Eur. Phys. J. Spec. Top. (2021) 230:495–503
https://doi.org/10.1140/epjs/s11734-021-00005-3

THE EUROPEAN PHYSICAL JOURNAL SPECIAL TOPICS

Review

Hundred years of the Saha equation and astrophysics

Gautam Gangopadhyay[a]

Department of Physics, University of Calcutta, 92 Acharya Prafulla Chandra Road, Calcutta 700009, India

Received 30 July 2020 / Accepted 23 December 2020 / Published online 29 March 2021

Abstract Saha ionization equation, discovered 100 years ago, is widely acknowledged to transform astrophysics from a qualitative to a quantitative science. It helped in clarifying the confusion prevailing in the first 2 decades of the twentieth century with regard to the physical conditions present in the stellar atmospheres and the abundance of elements in stars. Saha equation continues to be useful in areas far removed from the physical conditions in stellar atmospheres for which it was developed.

1 Introduction

It is now a 100 years that Meghnad Saha, working in the Department of Physics of the University of Calcutta, proposed his famous equation to describe the ionization in the stellar environment. Arguably the transformation of astrophysics from a qualitative to a quantitative science can be traced to the Saha ionization equation. In this article, we take a brief look at how, in the decade following the discovery of the equation, Saha and other astrophysicists used it to understand the stars. We also show that the equation continues to be useful, even a century after its discovery, in various areas of astrophysics which developed much later.

The article is arranged as follows. In Sect. 2 of this brief article, we discuss the scenario in astrophysics when the equation was proposed. Section 3 traces the development of the equation and its application by Saha. Immediate impact of the equation on other researchers in the field of stellar astrophysics is discussed in Sect. 4. For more recent applications of the equation, we select two areas which developed long after the ionization equation was developed. We briefly point out some of the applications of Saha equation in nuclear astrophysics at the present time with an emphasis on supernova nucleosynthesis in Sect. 5. The importance of Saha equation in understanding the recombination epoch is discussed in Sect. 6. Finally we summarize our discussions.

2 Historical background

The first 2 decades of the twentieth century saw great strides in observational astronomy. Particularly, spectral analysis of the Sun and stars yielded rich dividends. Three key problems were highlighted by the wealth of data. (1) There was the need to assign an absolute value of temperature to the stars and explain the difference among the spectra of the Sun and the stars. (2) The interpretation of the Harvard classification of stars remained a mystery. (3) The elemental composition of the stars was almost unknown. These problems were solved by the discovery of Saha equation. We very briefly summarize the prevailing situation in these areas prior to the publication of the equation.

Establishing the temperature scale was an urgent requirement. Various techniques for measurement of the temperature of stellar atmospheres were available to the astronomers, but frequently those did not agree among themselves. Spectrometric analyses of solar and stellar spectra, assuming the objects to be blackbodies, were compared with measured and theoretical blackbody spectral distribution to extract theoretical values for the temperature [1,2]. However, the deviation from black body spectra is severe in the case of the hottest stars and consequently, there were large errors in their measured temperature values. Observed stellar spectra were also compared with laboratory spectra of elements to assign temperature and pressure values to the stellar atmospheres. Comparison of laboratory spectra with those of red stars such as Arcturus and Betelgeuse, and also with sunspot spectra indicated that their temperature was lower than that general solar atmosphere [3]. Study also indicated that there were significant pressure variations in the atmospheres of different classes of stars [4]. However, no absolute value of temperature or pressure could be inferred. Temperature was generally obtained relative to that of the Sun but, the solar value obtained from observations was model-dependent and varied widely. A reasonable value for the solar atmosphere was obtained by Stefan [5] using his own law, but

[a] e-mail: ggphy@caluniv.ac.in (corresponding author)

acceptance was delayed and only in the second decade of the twentieth century a consensus emerged on its value.

Understanding the reasons behind stellar classification was another problem. The Harvard classification of stars was based on their line spectra. Spectral lines of ions and neutral atoms of helium and metals, and of atoms of hydrogen appear and disappear smoothly as one goes through the various classes of stars. There was an idea that different classes of stars perhaps had different surface temperatures. The Hertzsprung–Russel (HR) diagram clearly pointed to a relation between temperature and luminosity, though the absolute values of the temperature remained elusive [6]. The physical basis of the Harvard classification or the HR diagram was unclear. Milne noted that though the relation between HR diagram or the spectral type and the effective temperature was definite, the connection was empirical only and lacked logical foundation [7]. Russell, while presenting his version of what we now call the Hertzsprung–Russell diagram, had argued that the sequence was an order of advancing age [8]. This was the opinion of most of astrophysicists.

The abundance of elements in stars was generally considered to be known till Saha equation established that the prevailing opinion was completely erroneous. The constituents of the Sun had been inferred in two ways: estimating the intensities of the spectral lines of different elements in the solar spectrum, and counting the number of spectral lines of different elements in it. At the end of the nineteenth century, the general idea had been that the abundance of various elements in stars closely resembles their distribution in the earth's crust [9]. Spectroscopic evidence obtained from study of the Sun in the next two decades seemed to support this argument. For example, Russell noted that six of the eight elements, which are most common in the Earth's crust, are found to be among the elements which show the strongest lines in the solar spectrum [10]. Fowler also ventured the opinion that the composition of the Sun may be practically identical with that of the Earth [11]. Thus, the accepted opinion was that there were very little hydrogen and helium in the stars. However, the absence of some other elements in the solar spectra was still a puzzle.

There were other puzzles in the solar spectra. Helium was discovered in the Sun through spectroscopy of the emission spectrum of solar corona. Surprisingly it was nearly absent in the absorption spectrum. As also noted by Saha himself, lines of different elements and ions were seen at different heights in the solar chromosphere. The reasons behind these facts were not known.

Detailed discussions on the scenario in astrophysics prior to the discovery of Saha equation may be found in two articles by DeVorkin and Kenat [12,13].

3 Saha and the equation

Meghnad Saha proposed the ionization equation in Ref. [14] and followed it up in a series of articles. He explicitly introduced the Bohr atomic model to astrophysics by using it to explain ionization of atoms, and also to calculate the probability of an atom being in an excited state in a high-temperature environment. He exploited the idea of thermodynamic equilibrium and compared the phenomena of ionization to chemical dissociation. Using the language of physical chemistry, he called it the equation of the reaction-isobar for ionization. The reaction isobar K is related to the total pressure P and the fraction of ionized atoms x. In Ref. [14], he applied it to the case of ionization of atoms such as calcium, barium, helium, hydrogen, etc. Starting from Nernst's formula of reaction isobars, he showed that K may be written as a function of absolute temperature T as

$$\log K = \log \frac{x^2}{1-x^2} P = \frac{-U}{4.571T} + 2.5 \log T - 6.5, \qquad (1)$$

where U is the ionization energy per mole in calories and pressure is measured in atmospheric units. He used the Sackur–Tetrode–Stern relation for Nernst's chemical constant, which occurs in the equation for chemical equilibrium, to arrive at Eq. (1). It is important to note that the equation comes from the theory of chemical reaction in equilibrium. Saha assumed that the gas molecules in stellar atmosphere behave like a classical ideal gas. At high temperature, the atoms may be ionized; on the other hand the ions may pick up electrons to be neutralized. At equilibrium the forward and the reverse rates become equal.

In the first paper of the series [14], Saha pointed out that pressure has an unexpectedly strong effect on the ionization. Calcium atoms are only partially ionized in the photosphere due to higher pressure. However, they are completely ionized in the chromosphere, where the temperature is lower, but so also is the pressure. He also explained that it is possible to obtain ionized helium lines only in the hottest stars because helium has a high ionization potential .

In the following paper [15], Saha convincingly argued, from Eq. (1), that alkali metals like rubidium and caesium are completely ionized at solar temperature, and will show up only in the ultraviolet part of the solar spectrum. He suggested that spectrometric observations of sunspots, which are cooler, may reveal the presence of these elements. For rubidium. this was almost immediately confirmed by Russel from existing photographs [16]. Saha stressed on the importance of measuring ionization potential of various elements, which were not known at that time. It is a matter of historical interest that, despite his best efforts, he failed in his attempts to create an experimental set up to measure the ionization potential of elements.

In the third paper on the ionization equation [17], Saha tackled the problem of radiation from a gas at an elevated temperature. He pointed out that the line emission spectra depends strongly on the temperature and also on the degree of ionization. At elevated temperatures, it is possible for an atom to exist in an excited state. If such an atom absorbs a photon, its signature will be seen in the absorption spectra. These will be visible in stars with a high effective temperature. He also argued that some of the spectral lines are missing in the absorption spectra because the temperature of the gas is not high enough for atoms to exist is significant numbers in the states corresponding to those transitions.

In the fourth paper [18] of the series, Saha applied the equation to explain the Harvard classification of stars. He discussed excitation as well as single and multiple ionization of atoms, and showed how competition between these phenomena can explain the appearance and disappearance of spectral lines of various elements and ions, and thus lead to the observed spectral sequence as a function of stellar temperature. By considering the temperatures at which different spectral lines show marginal appearances, he could deduce an absolute temperature scale for all the classes of the stars.

This paper contains an idea about the composition of stars, which was carried forward, as we will see in the next section, not by Saha but by others. He suggested that whether lines of a particular element appear or disappear in the stellar spectra depends crucially on the physical conditions, i.e. temperature and pressure in the stellar atmosphere. Thus one is not justified in deciding on the constituents of a star based only on the presence of characteristic lines of elements and ions in its spectra. This pointed to the uniformity of composition of stellar atmospheres. He emphasized that the absence of Lymann series of hydrogen or corresponding series of helium in stellar spectra does not indicate that these elements are absent. However, he could not explain the unexpectedly strong hydrogen lines in hot stars and suggested that they may be from doubly ionized helium or triply ionized lithium.

In this way, Saha showed that an analysis of the stellar spectra can lead to measurements of the physical conditions in the stars and thus connected the Bohr atomic model, spectroscopy and the stars. It is no exaggeration to say that this ushered in the era of quantitative astrophysics. In the next section, we will consider the works of other astrophysicists in this regard.

Although the importance of the equation was clear, physicists hesitated to accept Saha's method of derivation, which utilized the concepts of physical chemistry. It became widely accepted only after Fowler and Milne rederived the equation using Boltzmann's statistical approach [19]. This work was based on Fowler and Darwin's work for the treatment of assembly of atoms in ground and excited states as well as ions in equilibrium [20,21]. A brief summary of the derivation is given below for completeness.

The number densities of i-th and $(i+1)$-th ionized state of the same element are given by n_i and n_{i+1}, respectively, while the density of free electrons is given by n_e. The energy to free an electron from the ground state of an ion or atom in the i-th ionization state is given by U_i. The energy of an electron of momentum p with respect to the same state is $U_i + p^2/2m$, the electron mass being m. Thus, at temperature T, the ratio of atomic population, in states of successive ionization, is given by the Boltzmann relation, including two spin states for the electron, as

$$\begin{aligned}\frac{n_{i+1}}{n_i} &= \frac{2G_{i+1}}{G_i}\frac{V_\mathrm{e}}{(2\pi\hbar)^3}\int_0^\infty \exp\left(-\frac{U_i+p^2/2m}{kT}\right)4\pi p^2\mathrm{d}p\\ &= \frac{2G_{i+1}}{G_i}\frac{1}{n_\mathrm{e}(2\pi\hbar)^3}(2\pi mkT)^{3/2}\exp(-U_i/kT), \quad (2)\end{aligned}$$

where G_i is the partition function for the i-th ionized state. The last step involved the relation between the volume available for an electron V_e and electron density n_e, i.e. $V_\mathrm{e} = 1/n_\mathrm{e}$. This is the more familiar form of the equation. Using the ideal gas law for pressure of electrons, $P = n_\mathrm{e}kT$, it can be reduced to Eq. (1) for neutral and singly ionized atoms after certain approximations.

It should be noted that the ionization equation was published earlier than Saha by Eggert [22] and Lindemann [23]. However, in a recent article Rai Choudhuri [24] has argued that, though Saha was not the first one who arrived at the equation, he definitely first understood clearly that the equation, with data from atomic physics, will unlock the stellar spectra. Cenadelil [25] also is of the opinion that Saha's work is a fundamental work in history because, although he was not the first to arrive at the equation, he first showed that it can be a bridge between observational data and quantum theory, and will lead to the spectral sequence and quantitative spectral analysis. He played a pioneering role in all these fields. The fact is also clear from the preceding paragraphs. In the next section, we summarize the work that immediately followed the publication of the equation.

4 Immediate impact on other astrophysicists

The publication of the equation drew immediate attention. As already pointed out, the examination of sunspot spectrum yielded the signature of rubidium [16]. Refinement of Saha's work was undertaken independently by Russel in Princeton, and Fowler and Milne in Cambridge. Russel [26] introduced the idea of the mixture of gases and multiple ionization stages. He showed that the ratio of the number of ionized to neutral atoms for two elements do not depend on pressure or relative abundance of the elements. He argued that the ionization potential is a periodic function of the atomic number. He also explicitly showed that the pressure in the ionization equation is actually electron pressure. Russel found that spectra of alkali metals with a single optical electron can easily be explained using the ionization equation. However, alkaline earth metals like barium, with two optical electrons were more difficult to explain [27]. Russel noted that sodium and barium have similar ionization potentials, yet the latter was more ionized. Seeking the solution of this puzzle, Russel was led to introduce the L–S coupling or Russel–Saunders coupling in atoms [28].

Fowler and Milne [19] argued that the maximum intensity of a line obviously does not depend on the abundance, i.e. given the statistical nature, there exists a maxima for atoms of a particular ionized state depending on the temperature and pressure alone. Russel and Stewart [29] and Fowler and Milne [19] independently showed that the pressure of gas in the Sun is much smaller than expected. Pressure in the photosphere is of the order of 10^{-2} atmosphere while in the reversing layer it goes down to 10^{-4} atmosphere, and finally comes down to 10^{-7} atmosphere or even lower in the chromosphere. These values were less than those assumed by Saha. These estimates established the importance of radiation pressure in the outer layers of a star. Fowler and Milne then went on to calculate temperatures from various stellar spectra and compared them to the Harvard classes.

The temperature scale was finally settled by the works of Menzel, and of Payne, both of whom used the Fowler-Milne modification of Saha's theory. Menzel [30] studied the spectra of 20 giant stars and suggested a provisional temperature scale. Payne showed that the temperature of the hottest stars are actually underestimated [31,32]. In her Ph.D. thesis, she studied a wide range of temperature and pressure and firmly established the Harvard classification as a temperature sequence [33]. The temperature scale proposed by her was widely accepted. By this time, Eddington had shown that the stars in the main sequence of HR diagram was not an evolutionary sequence but actually correspond to locus of equilibrium points reached by stars of different mass [34].

The problem of the excessively intense hydrogen lines in the hottest stars, as discussed in the last section, was also noted by Menzel [30]. Fowler and Milne [19] pointed out that the relative abundance of various elements in the stars can easily be obtained from the intensity of spectral lines by employing Saha's theory [19]. Earlier, applying Saha's work to the spectra of O-type stars, Plaskett could infer the overabundance of helium, but he tried to explain it away [35,36].

Payne, after she established a sufficiently reliable temperature scale, went on to determine the abundance of elements. In Ref. [37], she calculated the relative abundance of various elements in stars. She commented that the hydrogen and helium concentration were too high and 'almost certainly not real' and omitted their values from the final table. Influenced by the comments of Russel and Eddington, she downplayed the abundance of the two elements in her thesis also. By this time, Saha's suggestion that the composition of the atmospheres of different stars were actually uniform was commonly accepted.

The evidence for a high abundance of hydrogen in stellar atmospheres continued to mount, but efforts were made to explain away the data. For example, Rosseland suggested that hydrogen may be concentrated at the surface of the star, being expelled from the core [38]. Moore and Russel [39] studied the most widened lines in solar spectra and concluded that the widening in the solar atmosphere depends on the abundance of the element. However, in spite of their own evidence, they considered the calculated abundance of hydrogen as an anomaly. Unsöld [40,41] independently used the same approach and deduced a hydrogen abundance similar to the work of Payne. Yet he was sceptical of the result himself. Stewart followed Russel's arguments but concluded that the number of hydrogen atoms per unit column above the solar photosphere may be as high as 10^{22}, i.e. the atmosphere may be chiefly made of hydrogen [42]. Adams and Russel [43] were led to the great abundance of hydrogen but suggested that either hydrogen, being light, is expelled from the core by radiation pressure to the outer surface, or thermodynamic equilibrium may not hold in stellar atmosphere.

However, the evidence for overabundance of hydrogen continued to come from different investigations. Particularly, the signature of a huge abundance of hydrogen in the atmospheres of the red giant stars could not be explained away. Finally, Russel, in a seminal work, considered the evidence, including the work of Payne, and concluded that the amount of hydrogen in the stellar atmosphere is 'incredibly great [44]. He pointed out that this fact can explain a number of puzzling observations including electron pressure in the atmosphere. After this work, it was generally accepted that hydrogen is the chief constituent of the stars.

After it was established that hydrogen and helium constitute the chief ingredients of stellar matter, it was necessary to revisit various aspects of stellar structure. For example, Eddington recalculated the opacity and derived the the mass luminosity relation. He also came to the conclusion that hydrogen may be much more abundant than he had previously assumed [45]. It is obvious that opacity depends on the thermodynamic state of stellar plasma and Saha equation is essential to understand radiative transfer is stellar plasma in local thermal equilibrium.

Thus, within a decade of Saha's publication of the equation, based in his work, a temperature scale of stars was established, the pressure in the stellar atmospheres was reliably estimated, the Harvard stellar classification was put on a firm theoretical footing, and finally, the abundance of various elements in the stars, including the preponderance of hydrogen and helium, was calculated. Saha equation thus transformed astrophysics form a qualitative to a quantitative science.

In the next two sections, we discuss briefly the role played by the equation in two fields which developed much after the publication of the equation.

5 Applications in nuclear astrophysics

Saha Equation has proved to be useful in many fields; it is not possible to discuss all of them in the limited scope of this article. In this section we briefly introduce the nuclear astrophysics applications of Saha equation. This is not meant as a complete review but rather as a sample of various fields in nuclear astrophysics where this equation is employed.

Although the equation was designed for ionization of atoms, it can be applied in many similar situations, i.e. reaction rate equilibrium in high-temperature environments. Readers are referred to standard text books for details of nuclear Saha equation (see e.g. [46–48]) that is used in nuclear astrophysics. We follow subsection 3.1.6 of Ref. [47] for a brief derivation.

For a photonuclear reaction $0+1 \leftrightarrow \gamma+2$, where 0,1, and 2 refer to nucleons or nuclei, the rate of reaction is given by the difference between the forward and the reverse rates,

$$r = r_{01\to\gamma 2} - r_{\gamma 2\to 01} = \frac{N_0 N_1 \langle\sigma v\rangle_{01\to\gamma 2}}{1+\delta_{01}} - \lambda_\gamma(2) N_2. \tag{3}$$

Here, $\langle\sigma v\rangle$ is the Maxwellian averaged cross section (MACS) and $\lambda_\gamma(2)$ is the photodisintegration decay constant at elevated temperature T. Here N_i refers to the number density of nuclear species of type i. We have

$$\langle\sigma v\rangle_{01} = \left(\frac{8}{\pi m_{01}}\right)^{1/2} \times \frac{1}{(kT)^{3/2}} \int_0^\infty E\sigma_{01\to\gamma 2}\exp(-E/kT)\mathrm{d}E \tag{4}$$

$$\lambda_\gamma(2) = \frac{1}{\pi^2\hbar^3 c^2}\int_0^\infty \frac{E_\gamma^2}{\exp(E_\gamma/kT)-1}\sigma_{\gamma 2\to 01}\mathrm{d}E_\gamma. \tag{5}$$

The reduced mass of the system is given by m_{01}. At equilibrium, $r=0$ and we get

$$\frac{N_2}{N_0 N_1} = \frac{1}{1+\delta_{01}}\left(\frac{2\pi\hbar^2}{m_{01}kT}\right)^{3/2}\frac{G_2}{G_0 G_1}\exp(Q_{01\to\gamma 2}/kT). \tag{6}$$

This is the nuclear Saha equation and is applicable in nuclear statistical equilibrium (NSE).

Such a situation occurred in the early universe. In the Big Bang neutrons and protons were in chemical equilibrium (which is another form of Saha equilibrium). This ends when temperature drops below 1 MeV. In the first step of Big Bang nucleosynthesis, protons and neutrons combine to form deuteron. However, the temperature is still very high and the deuteron production reaction is in equilibrium with photodisintegration of deuterium, i.e. $n+p \leftrightarrow d+\gamma$. Hence, the density of protons, neutrons and deuterons are related through the nuclear Saha equation (6),

$$\frac{N_\mathrm{d}}{N_\mathrm{p}N_n} = \frac{3}{4}\left(\frac{4\pi\hbar^2}{M_N kT}\right)^{3/2}\exp(B_\mathrm{d}/kT), \tag{7}$$

where B_d is the deuteron binding energy and M_N is the nucleon mass. It is easy to see that initially the deuteron density remained low and further nucleosynthesis was not possible. This era ended only after the temperature dropped to 0.1 MeV when the reactions fell out of equilibrium and forward reaction started to dominate.

In a recent article Vovchenko et al. [49] have extended the Saha equation to nucleus–nucleus collisions in the Large Hadronic Collider (LHC). They argue that conditions equivalent to the above situation exist in heavy-ion collisions and hence, production of light nuclei and hypernuclei can be explained in terms of the Saha equation. After incorporating certain corrections, their results for $T_{kin} = 113 \pm 12$ MeV agree with the most central Pb-Pb collisions in LHC.

Saha equation was also useful in understanding the triple-α reaction in helium-burning phase. It involves an intermediate ^{8}Be formation by fusion of two α particles which then reacts with a third particle to form ^{12}C. The nucleus ^{8}Be is unstable and the reaction $\alpha+\alpha \leftrightarrow {}^8$Be is in thermal equilibrium in helium-burning phase which has a high temperature and high density. The fraction of ^{8}Be can easily be calculated from Eq. (6). However, the calculated ^{8}Be fraction was too small to explain the observed ^{12}C abundance. This led to the prediction that the reaction $\alpha+{}^8\mathrm{Be}\to{}^{12}\mathrm{C}$ to proceed through a 0^+ resonance of ^{12}C very close to the threshold energy [50,51] which was discovered subsequently [52]. Readers are referred to Ref. [53] for a recent review of triple alpha reaction and its implications for astrophysics.

Another interesting area where Saha equation is useful is the rapid proton capture or rp-process. This occurs in high-temperature proton rich environments when hydrogen-rich matter from a companion star falls on a compact object like a white dwarf or a neutron star. This dominates type I X-ray bursters and determine the crust composition of accreting neutron stars. Readers are referred to Ref. [54] for a recent review. The (p,γ) and the (γ,p) processes are in equilibrium and the process reaches nuclei near the proton drip line and then may wait for its β^+ decay for further progress. This nucleus where the process stalls is called a waiting point nucleus. The odd-even mass difference in nuclei indicates that there are two scenarios. Equilibrium may be established between the nuclei (Z,N) and its isotone $(Z+1,N)$ at a lower temperature. In this case, the destruction rate of the waiting point nucleus will be given by the sum of its beta decay rate and the proton capture rate of the $(Z+1,N)$ isotone that leads the population out of the equilibrium. At a higher temperature, the equilibrium will include the isotone $(Z+2,N)$

also. In this case, the destruction rate of the waiting point nucleus depends on its beta decay rate and the beta decay rate of the $(Z+2, N)$ isotone [55]. Thus, at a lower temperature, there is a pathway of two proton capture that establishes a net reaction flow from the nucleus (Z, N) to $(Z, N+2)$. Various waiting points in the rp-process have been studied in the literature, for example in Refs. [55–59]. For example, Fig. 1 shows the effective half lives of two waiting point nuclei. It is clear that a temperature window exists where two proton capture dominates It is also evident that the proton separation energy, occurring in the exponential in the Saha equation, plays a pivotal role in the rp-precess nucleosynthesis.

Saha equation serves another useful role as it indicates that the cross sections for many of the reactions are not important for abundance calculation in nucleosynthesis because they remain in equilibrium in astrophysical environments.

5.1 Supernova nucleosynthesis

In core collapse supernovae or type Ia supernovae, Saha equation may be applied in two situations. In the late advanced burning stages, many of the forward and reverse reaction rates become equal and a quasi-equilibrium exists. It is possible to determine the abundances in such cases through the application of the nuclear Saha equation. In the silicon burning stages in stellar interiors the temperature goes to even higher values [61–63]; in the pre-supernova stage it reaches 4–5 GK. At such a high temperature, photodisintegration of less tightly bound nuclei produces alpha particles as well as protons and neutrons which then react with the residual heavy nuclei. All the reactions now achieve equilibrium through strong and electromagnetic interactions. Iron group nuclei, having the highest binding energy, are expected to dominate the statistical equilibrium. In such a situation, the abundance of any isotope $Y(Z, N)$ can be obtained by repeated application of the Saha equation (subsection 5.6.2 of Ref. [47]) as

$$N_Y = \frac{N_{\mathrm{p}}^Z N_n^N}{2^A}\left(\frac{2\pi\hbar^2}{m_N kT}\right)^{3(A-1)/2} A^{3/2} G_Y \exp(B_Y/kT), \quad (8)$$

which has to be solved together with the equations of baryon number conservation and charge conservation. It is easy to see that, depending on the density, at extremely high temperature matter will predominantly consist of protons and neutrons. At a lower temperature it will be alpha-rich and still lower temperature prefers ^{56}Ni as the main constituent.

A direct derivation from statistical mechanical consideration yields the same result as Eq. (8) (see e.g. [64]). However, this approach allows modification of the equation in the NSE when some important effects are incorporated. For example, Banik et al. [65] have pointed out that inclusion of exclusion volume correction in the Equation of State at NSE will modify the Saha equation.

Another situation where the Saha equation may be applied is the heavy element synthesis through the r-process. This is expected to occur in a high neutron density ($N_n \geq 10^{21}\mathrm{cm}^{-3}$) and high temperature ($T \geq 1$ GK) environment. Such an environment is achieved in a supernova. In such a scenario, both neutron capture (n, γ) and photodisintegration (γ, n) processes are much faster than β-decay. For sufficiently large temperature and density, an equilibrium exists between the forward and the reverse processes along an isotopic chain. In this case we can get the abundance ratio of successive isotopes easily from Eq. (6) (see subsection 5.6.2 of Ref. [47]) as

$$\frac{N(^{A+1}_{Z}X)}{N(^{A}_{Z}X)} = N_n\left(\frac{2\pi h^2}{m_{An}kT}\right)^{3/2}\frac{1}{2}\frac{G_{Z,A+1}}{G_{Z,A}}\exp(Q_{n\gamma}/kT). \quad (9)$$

In the canonical r-process model [66], one assumes that the temperature remains high and the neutron density remains constant through out the period of r-process. The additional condition that the equilibrium between (n, γ) and (γ, n) holds throughout the time period of the process is called waiting point approximation. In this approximation, for a given T and N_n, the abundance of a given element is almost entirely concentrated on the isotopes which correspond to a neutron separation energy expressed in MeV,

$$Q_{n\gamma} \approx \left(34.075 - \log N_n + \frac{3}{2}\log T_9\right)\frac{T_9}{5.04}, \quad (10)$$

where T_9 is the temperature in GK and N_n is expressed in cm^{-3}. The isotopes with approximately this value of separation energy define the r-process path. At typical values of temperature and density, this corresponds to a value of 2–3 MeV, indicating that the r-process proceeds along the very neutron-rich side of the nuclear chart. As the process approaches the magic numbers $N = 82$ and $N = 126$, the separation energy increases leading to a pile up at these neutron numbers. This simple model is sufficient to explain the observed r-process abundance peaks at $A \sim 130$ and 195 [66,67]. This canonical r-process and waiting point approximation were introduced by Seeger et al. [68]. Further details are available in Ref. [66].

Such a high-temperature high neutron density environment is possible in supernova neutrino wind scenario. Although the canonical approximation is generally applied for site-independent studies, Saha equation is also useful beyond this approximation. For example Faroqi et al. [69] compared their r-process calculations to the results of Saha nuclear equation to check the extent of the (n, γ)–(γ, n) equilibrium in high-entropy wind of core-collapse supernovae. It has been suggested that the observed peak in the abundance in rare-earth region occurs during the freeze out phase when the equilibrium is weakened and reaction rates as

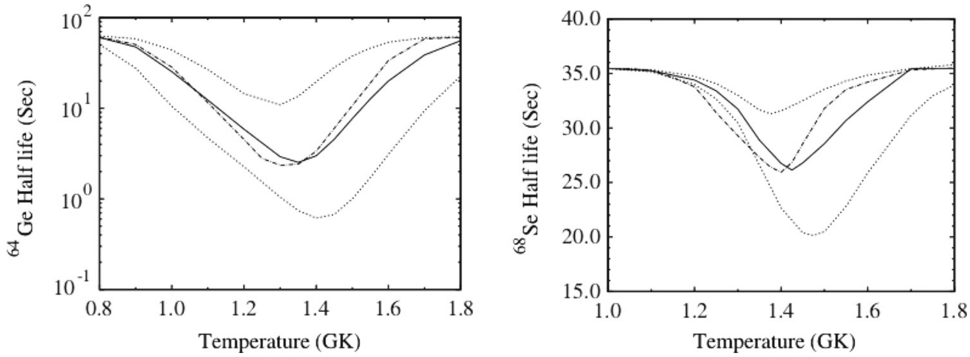


Fig. 1 Effective half-life values of ^{64}Ge and ^{68}Se as a function of temperature. The solid line represents the results of Ref. [59] and the dashed lines mark the two extremes for the errors in the Q-values of the reactions involved. The dash-dotted line shows the results obtained using the rates from another calculation [60]. Reprinted from Ref. [59] with kind permission form European Physical Journal A

well as β-decays become important. (See, for example, Mumpower et al. [70].) Even in this case, one usually starts with a seed nuclei distribution for NSE obtained from Saha equation.

Neutron star merger is another probable site for r-process nucleosynthesis. Observations of gravitational waves with LIGO and corresponding observations of the electromagnetic spectrum strongly suggest that r-process nuclei are formed in the event [71]. The merger creates a high temperature high neutron density environment where neutron capture will be in equilibrium with photodisintegration and Saha equation may be applied to determine the abundance of elements created in such events.

6 Recombination and Saha equation

Recombination is the epoch after the Big Bang when the nuclei first combined with electrons to form neutral atoms. Earlier the temperature was too high for atoms to exist. This is the epoch when matter and radiation decouple. Assuming the matter to consist of hydrogen only, for the photoionization reaction,

$$e + p \leftrightarrow H + \gamma, \tag{11}$$

it is possible, using Eq. (2), to write a relation involving the densities of electrons (n_e). protons (n_p) and neutral hydrogen atoms (n_H).

$$\frac{n_e n_p}{n_H} = \left(\frac{2\pi mkT}{2\pi\hbar^2}\right)^{3/2} \exp(-B/kT), \tag{12}$$

where $B = 13.6$ eV is the ionization energy of the hydrogen atom. The calculation shows that the decoupling temperature, i.e. the temperature at which the hydrogen ionization fraction drops to 0.5, is 3700 K which corresponds to a redshift of $Z = 1300$ [72].

The above calculation assumes direct recombination to the hydrogen ground state and thermal equilibrium between the ground and ionized state of Hydrogen atoms. There is a background radiation field at a few thousand K. It also assumes that recombination time scale is much lower than the universe expansion time scale. These assumptions are not valid during the middle and later stages of cosmic recombinations and therefore the resulting recombinations derived from the Saha equation are too fast compared to the actual situation. Peebles [73] and Zeldovitch et al. [74] relaxed these assumptions and re-derived the recombination history. For example, Peebles argued that direct recombination to the ground state of the hydrogen atom emits a photon which is likely to ionize another atom leaving no net change. He suggests that recombination forms neutral atoms in higher excited states. These atoms then decay to the $2s$ metastable state by a cascade of transitions, and finally to the ground state by the forbidden two photon transition. He also pointed out that another effect; electrons in the first excited states directly jump to the ground states and due to the universe's expansion a finite amount of the emitted photons get red shifted out of the L_α resonance line. Readers are referred to chapter 6 of Ref. [72] for details. A recent calculation [75] shows that the presence of excited state brings down the equilibrium temperature from 5000 to 4000 K.

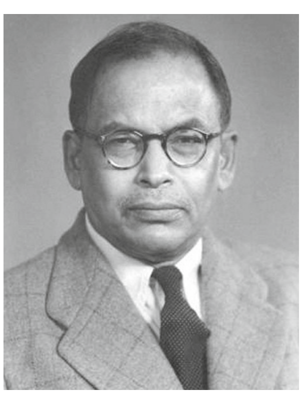
However, the Saha equation can be successfully used to estimate the first recombination of the He. The ionization energies of He and He^+ are 24.6 eV and 54.4 eV and recombination of helium takes place before hydrogen. The first recombination follows Saha ionization prescription as the rates are fast enough so that the recombination process is in equilibrium. Thus, one can find out a temperature for the first recombination from the Saha equation as above 15,000 K. For the second recombination, however, one needs to consider various other factors, including recombination in excited states, redshifting of resonance lines and absorption by existing neutral hydrogen atoms. Thus departure from the Saha equation is expected in this case. Readers are referred to recent articles [75,76] for more references.

7 Summary

Confusion prevailed in the first 2 decades of the twentieth century in the field of astrophysics and it was not possible to interpret the new observational data on the basis of existing knowledge. Temperature and pressure in stellar atmospheres and material composition of stars could not be reliably inferred. The situation changed dramatically with the discovery of Saha ionization equation. Saha combined atomic theory, thermodynamics and spectroscopy to pave the way to determine the physical conditions in the stellar atmospheres. particularly temperatures and pressure. He also explained the origin of the Harvard classification of stars. Modification and extension of Saha's work was immediately carried out by Russel, Milne, Payne and others. Starting from the equation, they established a firm temperature scale for stars, determined the pressure in the stellar atmospheres, and deduced the elemental composition of stars. Saha equation continues to be gainfully applied in nuclear astrophysics, where often an equilibrium between forward and reverse reactions is established in a high-temperature environment. It is useful to understand Big Bang Nucleosynthesis and rp-process. Advanced stellar burning stages often go through complete or partial nuclear statistical equilibrium which may be described by the equation. Finally r-process nucleosynthesis yields advanced burning stages; hence the equation is useful in understanding nucleosynthesis in supernova and neutron star merger. In the recombination era, Saha equation can be used to predict the recombination temperature. Studies of deviation from the predictions of Saha equation have yielded significant results.

Acknowledgements Discussions with Kanan K. Datta and A. Kundu are gratefully acknowledged. A part of this work has been carried out with financial assistance of the UGC-DRS(II) programme of the Department of Physics, University of Calcutta.

Appendix A: Brief biography of Meghnad Saha



Meghnad Saha (1893-1956) was born in East Bengal in British India, now a part of Bangladesh. He studied in Presidency college, Kolkata and then joined the newly formed University College of Science of the University of Calcutta as a research fellow. Working independently, he discovered the ionization equation. After a brief stint in London and Berlin, he joined the University of Calcutta in 1921 as a professor, and in 1923. shifted to The University of Allahabad. He came back to the University of Calcutta in 1938. Within the Department of Physics he started the Institute of Nuclear Physics, which later became an independent organization and now bears his name. He was instrumental in setting up the first cyclotron in Asia. Saha was involved in national planning and, after independence, was elected to the member of the Indian Parliament as an independent member.

References

1. E. Hertzsprung, Z. Wiss, Photogr. Photo. **4**, 43 (1906)
2. J. Wilsing, J. Scheiner, Publik. Ast. Obs. Potsdam **24**, 74 (1919)
3. See *e.g.* G.E. Hale, W.S. Adams, Astrophys. J. **23**, 400 (1906)
4. W.S. Adams, Astrophys. J. **33**, 64 (1911)
5. J. Stefan, Sitzber. Akad. Wiss. Wien **79**, 391 (1879)
6. H.N. Russel, Nature **93**, 227 (1914)
7. E.A. Milne, P. Phys, Soc. Lond. **36**, 94 (1924)
8. H.N. Russel, Nature **93**, 281 (1914)
9. H.A. Rowland, Am. J. Sci. **41**, 243 (1891)
10. H.N. Russell, Science **39**, 791 (1914)
11. A. Fowler, J. Brit. Astron. Assoc **28**, 197 (1918)
12. D.H. DeVorkin, R. Kenat, J. Hist. Astron. **14**, 102 (1983)
13. D.H. DeVorkin, R. Kenat, J. Hist. Astron. **14**, 180 (1983)
14. M.N. Saha, Philos. Mag. **40**, 472 (1920)
15. M.N. Saha, Philos. Mag. **40**, 809 (1920)
16. H.N. Russell, Publ. Astron. Soc. Pac. **33**, 202 (1921)
17. M.N. Saha, Philos. Mag. **41**, 267 (1921)
18. M.N. Saha, P.R. Soc, London **A99**, 135 (1921)
19. R.H. Fowler, E.A. Milne, Mon. Not. R. Astron. Soc. **83**, 403 (1923)
20. C.G. Darwin, R.H. Fowler, Philos. Mag. **44**, 450 (1922)
21. R.H. Fowler, Philos. Mag. **45**, 1 (1923)
22. J. Eggert, Phys. Z. **20**, 570 (1919)
23. F.A. Lindemann, Philos. Mag. **38**, 669 (1919)
24. A. Rai Choudhuri, Phys. News **48**(4), 27 (2018)
25. D. Cenadelli, J. Ast. Hist. Herit. **11**, 134 (2008)
26. H.N. Russell, Astrophys. J. **55**, 119 (1922)
27. H.N. Russell, Astrophys. J. **55**, 354 (1922)
28. H.N. Russell, F.A. Saunders, Astrophys. J. **61**, 38 (1925)
29. H.N. Russel, J.W.Q. Stewart, Astrophys. J. **59**, 197 (1924)
30. D.H. Menzel, Ph.D. Thesis. Harvard College Observatory Circular, No. 258 (1924)
31. C.H. Payne, Harvard College Observatory Circular,No. 252 (1924)
32. C.H. Payne, Harvard College Observatory Circular, No. 256 (1924)
33. C.H. Payne, *Stellar Atmosphere* (Harvard Observatory Monograph, Cambridge, Mas., 1925)

34. A.S. Eddington, Mon. Not. R. Astron. Soc. **84**, 308 (1924)
35. H.H. Plaskett, Pub. Am. Ast. Soc. **4**, 380 (1922)
36. H.H. Plaskett, Pub. Dom. Ast. Obs. **1**, 325 (1922)
37. C.H. Payne, P. Natl, Acad. Sci. USA **11**, 187 (1925)
38. S. Rosseland, Mon. Not. R. Astron. Soc. **85**, 541 (1925)
39. C.E. Moore, H.N. Russel, Astrophys. J. **63**, 1 (1926)
40. A. Unsöld, Zeit. Phys. **44**, 793 (1927)
41. A. Unsöld, Zeit. Phys. **46**, 765 (1928)
42. J.Q. Stewart, Pop. Ast. **36**, 346 (1928)
43. W.S. Adams, H.N. Russell, Astrophys. J. **68**, 9 (1928)
44. H.N. Russell, Astrophys. J. **70**, 11 (1929)
45. A.S. Eddington, Mon. Not. R. Astron. Soc. **92**, 471 (1932)
46. D.D. Clayton, *Principles of Stellar Evolution and Nucleosynthesis* (University of Chicago, Chicago, 1968)
47. C. Iliadis, *Nuclear Physics of Stars* (WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim, 2007)
48. B.E.J. Pagel, *Nucleosynthesi and Chemical Evolution of Galaxies*, 2nd edn. (Cambridge University Press, Cambridge, 2009)
49. V. Vovchenko, K. Gallmeister, J. Schaffner-Bielich, C. Greiner, Phys. Lett. B **800**, 135131 (2020)
50. F. Hoyle, D.N.F. Dunbar, W.A. Wenzel, W. Whaling, Phys. Rev. **92**, 1095 (1953)
51. F. Hoyle, Astrophys. J. Sup. **1**, 121 (1954)
52. W. Cook, W.A. Fowler, C.C. Lauritsen, T. Lauritsen, Phys. Rev. **107**, 508 (1957)
53. C.A. Bertulani, T. Kajino, Prog. Part. Nucl. Phys. **89**, 56 (2016)
54. A. Parikh, J. José, G. Sala, C. Iliadis, Rep. Prog. Nuc. Phys. **69**, 225 (2013)
55. B.A. Brown et al., Phys. Rev. C **65**, 0452 (2002)
56. J. Görres, M. Wiescher, F.-K. Thielemann, Phys. Rev. C **51**, 392 (1995)
57. H. Schatz et al., Phys. Rep. **294**, 167 (1998)
58. H. Schatz, L. Bildsten, A. Cumming, M. Wiescher, Astrophys. J. **524**, 1014 (1999)
59. C. Lahiri, G. Gangopadhyay, Eur. Phys. J A **47**, 87 (2011)
60. T. Rauscher, F.K. Thielemann, At. Data Nucl. Data Tabl. **75**, 1 (2000)
61. F. Hoyle, Mon. Not. R. Astron. Soc. **106**, 343 (1946)
62. E.M. Burbidge, G.R. Burbidge, W.A. Fowler, F. Hoyle, Rev. Mod. Phys. **29**, 547 (1957)
63. F.E. Clifford, R.J. Tayler, Mem. R. Astron. Soc. **69**, 21 (1965)
64. G. Wallerstein et al., Rev. Mod. Phys. **69**, 995 (1997)
65. S. Banik, M. Hempel, D. Bandyopadhyay, Astrophys. J. Suppl. S. **214**, 22 (2014)
66. M. Arnould, S. Goriely, K. Takahashi, Phys. Rep. **450**, 97 (2007)
67. C. Lahiri, G. Gangopadhyay, Int. J. Mod. Phys. E **21**, 1250042 (2012)
68. P.A. Seeger, W.A. Fowler, D.D. Clayton, Astrophys. J. Suppl. S. **11**, 121 (1965)
69. K. Farouqi, K.-L. Kratz, B. Pfeiffer, T. Rauscher, F.-K. Thielemann, J.W. Truran, Astrophys. J. **712**, 1359 (2010)
70. M.R. Mumpower, G.C. McLaughin, R. Surman, Phys. Rev. C **85**, 045801 (2012)
71. D.M. Siegel, Eur. Phys. J. A **55**, 203 (2019)
72. P.J.E. Peebles, *Principles of Physical Cosmology* (Princeton University Press, Princeton, 1993)
73. P.J.E. Peebles, Astrophys. J. **153**, 1 (1968)
74. Ya B. Zeldovich, V.G. Kurt, R.A. Syunyaev, Sov. Phys. JETP USSR **28**, 146 (1969)
75. A. Das, R. Ghosh, S. Mallik, Astrophys. J. **881**, 40 (2019)
76. J. Chluba, J. Fung, E.R. Switzer, Mon. Not. R. Astron. Soc. **423**, 3227 (2012)