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A re-visitation of space asymptotic theory in neutron transport

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Abstract The space asymptotic theory has constituted a powerful tool for the determination of neutron energy spectra in nuclear reactors, which are the basis of the generation of group constants for the neutronic core design. The method can provide a deep physical insight into the basics of reactor physics and may still give new ideas for modern computational methods. This contribution presents a re-visitation of the method, illustrating its most important general results, some of which may not be well known. In particular, the criticality theory and the space–energy separability theorem are presented. The validity of such theorem is extended also to the net neutron current. The procedure allows to generalize the Fick's law with a consistent definition of the energy-dependent diffusion coefficient. Some numerical examples are given in simple multigroup models to illustrate the relevant features of the theory.

1 Introduction

At the beginning of a paper devoted to a re-visitation of the spatial asymptotic theory in neutron transport the following question may be immediately asked: is it still worth doing it and to what objective? The computational power that is nowadays available with modern computers is such that the historical methods aiming at reducing the transport equation to a tractable form with limited computational resources seem completely useless and unworthy of any attention.

It is the authors' opinion that it is still worth knowing about the asymptotic approach to neutron transport for several reasons. The methodology is physically very rich and mathematically elegant, and it allows a deep understanding of transport phenomena in multiplying systems. Therefore it is certainly educationally very useful, as it may give a good opportunity to better comprehend the basic principles of nuclear reactor physics. Furthermore, its fundamentals and its results may throw some further light on the multiscale modern methodology for the simulation of nuclear reactor cores and still be of help for various applications.

The idea of the asymptotic reactor theory is rooted in the early developments of nuclear reactor physics. Some ideas leading to the asymptotic theory can be found in a work devoted to study the modelling of the neutron slowing-down process, where also the basics of the B_N method are illustrated [1]. Weinberg and Wigner observed that the theory of a homogeneous

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reactor can be reduced, at least approximately, to the theory of an infinite system [2]. They were thus able to give a consistent formulation of the first and second fundamental theorems of reactor physics, which concern the energy separability of the neutron distribution and the criticality theory. The validity of such theorems and their possible practical implications are discussed in some further works, e.g. [3–5]. The mathematical foundations of the theory are laid down by Case, Ferziger and Zweifel [6].

Some more recent works show how the theory can still play an important role for various applications; for instance, the idea presented in [7] to study the neutronics of lattices can be easily related to the asymptotic approach. In another work [8], an asymptotic-based leakage model is developed to account for the neutron transfer between homogeneous regions in the process of multigroup constant generation. Some applications in the time-dependent domain have also been attempted, leading to accurate full transport solutions and to a better understanding of the propagation phenomena in pulsed experiments [9].

The idea of approaching the solution of the transport equation using a Fourier transform goes back to the pioneering work by Case, De Hoffmann and Placzek [10]. In their work, a fully analytical solution to the transport equation is obtained in the infinite medium assuming the number of secondaries per collision to be smaller than one, which leads to a physically significant solution in the whole phase space domain. On the other hand, when assuming a number of secondaries per collision larger than one, the Fourier transform technique is still applicable, as shown in [10], leading to an oscillatory behaviour of the solution. As a consequence, the solution retains a physical meaning in a subdomain where it is nonnegative. This is the basis of spatial asymptotic theory and will be used throughout this work.

The following sections present the main aspects of asymptotic theory and its important consequences for the physical description of a chain-reaction nuclear reactor. The separability theorem is then established for both the neutron flux and the net current. The derivation of Fick's law allows obtaining a consistent expression for the energy-dependent diffusion coefficient.

2 The fundamentals of the method

Before entering a more mathematically consistent analysis, some intuitive considerations can help in the understanding of the asymptotic idea. For large nuclear reactors, where the neutron balance is dominated by fissions, the experimental observation shows that, "if one looks from far enough", only the global macroscopic behaviour of the neutron population may be detected. Therefore, only the macroscopic distribution of the neutrons is observed, which may be superimposed to a more detailed local microscopic behaviour that the observer cannot perceive. The interest is thus focused on the whole core, rather than on the fine structure at the pin-cell level. The spatially asymptotic observer is interested in the global behaviour of the reactor; for this observer the system can also be considered as materially homogeneous. In practical applications this is the case if cross sections are homogenized by means of a proper averaging process, at least on optically large spatial portions of the system. In elementary reactor theory based on the diffusion model for the homogeneous reactor one finds that the neutron population appears to be distributed as the fundamental Helmholtz eigenfunction for the geometry considered. However, in cases of practical interest in nuclear engineering, this is very close to reality also beyond the simple diffusion model, implying that the system is spatially "diffusive". One could thus think of solving the problem in an infinite system, accepting solutions that are not everywhere physically meaningful, as being negative, but yielding the almost correct solution if one restricts his interest to the spatial domain of the reactor. This approach allows the elimination of the problem of the boundary conditions, and hence it is expected that the results may deteriorate if the contribution of boundary effects becomes important. As a consequence, the theory that is being developed will turn out to be adequate only for optically large systems.

The possibility to extend the system to infinity has very important mathematical consequences. As we will see in the following, it is possible to preliminarily remove the spatial detail and thus to concentrate on the energy and angular aspects of the transport phenomenon.

The transport equation may be written using a proper kernel $P\left(\boldsymbol{r}' \rightarrow \boldsymbol{r}, E' \rightarrow E, \boldsymbol{\Omega}\right)$ so that

$$P\left(\boldsymbol{r}' \rightarrow \boldsymbol{r}, E' \rightarrow E, \boldsymbol{\Omega}\right) \mathrm{d}\boldsymbol{r} \mathrm{d}E \mathrm{d}\boldsymbol{\Omega} \tag{1}$$

describes the probability for a neutron isotropically generated by fission with energy E' at the spatial point $\boldsymbol{r}'$ to suffer a last collision within $\mathrm{d}\boldsymbol{r}$, resulting in its emission with energy between E and $E + \mathrm{d}E$ and direction within the solid angle $\mathrm{d}\boldsymbol{\Omega}$ around the direction $\boldsymbol{\Omega}$. In the transfer from point $\boldsymbol{r}'$ to $\boldsymbol{r}$ through streaming and scatterings it is assumed that the neutron is not further multiplied. The transport equation can thus be written as:

$$\begin{aligned}
&\boldsymbol{\Omega} \cdot \nabla \phi (\boldsymbol{r}, E, \boldsymbol{\Omega}) + \Sigma (E) \phi (\boldsymbol{r}, E, \boldsymbol{\Omega}) \\
&\quad = \int_{\Re^3} \mathrm{d}\boldsymbol{r}' \left\{ \int \mathrm{d}E' \chi \left(E'\right) \left[\int \mathrm{d}E'' \nu \Sigma_f \left(E''\right) \oint \mathrm{d}\boldsymbol{\Omega}' \phi \left(\boldsymbol{r}', E'', \boldsymbol{\Omega}'\right) \right] \right. \\
&\qquad \left. \times P \left(\boldsymbol{r}' \rightarrow \boldsymbol{r}, E' \rightarrow E, \boldsymbol{\Omega}\right) \right\} \\
&\qquad + \frac{\chi (E)}{4\pi} \int \mathrm{d}E' \nu \Sigma_f \left(E'\right) \oint \mathrm{d}\boldsymbol{\Omega}' \phi \left(\boldsymbol{r}', E', \boldsymbol{\Omega}'\right).
\end{aligned} \tag{2}$$

Above Eq. (2) is a homogeneous (source-free) equation. The existence of a nontrivial solution that does not vanish everywhere in phase space can be studied introducing an eigenvalue. For instance, a factor $1/k$ can be introduced to multiply both terms in the r.h.s. of Eq. (2). In this case, k takes the meaning of multiplication factor and its determination is known as the criticality problem of nuclear reactor physics. However, in the present treatment, we prefer to discuss the problem of the existence through a different point of view, leading to an alternative formulation of the criticality problem involving a quantity that is known as the material buckling of the system.

The idea behind asymptotic theory amounts to assuming an infinite medium for which the structure of the transfer kernel simplifies, as it only depends on the distance between the point at which the neutron is generated by fission and the point at which it is observed to reach the energy considered. Furthermore, the emission direction shows an azimuthal symmetry; hence, it is possible to explicitly write down the infinite medium kernel averaged on the fission spectrum as:

$$P_\infty \left(\left|\boldsymbol{r} - \boldsymbol{r}'\right|, E, \frac{\left(\boldsymbol{r} - \boldsymbol{r}'\right) \cdot \boldsymbol{\Omega}}{\left|\boldsymbol{r} - \boldsymbol{r}'\right|} \right) \equiv \int \mathrm{d}E' \chi(E') P(\boldsymbol{r}' \rightarrow \boldsymbol{r}, E' \rightarrow E, \boldsymbol{\Omega}) \tag{3}$$

Introducing the fission emission density as:

$$M (\boldsymbol{r}) = \int \mathrm{d}E \nu \Sigma_f (E) \oint \mathrm{d}\boldsymbol{\Omega} \phi (\boldsymbol{r}, E, \boldsymbol{\Omega}), \tag{4}$$

it is possible to write the transport equation in the following form:

$$\begin{aligned}&\mathbf{\Omega}\cdot\nabla\phi\left(\boldsymbol{r},E,\mathbf{\Omega}\right)+\Sigma\left(E\right)\phi\left(\boldsymbol{r},E,\mathbf{\Omega}\right)\\&\quad=\int_{\Re^3}\mathrm{d}\boldsymbol{r}'M(\boldsymbol{r}')P_\infty\left(|\boldsymbol{r}-\boldsymbol{r}'|,E,\frac{(\boldsymbol{r}-\boldsymbol{r}')\cdot\mathbf{\Omega}}{|\boldsymbol{r}-\boldsymbol{r}'|}\right)+\frac{\chi(E)}{4\pi}M(\boldsymbol{r}).\end{aligned}\tag{5}$$

3 The transport equation in the Fourier transformed space

The integral term in the r.h.s. of Eq. (5) is of convolution type, and it is therefore suggesting the use of the Fourier transform [11] to deal with the problem. In the following the Fourier transform operator shall be denoted by the symbol $\mathcal{F}$. From the properties of the kernel appearing in Eq. (3), it is straightforward to demonstrate that its Fourier transform takes the form:

$$\mathcal{F}\left\{P_\infty\left(|\boldsymbol{r}|,E,\frac{\boldsymbol{r}\cdot\mathbf{\Omega}}{|\boldsymbol{r}|}\right)\right\}=P_F\left(k,E,\frac{\boldsymbol{k}\cdot\mathbf{\Omega}}{k}\right).\tag{6}$$

Applying the Fourier transform to all the terms of Eq. (5) and making use of the convolution theorem, the following equation is obtained:

$$\left[\Sigma\left(E\right)-i\mathbf{\Omega}\cdot\boldsymbol{k}\right]\phi_F\left(\boldsymbol{k},E,\mathbf{\Omega}\right)=M_F\left(\boldsymbol{k}\right)\left[(2\pi)^{3/2}P_F\left(k,E,\frac{\boldsymbol{k}\cdot\mathbf{\Omega}}{k}\right)+\frac{\chi(E)}{4\pi}\right].\tag{7}$$

It is now possible to divide both sides by the term $[\Sigma\left(E\right)-i\mathbf{\Omega}\cdot\boldsymbol{k}]$, multiply by $\nu\Sigma_f\left(E\right)$ and integrate over angle and energy, in order to reconstruct the Fourier transform of the fission emission density on the l.h.s. of the equation. At last one finds:

$$\left\{1-\int\mathrm{d}E\nu\Sigma_f\left(E\right)\oint\mathrm{d}\mathbf{\Omega}\frac{(2\pi)^{3/2}P_F\left(k,E,\frac{\boldsymbol{k}\cdot\mathbf{\Omega}}{k}\right)+\frac{\chi(E)}{4\pi}}{\Sigma\left(E\right)-i\mathbf{\Omega}\cdot\boldsymbol{k}}\right\}M_F\left(\boldsymbol{k}\right)=0.\tag{8}$$

The consequences of this result are very important. It is obvious that $M_F\left(\boldsymbol{k}\right)$ cannot vanish for all values of $\boldsymbol{k}\in\Re^3$, because this would imply an everywhere vanishing solution for the angular flux in the transformed space as well as in the real geometrical space. As a consequence, its coefficient must vanish at some points in the transformed space to assure that $M_F\left(\boldsymbol{k}\right)$ can take there some non-zero values, explicitly:

$$1=\int\mathrm{d}E\nu\Sigma_f\left(E\right)\int_{-1}^{1}\mathrm{d}\mu\frac{(2\pi)^{5/2}P_F\left(k,E,\mu\right)+\frac{\chi(E)}{2}}{\Sigma\left(E\right)-i\mu}.\tag{9}$$

This equation constitutes a criticality condition, and we shall see that the value of k for which it is satisfied establishes the wavelength of the spatial solution and hence the spatial domain in which it is physical meaningful.

In order to investigate on the nature of Eq. (9), let us consider the auxiliary problem of the determination of the solution φ of the transport equation in an infinite nonmultiplying medium injected by a fission point source:

$$\mathbf{\Omega}\cdot\nabla\varphi\left(\boldsymbol{r},E,\mathbf{\Omega}\right)+\Sigma\left(E\right)\varphi\left(\boldsymbol{r},E,\mathbf{\Omega}\right)=P_\infty\left(r,E,\frac{\boldsymbol{r}\cdot\mathbf{\Omega}}{r}\right)+\frac{\chi(E)}{4\pi}\delta\left(\boldsymbol{r}\right).\tag{10}$$

If the Fourier transform is taken, one obtains:

$$[\Sigma(E) - i\boldsymbol{\Omega}\cdot\boldsymbol{k}]\,\varphi(\boldsymbol{k}, E, \boldsymbol{\Omega}) = P_F\left(k, E, \frac{\boldsymbol{k}\cdot\boldsymbol{\Omega}}{k}\right) + \frac{\chi(E)}{(2\pi)^{3/2}\,4\pi}. \tag{11}$$

By dividing by $[\Sigma(E) - i\boldsymbol{\Omega}\cdot\boldsymbol{k}]$ and integrating over direction it is immediate to find the following expression for the total flux φ_0:

$$\varphi_0(k, E) = \frac{1}{(2\pi)^{3/2}}\int_{-1}^{1} \mathrm{d}\mu \frac{(2\pi)^{5/2} P_F(k, E, \mu) + \dfrac{\chi(E)}{2}}{\Sigma(E) - i\mu}. \tag{12}$$

In conclusion, Eq. (9) is perfectly equivalent to the following condition involving the Fourier transform of the solution of problem (10):

$$(2\pi)^{3/2}\int \mathrm{d}E\,\nu\Sigma_f(E)\,\varphi_0(k, E) = 1. \tag{13}$$

It can be verified that no matter what approximate energy model is adopted (see a following section for examples), φ_0 is a monotonous decreasing function of the transformation parameter k; hence, also the energy integral appearing in the r.h.s. of Eq. (12) is a decreasing function of k. Consequently, if

$$(2\pi)^{3/2}\int \mathrm{d}E\,\nu\Sigma_f(E)\,\varphi_0(0, E) > 1, \tag{14}$$

a unique real value of k shall satisfy Eq. (13). Such value is known as the *material buckling* of the problem and is denoted by B_M. Condition (14) assumes the same physical role as the requirement in elementary criticality theory that the infinite medium multiplication factor must be larger than unity in order to allow to obtain a critical finite system. Criticality conditions in the multigroup model are also discussed in the following.

4 On the space–energy form of the neutron flux and its separability theorem

The discussion carried out in the previous section leads to very important consequences for the space–energy form of the solution of the transport equation. It has been concluded that only for $k = B_M$, i.e. at the points belonging to a spherical surface of radius B_M, the transform of the fission emission density can take nonzero values. Henceforth, to yield a nonzero solution in the geometrical space, on such surface it must be singular, namely it shall be of the following form:

$$M_F(\boldsymbol{k}) = \frac{\delta(k - B_M)}{B_M^2} U(B_M\boldsymbol{\omega}), \tag{15}$$

where $\boldsymbol{\omega}$ is the direction of the vector $\boldsymbol{k}$ in the transformed space and the term B_M^2 is irrelevant being the problem homogeneous and it is introduced to simplify algebra. The function U is quite arbitrary, and it is chosen in order to establish the symmetries and describe the geometrical configuration of the system. In "Appendix A" some simple cases are illustrated.

Applying the inverse Fourier transformation to the angular flux as obtained from Eq. (7), having substituted expression (15), it is possible to write an explicit expression for the angular flux:

$$\phi\left(\boldsymbol{r}, E, \boldsymbol{\Omega}\right)=\frac{1}{4\pi}\oint \mathrm{d}\boldsymbol{\omega}U\left(B_M\boldsymbol{\omega}\right)\frac{4\pi P_F\left(B_M, E, \boldsymbol{\omega}\cdot\boldsymbol{\Omega}\right)+\dfrac{\chi\left(E\right)}{\left(2\pi\right)^{3/2}}}{\Sigma\left(E\right)-iB_M\boldsymbol{\omega}\cdot\boldsymbol{\Omega}}e^{-iB_M\boldsymbol{\omega}\cdot\boldsymbol{r}}. \quad (16)$$

The total flux can be obtained by angular integration as:

$$\Phi\left(\boldsymbol{r}, E\right)=\frac{1}{4\pi}\oint \mathrm{d}\boldsymbol{\omega}U\left(B_M\boldsymbol{\omega}\right)e^{-iB_M\boldsymbol{\omega}\cdot\boldsymbol{r}}\oint d\boldsymbol{\Omega}\frac{4\pi P_F\left(B_M, E, \boldsymbol{\omega}\cdot\boldsymbol{\Omega}\right)+\dfrac{\chi(E)}{\left(2\pi\right)^{3/2}}}{\Sigma\left(E\right)-iB_M\boldsymbol{\omega}\cdot\boldsymbol{\Omega}}. \quad (17)$$

Carrying out the integration over $\boldsymbol{\Omega}$ first and noticing that the results of the first integral shall not depend on $\boldsymbol{\omega}$, one can write the following important result for the total flux:

$$\Phi\left(\boldsymbol{r}, E\right)=\Psi\left(E; B_M\right)\oint \mathrm{d}\boldsymbol{\omega}U\left(B_M\boldsymbol{\omega}\right)e^{-iB_M\boldsymbol{\omega}\cdot\boldsymbol{r}}\equiv\Psi\left(E; B_M\right)f\left(\boldsymbol{r}; B_M\right), \quad (18)$$

which constitutes an explicit and general expression of the space–energy separability theorem of reactor physics (first fundamental theorem [2]). The function Ψ is the flux energy spectral function that is only dependent on the material buckling, defined as:

$$\Psi\left(E; B_M\right)=\frac{1}{4\pi}\oint \mathrm{d}\boldsymbol{\Omega}\frac{4\pi P_F\left(B_M, E, \boldsymbol{\omega}\cdot\boldsymbol{\Omega}\right)+\dfrac{\chi(E)}{\left(2\pi\right)^{3/2}}}{\Sigma\left(E\right)-iB_M\boldsymbol{\omega}\cdot\boldsymbol{\Omega}}. \quad (19)$$

It can be immediately proven that the spatial shape f of the total flux, and, hence, the total flux itself fulfils the Helmholtz equation, namely:

$$\nabla^2\Phi\left(\boldsymbol{r}, E\right)+B_M^2\Phi\left(\boldsymbol{r}, E\right)=0. \quad (20)$$

The validity of the results established by Eqs. (18–20) is quite general, as they have been derived without any approximation of the exact transport model beyond the space asymptotic assumption. The validity, obviously, extends when applying all the approximations that are usually adopted in practical applications, such as the energy discretization (multigroup approach), angular expansions (e.g. spherical harmonics approximations), angular discretizations (discrete ordinate schemes) and/or spatial discretization.

At last, it is easy to write down an explicit expression for the fission density, upon inverse transformation of Eq. (15), as

$$M\left(\boldsymbol{r}\right)=\oint \mathrm{d}\boldsymbol{\omega}U\left(B_M\boldsymbol{\omega}\right)e^{-iB_M\boldsymbol{\omega}\cdot\boldsymbol{r}}. \quad (21)$$

A useful reciprocity relationship is also easily demonstrated using Eq. (16), as:

$$\phi\left(\boldsymbol{r}, E, \boldsymbol{\Omega}\right)=\phi\left(-\boldsymbol{r}, E, -\boldsymbol{\Omega}\right). \quad (22)$$

5 On the space–energy form of the neutron current and its separability theorem

While the separability theorem for the neutron flux is well established and known, not so much interest seems to have been focused towards the behaviour of the neutron current in the space–energy domain. However, it is worthwhile to investigate the consequences of the previous results also in relation to the neutron current. From the definition of the net current:

$$\boldsymbol{J}\left(\boldsymbol{r}, E\right)=\oint \mathrm{d}\boldsymbol{\Omega}\phi\left(\boldsymbol{r}, E, \boldsymbol{\Omega}\right)\boldsymbol{\Omega} \quad (23)$$

it is possible to write:

$$\boldsymbol{J}(\boldsymbol{r}, E) = \frac{1}{4\pi} \oint \mathrm{d}\boldsymbol{\omega} U(B_M \boldsymbol{\omega}) e^{-iB_M \boldsymbol{\omega}\cdot\boldsymbol{r}} \left\{ \oint \mathrm{d}\boldsymbol{\Omega} \frac{4\pi P_F(B_M, E, \boldsymbol{\omega}\cdot\boldsymbol{\Omega}) + \frac{\chi(E)}{(2\pi)^{3/2}}}{\Sigma(E) - iB_M \boldsymbol{\omega}\cdot\boldsymbol{\Omega}} \boldsymbol{\Omega} \right\}. \tag{24}$$

The integration over $\boldsymbol{\Omega}$ is carried out firstly, and by a reasoning similar to the one made for the total flux, one obtains:

$$\boldsymbol{J}(\boldsymbol{r}, E) = -\frac{\Sigma(E)}{B_M^2} \left\{ \frac{4\pi \mathbb{P}_F(B_M, E) + \sqrt{\frac{2}{\pi}}\chi(E)}{4\pi\Sigma(E)\Psi(E; B_M)} - 1 \right\} \nabla \Phi(\boldsymbol{r}, E) \equiv -\mho(E; B_M) \nabla f(\boldsymbol{r}; B_M), \tag{25}$$

where

$$\mathbb{P}_F(B_M, E) = \oint \mathrm{d}\boldsymbol{\Omega} P_F(B_M, E, \boldsymbol{\omega}\cdot\boldsymbol{\Omega}), \tag{26}$$

independent of unit vector $\boldsymbol{\omega}$, and the current spectral function $\mho$ of the net current is obviously defined as:

$$\mho(E; B_M) \equiv \frac{\Sigma(E)}{B_M^2} \left\{ \frac{4\pi \mathbb{P}_F(B_M, E) + \sqrt{\frac{2}{\pi}}\chi(E)}{4\pi\Sigma(E)\Psi(E; B_M)} - 1 \right\} \Psi(E; B_M). \tag{27}$$

Through the discussion above it is thus verified that also the net neutron current is separable in space and energy. Furthermore, it turns out to be proportional to the gradient of total flux, Eq. (25), establishing a Fickian feature for the asymptotic transport process. A straightforward definition of the diffusion coefficient stems from Eq. (25):

$$D(E; B_M) = \frac{\Sigma(E)}{B_M^2} \left\{ \frac{4\pi \mathbb{P}_F(B_M, E) + \sqrt{\frac{2}{\pi}}\chi(E)}{4\pi\Sigma(E)\Psi(E; B_M)} - 1 \right\} = \frac{\mho(E; B_M)}{\Psi(E; B_M)}. \tag{28}$$

The above definition may be practically useful, for instance, to obtain group diffusion constants for nuclear reactor full core simulations. Some further elaborations and numerical examples are found in the next section.

6 Some applications and numerical illustrations

In this section some applications are presented, referring to the multigroup model that allows a simple analytical approach. In particular, the theory is applied to the one- and two-group cases and some parametric numerical results are presented.

6.1 The study of the criticality condition in the one-group model

In order to derive the criticality condition in the one-group model, Eq. (13) is used, namely:

$$(2\pi)^{3/2} \nu \Sigma_f \varphi_0(k) = 1. \tag{29}$$

It is then required to determine the total neutron flux in response to a localized fission source by solving the following transport equation in the transformed space:

$$\boldsymbol{\Omega} \cdot \nabla \varphi(\boldsymbol{r}, \boldsymbol{\Omega}) + \Sigma \varphi(\boldsymbol{r}, \boldsymbol{\Omega}) = P_{\infty}\left(r, \frac{\boldsymbol{r} \cdot \boldsymbol{\Omega}}{r}\right) + \frac{1}{4\pi} \delta(\boldsymbol{r}). \tag{30}$$

It is straightforward to obtain

$$[\Sigma - i\boldsymbol{\Omega} \cdot \boldsymbol{k}] \varphi(\boldsymbol{k}, \boldsymbol{\Omega}) = \frac{\Sigma_s}{4\pi} \varphi_0(k) + \frac{1}{(2\pi)^{3/2} 4\pi}, \tag{31}$$

and consequently:

$$\varphi_0(k) = \frac{1}{(2\pi)^{3/2}} \oint \mathrm{d}\boldsymbol{\Omega} \frac{1}{\Sigma(E) - i\boldsymbol{k} \cdot \boldsymbol{\Omega}} \left[\frac{\Sigma_s}{4\pi} \varphi_0(k) + \frac{1}{(2\pi)^{3/2} 4\pi}\right], \tag{32}$$

and, performing the integration:

$$\varphi_0(k) = \frac{1}{(2\pi)^{3/2}} \frac{\frac{1}{k} \tan^{-1} \frac{k}{\Sigma}}{1 - \frac{\Sigma_s}{k} \tan^{-1} \frac{k}{\Sigma}}. \tag{33}$$

At last, using Eq. (29), the final form of the criticality condition takes the form:

$$\nu \Sigma_f \frac{\frac{1}{k} \tan^{-1} \frac{k}{\Sigma}}{1 - \frac{\Sigma_s}{k} \tan^{-1} \frac{k}{\Sigma}} \equiv f(k) = 1, \tag{34}$$

which allows to determine the critical buckling B_M and has a validity in the transport model within the space asymptotic theory. In the limiting case of small values of k/Σ, one retrieves the well-known diffusion result:

$$B_M^2 = \frac{k_{\infty} - 1}{L^2}, \tag{35}$$

with the usual definitions of the diffusion parameters. Observing Eq. (34), one can immediately notice a similarity with the eigenvalue equation for the infinite medium characterized by a number of secondaries per collision smaller than one [12]. However, in such a case, the two solutions will turn out to be purely imaginary, yielding the diffusive portion of the solution in the infinite medium. On the contrary, Eq. (34) yields a real solution for k.

A plot of the critical function f is reported in Fig. 1 for different values of the number of neutrons emitted per scattering $c = \Sigma_s/\Sigma$, clearly showing that it is monotonously decreasing with k, as anticipated. The critical values of k obtained by diffusion theory are also marked in the graph. As can be seen in the graph and better shown in Table 1, the difference between diffusion and the asymptotic transport model increases as c is reduced.

It is also possible to obtain a consistent formula for the diffusion coefficient applying formula (28) for the specific case and obtaining the explicit expression:

$$D(B_M) = \frac{\Sigma}{B_M^2} \left[\frac{1}{\frac{\Sigma}{B_M} \tan^{-1} \frac{B_M}{\Sigma}} - 1\right]. \tag{36}$$

The above formulation can also be retrieved as a limit of the diffusion coefficient that may be defined within the spherical harmonics method, as it has been recently shown [13].

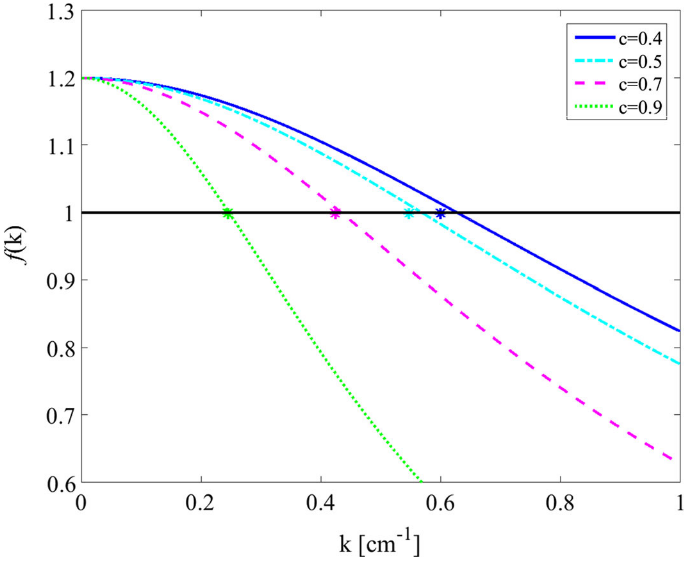

Fig. 1 Behaviour of the critical function f for different values of the number of secondaries per scattering c (the total cross section Σ is assumed unitary and $k_\infty = 1.2$). The critical condition in diffusion theory is identified by the marker *

Table 1 Comparison between the critical bucklings evaluated using diffusion theory (k_{diff}) and asymptotic transport (k_{asymp}) for different values of c (the total cross section Σ is assumed unitary)

c	$k_{\text{diff}}\,(\text{cm}^{-1})$	$k_{\text{asymp}}\,(\text{cm}^{-1})$
0.1	0.73485	0.78619
0.2	0.69282	0.73599
0.3	0.64807	0.68352
0.4	0.60000	0.62822
0.5	0.54772	0.56926
0.6	0.48990	0.50536
0.7	0.42426	0.43434
0.8	0.34641	0.35191
0.9	0.24495	0.24690
0.95	0.17320	0.17390

6.2 The two-group case

Let us consider a problem characterized by only thermal fissions, emitting neutrons only in the fast group ($\chi_1 = 1$). The emissions from scattering are assumed to be isotropic, and up-scattering is absent. To apply Eq. (13) the following system of coupled equations has to be solved in the Fourier transformed space:

$$\begin{cases} \boldsymbol{\Omega} \cdot \nabla \varphi_1 (\boldsymbol{r}, \boldsymbol{\Omega}) + \Sigma_1 \varphi_1 (\boldsymbol{r}, \boldsymbol{\Omega}) = \dfrac{\Sigma_{11}}{4\pi} \varphi_{1,0} + \dfrac{1}{4\pi} \delta (\boldsymbol{r}) \\ \boldsymbol{\Omega} \cdot \nabla \varphi_2 (\boldsymbol{r}, \boldsymbol{\Omega}) + \Sigma_2 \varphi_2 (\boldsymbol{r}, \boldsymbol{\Omega}) = \dfrac{\Sigma_{22}}{4\pi} \varphi_{2,0} + \dfrac{\Sigma_{21}}{4\pi} \varphi_{1,0}. \end{cases} \tag{37}$$

Table 2 Two-group material data. The corresponding value of k_∞ is 1.08428

	$g = 1$	$g = 2$
$\Sigma_{t,g}$ (cm^{-1})	3.54e−01	4.09e−01
$\Sigma_{a,g}$ (cm^{-1})	9.74e−04	7e−03
$\Sigma_{s,g\to g+1}$ (cm^{-1})	3.08e−03	–
$\Sigma_{r,g}$ (cm^{-1})	4.058e−03	7e−03
$\Sigma_{s,gg'}$ (cm^{-1})	$\begin{bmatrix} 3.49942e{-}01 & 0 \\ 3.08e{-}3 & 4.02e{-}01 \end{bmatrix}$	
$\nu\Sigma_{f,g}$ (cm^{-1})	0	1e−02
χ_g	1	0

Using the results derived in the previous subsection it is possible to obtain for the total flux in the fast group:

$$\varphi_{1,0}(k) = \frac{1}{(2\pi)^{3/2}} \frac{\frac{1}{k}\tan^{-1}\frac{k}{\Sigma_1}}{1 - \frac{\Sigma_{11}}{k}\tan^{-1}\frac{k}{\Sigma_1}}, \tag{38}$$

and, using this result in the second equation, the total flux for the thermal group is also obtained:

$$\varphi_{2,0}(k) = \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{\frac{\Sigma_{21}}{k}\tan^{-1}\frac{k}{\Sigma_1}}{1 - \frac{\Sigma_{11}}{k}\tan^{-1}\frac{k}{\Sigma_1}} \frac{\frac{1}{k}\tan^{-1}\frac{k}{\Sigma_2}}{1 - \frac{\Sigma_{22}}{k}\tan^{-1}\frac{k}{\Sigma_2}}. \tag{39}$$

In conclusion, for the particular physical problem of interest, the criticality condition shall be written in the following form:

$$\nu\Sigma_{f,2} \frac{\frac{\Sigma_{21}}{k}\tan^{-1}\frac{k}{\Sigma_1}}{1 - \frac{\Sigma_{11}}{k}\tan^{-1}\frac{k}{\Sigma_1}} \frac{\frac{1}{k}\tan^{-1}\frac{k}{\Sigma_2}}{1 - \frac{\Sigma_{22}}{k}\tan^{-1}\frac{k}{\Sigma_2}} \equiv f(k) = 1. \tag{40}$$

The above results can be easily generalized by induction to any number of energy groups.

Some numerical results are now presented for a system characterized by the material data given in Table 2. The plot of the two-group critical function f is reported in Fig. 2. Again the monotonicity of the function is demonstrated. The value taken at $k = 0$ corresponds to the value of k_∞ characterizing the material.

6.3 The neutron spectral functions for the flux and current and the diffusion coefficient in the two-group model

The spectral functions can be easily written down in the two-group model by preliminarily constructing the Fourier transform of the transport kernel as:

$$P_{g,F}(B_M, \boldsymbol{\omega}\cdot\boldsymbol{\Omega}) = \frac{1}{4\pi}\Sigma_{g1}\varphi_{1,0}(B_M) = \frac{1}{4\pi}\mathbb{P}_F(B_M, E), \quad g = 1, 2. \tag{41}$$

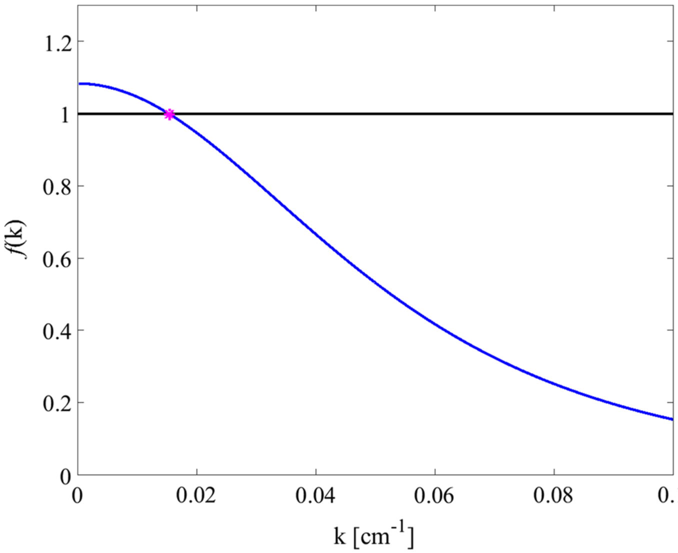

Fig. 2 Behaviour of the critical function f in the two-group model. The critical condition in diffusion theory is identified by the magenta marker

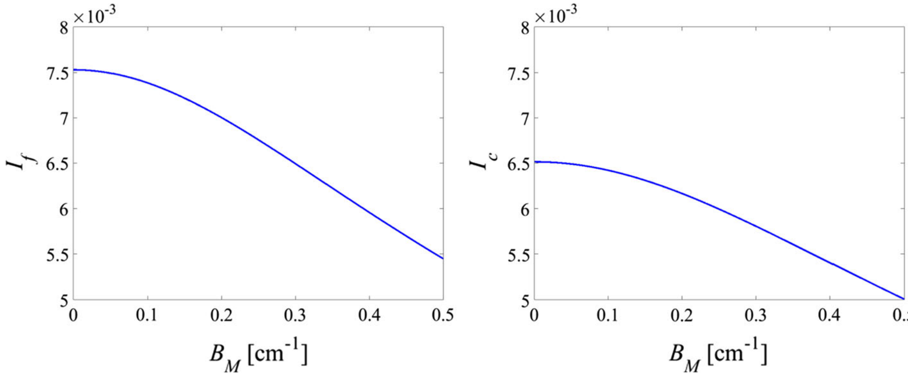

Fig. 3 Spectral indexes for the flux (left) and for the current (right) in two-group theory as functions of B_M

From definition (19) one can write the following expressions:

$$\Psi_1 \left(B_M\right) = \frac{1}{(2\pi)^{3/2}} \frac{1}{B_M} \tan^{-1} \frac{B_M}{\Sigma_1} \left[\frac{\Sigma_{11}}{B_M} \tan^{-1} \frac{B_M}{\Sigma_1} - 1 \right], \tag{42}$$

$$\Psi_2 \left(B_M\right) = \frac{1}{(2\pi)^{3/2}} \frac{1}{B_M} \tan^{-1} \frac{B_M}{\Sigma_2} \left[\frac{\Sigma_{21}}{B_M} \tan^{-1} \frac{B_M}{\Sigma_1} \right]. \tag{43}$$

In Fig. 3 the spectral indexes for the flux $I_f \equiv \Psi_2/\Psi_1$ and for the current $I_c \equiv \Psi_2/\Psi_1$ are represented. As can be seen, larger differences are evidenced for smaller values of k.

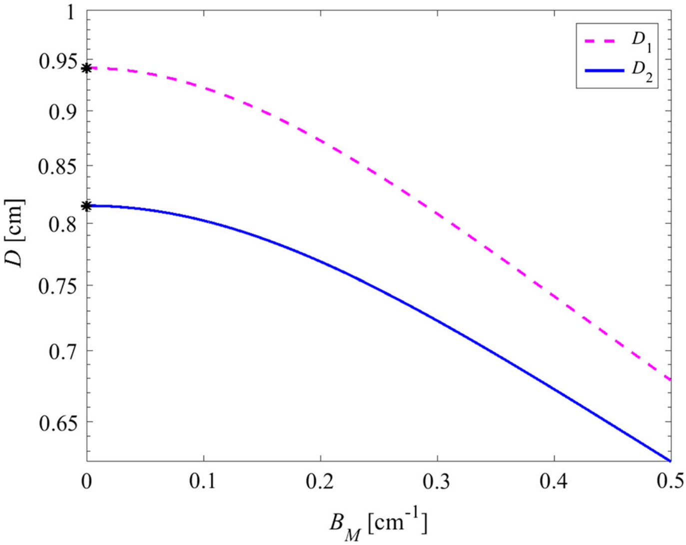

Fig. 4 Behaviour of the two-group diffusion coefficients as functions of B_M. The P_1-diffusion values are identified with the black marker

The diffusion coefficients for each group are easily derived as:

$$D_g(B_M) = \frac{\Sigma_g}{B_M} \tan^{-1} \frac{B_M}{\Sigma_1} \left[\frac{1}{\frac{\Sigma_g}{B_M} \tan^{-1} \frac{B_M}{\Sigma_g}} - 1 \right], \quad g = 1, 2. \tag{44}$$

Equation (44) has the same structure as the one for the monokinetic case for both groups. This result is physically consistent for the specific case considered in which all fission neutrons are emitted in group 1 and there is no up-scattering from group 2.

Figure 4 illustrates the behaviour of the two-group diffusion coefficients. The graphs show a consistent difference with respect to the values provided by the use of the standard P_1-diffusion model, which is obtained in the limit $B_M \to 0$. This effect may result in a relevant impact for the accuracy of the neutronic calculations performed with the diffusion model for nuclear reactor cores [14].

7 Conclusions

The space asymptotic approach is presented in this paper for applications to the study of the neutron transport in critical nuclear reactors. The implications of the asymptotic assumption are briefly discussed on a physical ground. The transport equation is then analysed applying the Fourier integral transform. This procedure allows to establish important theoretical results for the transport model. At first, the criticality theory can be given a consistent formulation, leading to an explicit and general formula, valid for any energy approach. The classic space–energy separability theorem stems quite naturally from the theory. It is then shown that the separability property can be extended also to the net neutron current. Furthermore, the expression obtained for the current leads to a straightforward definition of the energy-dependent diffusion coefficient.

To better highlight the features of the asymptotic procedure, some numerical analyses are presented in the one- and two-group models. Some results for the critical parameters and for the energy spectra for the neutron flux as well as for the current are illustrated. In particular, the results presented in two-group theory suggest that the alternative definitions of the multigroup diffusion coefficient may be more adequate than the standard ones for the simulations of nuclear reactor cores. Such definition may prove to be useful in energy collapsing procedures.

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Appendix A: Choice of the function *U* for simple geometries

We have remarked that a function written in integral form as

$$\oint \mathrm{d}\boldsymbol{\omega} U\left(B_M \boldsymbol{\omega}\right) e^{-i B_M \boldsymbol{\omega}\cdot \boldsymbol{r}} \tag{A1}$$

satisfies the Helmholtz equation, as can be readily proven by explicitly evaluating its Laplacian. Therefore, formula (A1) represents an integral formulation of the solutions of the Helmholtz problem. The function U is arbitrary, and its choice allows to fix the geometrical configuration under consideration. In particular, if U is chosen among real functions, it must be an even function of the direction vector in the transformed space $\boldsymbol{\omega}$ to guarantee the solution to be real in the geometry space, as it is immediately observing Eq. (A1). In a more general case, if it is a complex function, its real part must be even, while its imaginary part must be odd. For simple geometries it is easy to induce the form of U by intuitive symmetry considerations, as is seen in the following. For more complex geometries the process leading to the choice of U can be rather involved. Considering that the integral in Eq. (A1) must vanish on the boundary of the spatial domain, U can be determined as the solution of an integral equation that, in principle, can be numerically solved.

Spherical symmetry—To obtain the solution for the spherical geometry, the function U is chosen to take constant values on all points of the sphere with radius B_M. Using Eq. (A1), one immediately obtains:

$$f\left(\boldsymbol{r}; B_M\right) = \frac{\sin\left(B_M r\right)}{r}. \tag{A2}$$

The value of the material buckling B_M establishes the wavelength of the spatial spherical wave.

Plane symmetry—The solution for a slab geometry is obtained taking a U function that vanishes everywhere except at two opposite poles, where, of course, it must be singular, as:

$$U\left(B_M \boldsymbol{\omega}\right) = \frac{1}{2}\left[\delta\left(\boldsymbol{\omega} - \boldsymbol{\omega_w}\right) + \delta\left(\boldsymbol{\omega} + \boldsymbol{\omega_w}\right)\right], \tag{A3}$$

where $\boldsymbol{\omega_w}$ is the wave vector. Using Eq. (A1) it is immediate to obtain:

$$f\left(\boldsymbol{r};B_M\right)=\cos\left(B_M\boldsymbol{\omega_w}\cdot\boldsymbol{r}\right). \tag{A4}$$

Cylindrical symmetry—The spatial shape for a cylindrical domain can be retrieved assuming U to vanish everywhere on the sphere except on two symmetrically located circles, where it must be singular, as

$$U\left(B_M\boldsymbol{\omega}\right)=\frac{1}{4\pi}\left[\delta\left(\mu-\mu_w\right)+\delta\left(\mu+\mu_w\right)\right], \tag{A5}$$

where μ denotes the cosine of the latitude angle in the transformed space and μ_w is the cylindrical wave characteristic parameter. Inserting expression (A5) for U in Eq. (A1), recalling the integral representation of the Bessel function [15], one obtains, as expected:

$$f\left(\boldsymbol{r};B_M\right)=\cos\left(B_M\mu_w z\right)\mathrm{J}_0\left(B_M\sqrt{1-\mu_w^2}\rho\right), \tag{A6}$$

having indicated with z and ρ the axial and radial coordinates in the geometrical space, respectively. Quite obviously, if μ_w takes a unitary value one gets back the plane geometry solution, since the two circles collapse into two diametrically opposite points. On the contrary, assuming $\mu_w = 0$, the two circles merge into the equatorial line and one gets the purely cylindrical spatial wave.

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