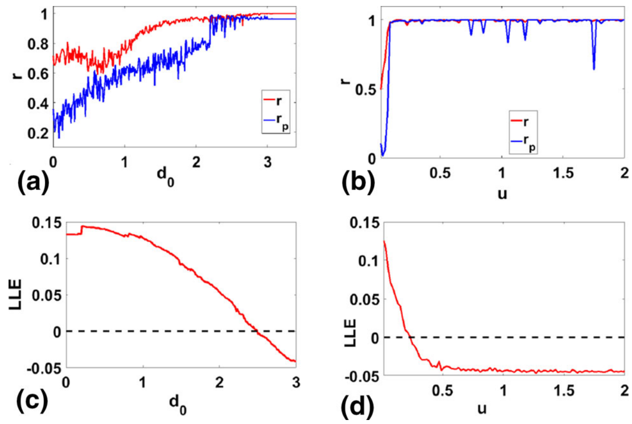

Fig. 1 Order parameter for the system of oscillators (red) and mobile agents (blue) as a function of: **a** d_0 for the values $u = 0.5$, $k = 1.0$, $D_0 = 2$ and **b** the speed u for the values $d_0 = 2.5$, $k = 1.5$, $D_0 = 2$. **c** Largest Lyapunov exponent as function of d_0 with the same parameters' values as in **a** and **d** LLE as a function of u with $d_0 = 2.5$, $k = 1.5$, $D_0 = 2$

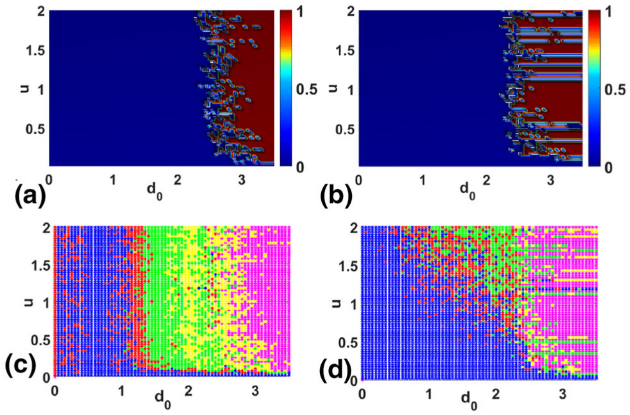

Fig. 2 Dynamics of the whole network as function of d_0 and u. Order parameter of: **a** the oscillators and **b** agents. **c** and **d** Dynamics of the number of states in the oscillators' and the agents' systems respectively for $k = 0.5$ and $D_0 = 2$

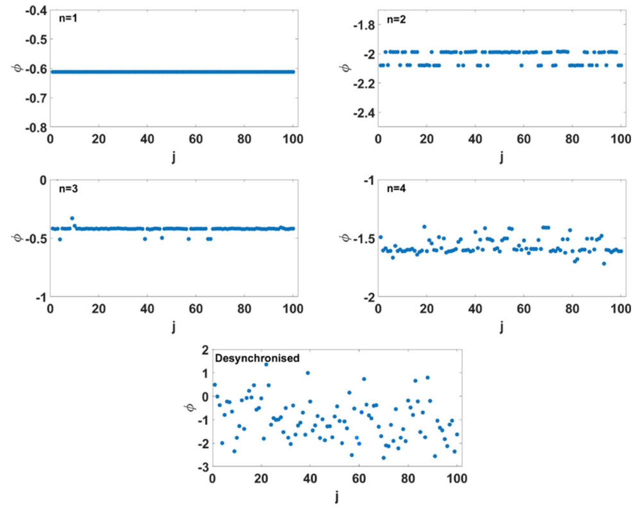

Fig. 3 Snapshots of the oscillators' phase dynamics for different number of states as described in Fig. 2c

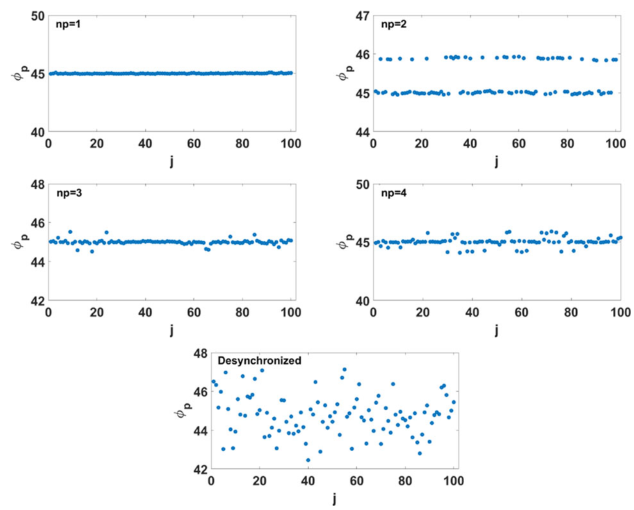

Fig. 4 Snapshots of the agents' phase dynamics for different number of states as described in Fig. 2d

To appreciate the impact of the speed and the vision range control of the agents on the dynamics of the whole network, we evaluate the order parameter [27] of all agents and oscillators to characterize the phase synchronization of the coupled systems. Based on this method and these parameters, we present in Fig. 2a, b the order parameter of the oscillators and the agents respectively as a function of u and d_0. For $k = 0.5$ and $D_0 = 2$, theses figures show two different dynamics: red color characterizes complete phase synchronization in the oscillators and agents. On the other hand, the blue color shows where the synchronization does not exist. It should be noted that, these regions where the synchronization does not exist can posses other interesting dynamics such as clusters formation characterized here by the number of states. This number of different states defines the number of clusters existing in a considered network for a chosen set of parameters. Therefore, Fig. 2c, d show the evolution of the number of states or the number of clusters in the oscillators and agents respectively. We can identify in this figure five domains defined by five colors: blue) parameters' region where the number of states is higher than or equal to five; red) domain where the number of states (clusters) is equal to four ($n = 4$) in both oscillators and agents; green) region with three ($n = 3$) clusters; yellow) region with two ($n = 2$) clusters; pink) region where we have one ($n = 1$) cluster. The result for $n = 1$ is confirmed by the order parameter shown in Fig. 2a, b. According to these results, it is clear that the parameters d_0 and u lead the network to a synchronization state passing by clusters formation. To better appreciate these results, we have represented the oscillators' phase dynamics' snapshots in Fig. 3 as well as the agents' phase dynamics' snapshots in (in Fig. 4) as a function of their indices (and irrespective of their spatial position) for every number state of the system mentioned above (i.e. for $n = 1$, $n = 2$, $n = 3$, $n = 4$ and for n greater than or equal to 5 or desynchronized).

3.2 Phase parameters' influence on the network's dynamics

This section deals with the investigation of the oscillators' parameters' influence on the whole network. These parameters are the coupling coefficient of the oscillators k and D_0, which controls the vision range of the oscillators. As done previously (Sect. 3.1), we proceed through the order parameter to measure the level of the synchronization between both oscillators and agents. The results of this investigation as a function of the vision range of the oscillators is presented in Fig. 5a. In this figure, the curve in red shows the order parameter of the oscillators and the curve in blue that of the agents. This result suggests that depending on the value of D_0, phase

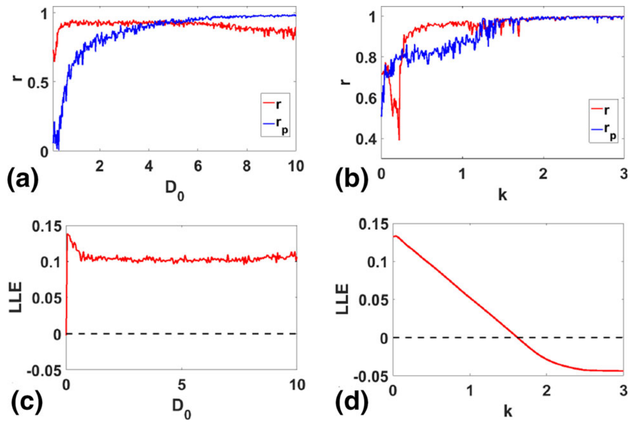

Fig. 5 Synchronization dynamics in the oscillator's network (red) and in the moving network (blue): **a** as a function of the values of the oscillators' vision range D_0 ($u = 1$, $k = 0.5$, $d_0 = 2$); **b** as a function of the values of the coupling coefficient k ($u = 1$, $D_0 = 2$, $d_0 = 2$); **c** and **d** evolution of the Largest Lyapunov Exponent in the oscillator's network, as a function of D_0 for $k = 0.5$ and of k for $D_0 = 2$ respectively

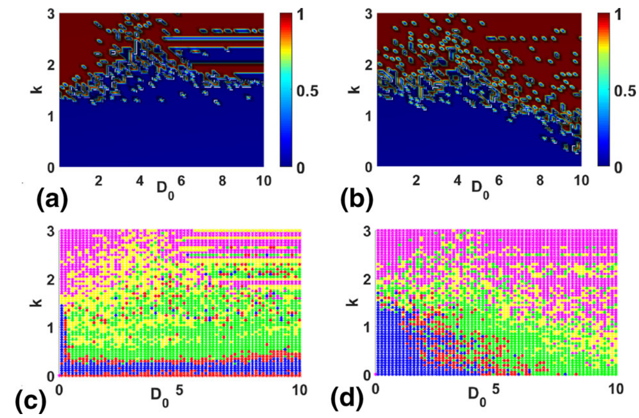

Fig. 6 Dynamics of the whole network as a function of k and D_0 for the values of: $u = 1$ and $d_0 = 2$. **a** order parameter of the oscillator's network; **b** order parameter of the moving agents' network. **c** and **d** number of states of the oscillators' and the agents' networks respectively

synchronization can occur in the system of oscillators. But this parameter leads the agents towards a synchronization state, thus proving that the vision range of the oscillators has a particular impact on the synchronization of the whole network. Using the same logic, we present the order parameter of the oscillators (red) and of the agents (blue) as a function of the coupling coefficient k in Fig. 5b. This figure shows that the synchronization occurs in both, oscillators and agents. To understand the synchronization phenomenon, we study the emergence and stability of the synchronization of the oscillators using the master stability function (See 1). We present the Largest Lyapunov Exponent (LLE) as a function of D_0, for parameter values $u = 1$, $k = 2$ and $d_0 = 2$ in Fig. 5c, where we notice that the Largest Lyapunov Exponent is always greater than zero, which corresponds to a class I master stability function [25]. According to this result, complete synchronization does not exist between the oscillators when varying D_0, for $k = 0.5$. When varying k, we observe a type II master stability function according to Boccaletti et al. [25] (Fig. 5d). This parameter leads the oscillators to a state of complete synchronization above the critical value k_c ($k_c = 1.635$). Thus, we see that the coupling parameter (k) has a great impact on the whole network.

To get an insight on the impact of the oscillators parameters on the whole network, we plotted the order parameter of the oscillators and the agents as a function of D_0 and the coupling coefficient, k, simultaneously, which is shown in Fig. 6a, b for the oscillators and agents respectively, with the parameter values $u = 1$, $d_0 = 2$. In this figure, the blue color shows where phase synchronization does not exist in the oscillators nor in the agents, while the red color characterizes complete phase synchronization in both oscillators and agents. We find that phase synchronization is unstable in both oscillators and agents. This instability is characterized by the spots observed in the red colored region. To understand the dynamics of the region where synchronization does not exist (blue), we used the number of states to determine if clusters existed in the whole network. Figure 6c, d present the distribution of the number of states in the oscillators and agents respectively in the (k, D_0) space. As before, we identified five domains defined by five colors in these figures. The blue domain represents the region of parameters where the number of states is greater than or equal to five. In red, we identify the domain where the number of states (clusters) is equal to four ($n = 4$) in both oscillators and agents; in green we show the region where we have three ($n = 3$) clusters and finally the yellow color delimits the region where we have two ($n = 2$) clusters while pink shows the region where we have only one ($n = 1$) cluster. The result for $n = 1$ is confirmed by the order parameter red shown in Fig. 6a, b. These results show that the emergence of synchronization based on the vision range of oscillators and the coupling coefficient passes through the formation of clusters.

3.3 Agents' and oscillators' vision range's mutual influence on the network's dynamics

According to Mahji et al. [19], it has been demonstrated that, considering the vision range of the agents. the oscillators may achieve complete synchronization. But they did not take into account the effect of the vision range of the oscillators on the dynamics of the agents. In this section we investigate the impact of both vision range of the agents as well as vision range of the oscillators on the dynamics of the whole network. For this study, we evaluate the order parameter in the agents which we use to characterize phase synchronization in the agents. For the case of the oscillators, we evaluated the same order parameter and the Master Stability Function to characterize phase and complete synchronization.

We explore the impact of the control parameters D_0 and d_0 on the dynamics of the oscillators of the network (see Fig. 7). For a simultaneous variation of d_0 between 0 and 3, and D_0 between 0 and 10, and using the MSF we identify a zone of complete synchronization (CS), the zone of phase synchronization (PS) and the zone where there is no synchronization (see fig. 7a). Snapshots are provided to appreciate all three phases: desynchronization, phase synchronization and complete synchronization.

In Fig. 7b we show the results for the dynamics of the agents: the order parameter is evaluated for the agents for the same range of the value D_0 and d_0 used in Fig. 7a. Here we distinguish two regions: PS, where we have phase synchronization (in green) and DS, where the synchronization does not exist (blue). Based on Fig. 7a, b, we conclude that complete synchronization in the oscillators does not automatically imply phase synchronization in the agents. On the other hand, we can have regions where phase synchronization in the agents and oscillators coexist.

Following the reasoning for the oscillators (Fig. 8a), we evaluate the Master Stability Function as a function of the coupling coefficient k and speed of agents u simultaneously to identify the values of k and u which lead oscillators to complete synchronization (CS). We also evaluate the order parameter to identify the values of these parameters which lead oscillators to phase synchronization (PS) and where synchronization does not exist (DS). The snapshot of the dynamics of the oscillators for some chosen zones is represented in Fig. 8a.

Figure 8b shows the order parameters for the agents. This figure presents two different zones: the zone in green presents the values of k and u which lead agents to phase synchronization (PS) and secondly the zone in blue is where synchronization does not exist (DS). This figure also shows the agents' phase dynamics' snapshot for each zone.

Following the same scheme, Fig. 9 shows the dynamics of the whole system as function of the speed u and the vision ranges delimiting parameters for the agents and oscillators (d_0 and D_0). In this case, we assume both vision range control parameters are identical ($d = d_0 = D_0$). Figure 9a presents the dynamics of the oscillators as a function of the values of the speed u and the vision ranges d. This figure shows three domains, the domain where synchronization does not exist (DS), where we obtain phase synchronization (PS) and where we have complete synchronization (CP). Figure 9b shows the dynamics of the order parameter of the agents. The region in green indicates where the agents are in a synchronous state and the blue color indicates where synchronization does not exist.

4 Conclusion

In this work, we proposed a mobile system consisting of a network of agents with an oscillator associated to it which has the same index numbers and represents its

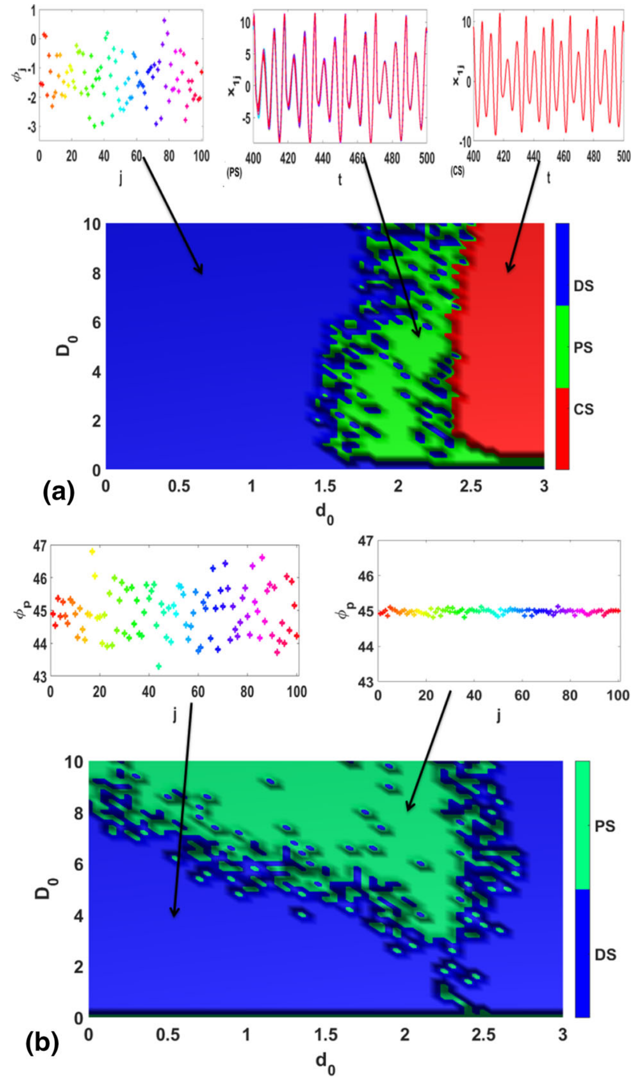

Fig. 7 Order parameter of oscillators (**a**) and moving agents (**b**) as a function of d_0 and D_0 ($u = 1.5$, $k = 1.5$). DS, PS and CS indicate the regions where the network's elements are in a disordered phase state, phase synchronization and complete synchronization respectively

internal dynamics, moving in a two dimensional space whose topology changes according to the proximity of the state of the oscillators and a network of oscillators, whose dynamics changes according to the proximity of the mobile agents. We have studied the influence of all parameters in the complete system on the synchronization dynamics of each part of the system. Using the order parameter, we have shown the existence of synchronization in the oscillators system (as mentioned in the literature [19]) and in the agents system moving in space under variation of the mobility parameters (d_0 and u). This result is the same when we vary the parameters related to the oscillators' system (D_0 and k). The analysis of the transition to synchronization allowed us to show that the synchronization process in both the oscillators' and agents' systems goes through clusters formation. We also showed the stability of synchroniza-

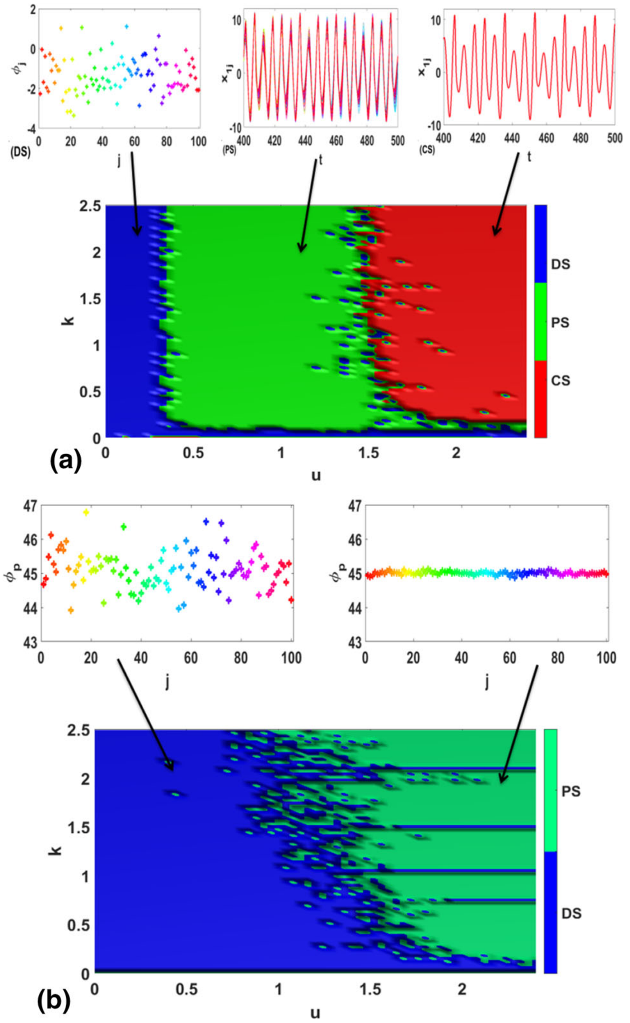

Fig. 8 Order parameter of oscillators (**a**) and agents (**b**) as a function of the coupling coefficient k and the velocity u ($d_0 = 2$, $D_0 = 3$). The surfaces DS, PS and CS indicate respectively the regions where the network's elements are in a disordered phase state, phase synchronization and complete synchronization

tion in the oscillator system using the master stability function and obtained a class II master stability function according to Boccaletti et al. [25] with the coupling coefficient (k), agents vision range control (d_0) and velocity (u). With regard to the oscillators' vision range control, depending on the range of values of the coupling constant ($k < k_c$), it does not lead the oscillators' system to a stable synchronized state and we obtained a class I master stability function. The mobile network is more sensitive to the effects of the viewing distance of the oscillators. These results are essential not only because they contribute to a better understanding of the phenomenon of synchronization in mobile systems, but above all because they show that it is possible to establish a link between the way agents move and their internal dynamics. Overall, these results find application in the coordination of group movements.

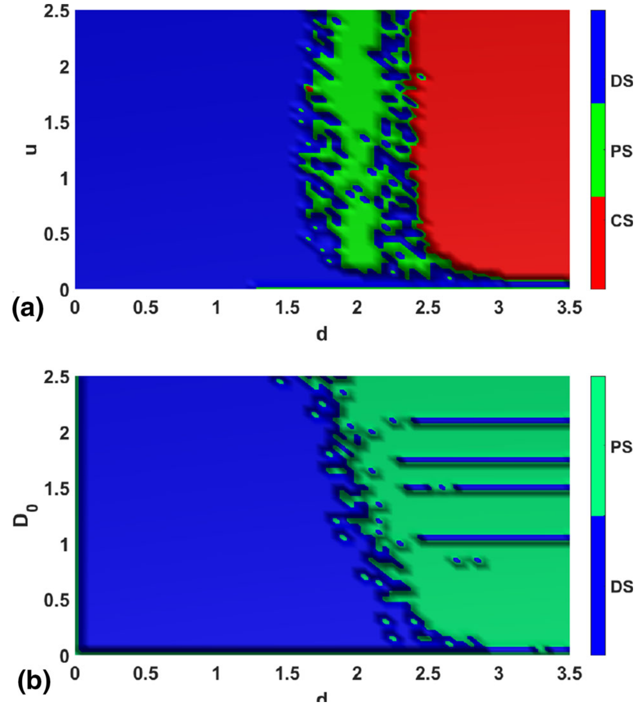

Fig. 9 Dynamics of oscillators (**a**) and moving agents (**b**) as a function of the speed u and for the same value d for d_0 and D_0 ($d = d_0 = D_0$) for $k = 1.0$. The surfaces DS, PS and CS indicate respectively the regions where the network's elements are in a disordered phase state, phase synchronization and complete synchronization

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Appendix A: Evaluation of Order parameter

The Order parameter, defined by Kuramoto and Battogtokh [27], is a good tool used to analyse phase synchronization in coupled systems as well as in networks. The computation of the Order parameter needs the phase of each system. To calculate this phase, we consider an arbitrary time signal $s(\tau)$ and $\tilde{s}(\tau)$ being its Hilbert transform. We have:

$$\psi(\tau) = s(\tau) + j\tilde{s}(\tau) = R(\tau)\, e^{j\varphi(\tau)} \tag{A.1}$$

where $R(\tau)$ is the amplitude and $\varphi(\tau)$ the phase of the variable $s(\tau)$. If we denote by $\varphi_i(\tau)$ the instantaneous phase, then it can be determined by:

$$\varphi_i(\tau) = \tan^{-1}\left(\frac{\tilde{s}_i(\tau)}{s_i(\tau)}\right) \tag{A.2}$$

The Order parameter for a system with N oscillators is expressed as:

$$r = \frac{1}{N}\sum_{i=1}^{N} e^{j\varphi_i} \tag{A.3}$$

Where $j^2 = -1$. Phase synchronization is effective when the value of $r = 1$ and when $r = 0$, the network is completely desynchronized.

Appendix B: Evaluation of Master Stability Function

Developed by Pecora and Carroll in 1998, the Master Stability Function (MSF) is the best tool used in a network of coupled identical systems to demonstrate the stability of the complete synchronization of the network [24,26]. To facilitate the comprehension of some results presented in this paper, we develop here some important points of this method. Let us consider an isolated system (oscillator) defined by Eq. B.1:

$$\dot{x}_i = F(x_i) \tag{B.1}$$

Where x_i is a vector of m-components used to describe the state of the ith oscillator; F is a function defined from $R^m \longrightarrow R^m$ used to define the local synchrony of the oscillators. Taking into account the interaction or the connection between the N oscillators of the network, Eq. B.1 is not sufficient to describe the dynamics of the network. Taking into account the interactions the governing law of the i^{th} oscillator is given by:

$$\dot{x}_i = F(x_i) + k\sum_{j=1}^{N} G_{ij} H(x_j) \tag{B.2}$$

In this equation (Eq. B.2), k represents the coupling strength; $H : R^m \longrightarrow R^m$ is an arbitrary coupling function and G is a Laplacian matrix. The oscillators of the network are synchronized if all oscillators converge toward the same state s such as $x_1 = x_2 = ... = x_N = s$. It should be noted that, for the N nodes we can design N state variables, N coupling functions and N local functions F contained into the matrix described respectively by Eqs. B.3, B.4 and B.5.

$$x = [x_1;\ x_2;\ \ldots;\ x_N] \tag{B.3}$$

$$H(x) = [H(x_1);\ H(x_2);\ \ldots;\ H(x_N)] \tag{B.4}$$

$$F(x) = [F(x_1);\ F(x_2);\ \ldots;\ F(x_N)] \tag{B.5}$$

Based on Eqs. B.3, B.4 and B.5, B.2 can be expressed in compact form as:

$$\dot{x} = F(x) + kG \otimes H(x) \tag{B.6}$$

where $\otimes$ is the Kronecker product. For the case of our network, we have:

$$F(x_i) = \begin{cases} -y_i - z_i \\ x_i + ay_i \\ b + (x_i - c)\, z_i \end{cases} \quad \text{and} \quad H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{B.7}$$

Initially, we suppose that δx_i is a small perturbation of the i^{th} oscillators of the network. After perturbation, the state variable of the i^{th} oscillators becomes $x_i = s + \delta x_i$. For the whole network, the collection of the variation of the N oscillators is expressed by $\delta x = [\delta x_i;\ \delta x_i;\ \ldots;\ \delta x_i]$. Replacing the perturbation in Eq. B.2 and using the Taylor expansion theorem for $F(s + \delta x_i)$ and $H(s + \delta x_i)$ to first order, the following variational equation is obtained:

$$\dot{\delta x_i} = DF(s)\,.\delta x_i + k\sum_{j=1}^{N} G_{ij} DH(s)\,.\delta x_j \tag{B.8}$$

where $DF(s)$ and $DH(s)$ are the $N \times N$ Jacobian matrices of the corresponding vector functions evaluated at the synchronous state $s(t)$. Equation B.8 is used to explore if the synchronous state is stable or unstable. According to Peccora and Carroll the use of tensor notation leads to Eq. B.9:

$$\dot{\delta x} = [1_N \otimes DF(s) + kG \otimes DH(s)].\delta x \tag{B.9}$$

Due to the degree of Eq. B.9, the solution can be in the form $\delta x_i \sim exp(\lambda_i t)$ where exponent λ helps to us to know if the perturbation grows ($\lambda > 0$) or decays ($\lambda < 0$). So, the digitalization of the second term of Eq. B.9 helps us to obtain the following variational equation expressed as:

$$\dot{\delta x_k} = [DF(s) + k\alpha_k DH(s)]\,.\delta x_k \tag{B.10}$$

α_k is the eigenvalue of the matrix G, $k = 1, 2, \ldots, N$. Finally, these steps help us to design the following Master Stability Eq. B.11:

$$\dot{\delta x} = [DF(s) + \alpha_k DH(s)]\,.\delta x \tag{B.11}$$

ased on refs [24,28] the synchronization is stable if the largest Lyapunov Exponent computed from Eq. B.11 which corresponds to the largest eigenvalue is negative and unstable otherwise.

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