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Effects of a harmonic-type potential on a scalar field in a background of CPT-odd Lorentz symmetry violation

R. L. L. Vitória[1,a] and H. Belich[2,b]

[1] Faculdade de Física, Universidade Federal do Pará, Av. Augusto Corrêa, Guamá, Belém, PA 66075-110, Brazil
[2] Departamento de Física e Química, Universidade Federal do Espírito Santo, Av. Fernando Ferrari, 514, Goiabeiras, Vitória, ES 29060-900, Brazil

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Abstract. We have investigated the relativistic quantum dynamics of a scalar field in an anisotropic background governed by the Lorentz symmetry violation. This analysis is done through a non-minimal coupling in the Klein–Gordon equation, which is inspired by the CPT-odd gauge sector of the standard model extension. From the elaboration of a vector and electromagnetic field configuration as a possible Lorentz break scenario, we induced an electric field which influences the scalar field. In this scenario, we show that it is possible to determine two energy profiles for the system, that is, a description of the relativistic quantum dynamics of the scalar field in the xy-plane and in all space-time.

1 Introduction

Although the standard model (SM) is the best theory available to date to describe fundamental interactions, with the exception of gravitational interaction, there are some contradictions or lack of explanations about the phenomena described by the SM. For example, there is observational evidence that the fine-structure constant is slowly changing [1,2], a constant that is provided by the interaction between matter and light well described by quantum electrodynamics [3], one of the pillars of SM, there is evidence that neutrinos have mass [4], which is contrary to what the SM describes. In addition, more recent experimental data, in the context of particle physics, intensify more questions about the need for SM extensions, that is, recently, a muon is used instead of an electron orbiting the atomic nucleus of the hydrogen atom, pointing out that the proton radius is little different from the theoretical predictions [5]. Another question is the evidence that some particles detected in the Antarctic, despised as anomalies because they do not fit into theories, are real and must be taken into account [6,7].

Based on these pertinent questions about the SM, there is a need for complementary or alternative theories to the SM. In this perspective, one of the theories that has attracted a lot of attention from the scientific community is the Lorentz symmetry violation (LSV), whose main characteristic is the presence of anisotropies in space-time governed by background fields of a vector and tensorial nature [8,9]. With the vast study on LSV [10–15] and its extensive applicability in theoretical physics [16–25], the field theory that describes the fundamental particles in an anisotropic background has come to be known as the standard model extension (SME) [26,27].

In the SME, there are several sectors corresponding to the nature of the interaction particles [26,27]. For example, there is the gauge sector, which is divided into even and odd CPT sectors. This terminology is associated with CPT symmetry. In the even case, this means that the Lorentz symmetry is broken, however, the CPT symmetry is preserved; in the odd case, it means that both symmetries are violated [28]. These two sectors have been extensively analyzed in the context of relativistic quantum dynamics. In the case of the CPT-even gauge sector, there are studies on a scalar field subject to induced central potentials [29–32] and on the Klein–Gordon oscillator [33,34]. On the other hand, the CPT-odd gauge sector has been extensively analyzed in the relativistic quantum dynamics of the Dirac field, for example, on the Dirac oscillator [35] and on a Landau-type quantization induced by the LSV [36]. Recently, the relativistic quantum dynamics of a scalar field has been analyzed only in Ref. [37], although the analysis is incomplete, as the authors disregarded quadratic terms from the non-minimum coupling, thus causing the loss of possible effects of LSV on the quantum system in the particular field configuration scenario considered. Given this, our aim is to analyze the non-minimal coupling of the CPT-odd gauge sector [28] in its complete form in the Klein–Gordon equation and, from this theoretical point of view, address numerous possible scenar-

[a] e-mails: ricardo.vitoria@pq.cnpq.br; ricardo-luis91@hotmail.com (corresponding author)
[b] e-mail: belichjr@gmail.com

ios of vector and electromagnetic field configurations in an anisotropic space-time characterizing the breaking of Lorentz symmetry.

In the present analysis, we have investigated the effects of a central harmonic-type potential induced by the LSV through a background vector and electromagnetic field configuration, which correspond to a vector field of azimuth nature and an electric field that varies linearly with the axial distance, respectively, from which we determine the relativistic energy profile of this anisotropic quantum system. We complement our analysis with the presence of a hard-wall confining potential.

The structure of this paper is as follows: in Sect. 2, we have done the complete analysis of the Klein–Gordon equation modified by the non-minimum coupling of the CPT-odd gauge sector; in Sect. 3 we have analyzed a possible LSV scenario through a choice of vector and electromagnetic field configuration from which, in plane (3.1) we induce a harmonic-type potential and thereby determine the relativistic energy profile. This analysis is also carried out in the presence of a rigid wall potential; in Sect. 3.2, we investigated the effects of the harmonic-type potential induced by LSV on a scalar field for $k \neq 0$ from which we determine the lowest energy state of the system; in Sect. 4, we present our conclusions.

2 General background

Despite the great success in explaining the origin of fundamental particles by quantized fields, we need to extend the standard model in order to achieve a more fundamental theory. The search for manifestations of primordial fields that may arise from manifestations of anisotropies in space is our objective in this section. Such proposal can be realized through non-minimal couplings with background fields that can be felt by local physical systems. Based on Ref. [38], recently Vitória and Belich [37] proposed the non-minimal coupling

$$\partial_\mu \rightarrow \partial_\mu - ig\tilde{F}_{\mu\nu}v^\nu \tag{1}$$

into Klein–Gordon equation in order to describe the relativistic quantum dynamics of a scalar particle in possible scenarios of Lorentz symmetry breaking, where g is a coupling constant, $\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ is the dual electromagnetic tensor, with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and v_μ is vector field that governs the LSV. However, this modification in the Klein–Gordon equation was not complete, as the authors disregarded the quadratic terms from the non-minimal coupling, causing a likely loss of physical effects which may be important in the final results. Therefore, let us consider the Klein–Gordon equation for a massive free scalar field ϕ $(c = \hbar = 1)$

$$\Box\phi - m^2\phi = 0, \tag{2}$$

where m is rest mass of the scalar field and $\Box \equiv \partial_\mu\partial^\mu$. By considering the Minkowski space-time with cylindrical symmetry described by the metric

$$\mathrm{d}s^2 = -dt^2 + \mathrm{d}\rho^2 + \rho^2\mathrm{d}\varphi^2 + \mathrm{d}z^2, \tag{3}$$

with $\rho = (x + y)^{1/2}$, Eq. (2) is rewritten as follows

$$-\partial_t^2\phi + \partial_\rho^2\phi + \frac{1}{\rho}\partial_\rho\phi + \frac{1}{\rho^2}\partial_\varphi^2\phi + \partial_z^2\phi - m^2\phi = 0, \tag{4}$$

which describes the relativistic quantum dynamics of free scalar field in the Minkowski space-time.

Then, by substituting Eq. (1) into Eq. (2) we obtain

$$(\partial_\mu - ig\tilde{F}_{\mu\alpha}v^\nu)(\partial^\mu - ig\tilde{F}^{\mu\beta}v_\beta) - m^2\phi = 0, \tag{5}$$

or

$$\begin{aligned}&\Box\phi - 2ig(\partial_\mu\phi)\tilde{F}^{\mu\alpha}v_\alpha - ig(\partial_\mu\tilde{F}^{\mu\alpha})v_\alpha\phi \\ &\quad - g^2v^\alpha v_\alpha\tilde{F}_{\mu\alpha}\tilde{F}^{\mu\alpha}\phi - m^2\phi = 0.\end{aligned} \tag{6}$$

We can go further with Eq. (6), because the term $\partial_\mu\tilde{F}^{\mu\alpha}$ are the homogenous Maxwell equations. Thus, Eq. (6) becomes

$$\Box\phi - 2ig(\partial_\mu\phi)\tilde{F}^{\mu\alpha}v_\alpha - g^2v^\alpha v_\alpha\tilde{F}_{\mu\alpha}\tilde{F}^{\mu\alpha}\phi - m^2\phi = 0. \tag{7}$$

As mentioned before, in Ref. [37] the third term of Eq. (7) is disregarded, which makes the analysis incomplete. From now on, our analysis will take into account the effects of this once neglected term. We can write Eq. (7) in terms of the electric and magnetic field in explicit form, that is,

$$\begin{aligned}&\Box\phi + 2ig(\mathbf{v}.\mathbf{B})\partial_0\phi + 2igv_0(\mathbf{B}.\boldsymbol{\nabla})\phi - 2ig(\mathbf{v}\times\mathbf{E}).\boldsymbol{\nabla}\phi \\ &\quad + g^2v_0^2\mathbf{B}^2\phi - g^2(\mathbf{v}.\mathbf{B})^2\phi - g^2[(v_2^2+v_3^2)E_1^2 + (v_1^2+v_3^2)E_2^2 \\ &\quad + (v_1^2+v_2^2)E_3^2]\phi - m^2\phi = 0.\end{aligned} \tag{8}$$

We can note that, by considering all terms from the non-minimal coupling of the Klein–Gordon equation, the equation is drastically modified, mainly with the presence of the quadratic terms of the electric and magnetic fields, which can provide quite interesting physical effects in the scope of symmetry breaking of Lorentz. In addition, Eq. (8) represents a general case which can be analyzed in particular scenarios through less complex vector and electromagnetic field configurations; by comparing Eq. (8) with Eq. (07) of Ref. [37], we can see the extra term coming from the consideration of the quadratic term of the LSV, $-g^2[(v_2^2+v_3^2)E_1^2 + (v_1^2+v_3^2)E_2^2 + (v_1^2+v_2^2)E_3^2]$. From now on, we will analyze one of these possible anisotropic scenarios governed by a vector field associated with the SME CPT-odd sector.

3 Effects of a harmonic-type potential induced by the LSV on a scalar field

Now we will establish a scenario for measuring fields that generate anisotropies through non-minimum couplings. Let us consider a background determined by the vector v_μ and electromagnetic field $F_{\mu\nu}$ configuration in the space-time described by the line element given in Eq. (3)

$$\begin{aligned} v_\mu &= (0,0,v_\varphi,0);\\ \overrightarrow{E} &= \frac{\vartheta\rho}{2}\hat{\rho};\\ \overrightarrow{B} &= 0, \end{aligned} \tag{9}$$

where v_φ is a constant and ϑ is a constant associated with a volumetric distribution of electric charges. This electric field configuration has been studied in induced electric dipole moment systems [39–42] and in LSV possible scenarios [28,43]. It is noteworthy that the field configuration given in Eq. (9) differs from the configuration given in Ref. [37]. This difference only occurs in the distribution of electrical charge; in Ref. (37) we consider a Coulomb-type electric field. Then, for this possible scenario given in Eq. (9), Eq. (8) becomes

$$\begin{aligned} &-\partial_t^2\phi+\partial_\rho^2\phi+\frac{1}{\rho}\partial_\rho\phi+\frac{1}{\rho^2}\partial_\varphi^2\phi\\ &\quad+\partial_z^2\phi+igv_\varphi\vartheta\rho\partial_z\phi-\frac{g^2v_\varphi^2\vartheta^2}{4}\rho^2\phi-m^2\phi=0. \end{aligned} \tag{10}$$

In order to solve Eq. (10), let us consider the general solution in terms of the eigenvalues $l=0,\pm1,\pm2,\ldots$ and $-\infty<k<\infty$, that is, angular momentum $\hat{L}_z=-i\partial_\varphi$ and linear momentum $\hat{p}_z=-i\partial_z$, respectively, given in the form

$$\phi(t,\rho,\varphi,z)=R(\rho)e^{-i(\mathcal{E}t-l\varphi-kz)}. \tag{11}$$

By substituting Eq. (11) into Eq. (10), we have the radial wave equation

$$\begin{aligned} &\frac{\mathrm{d}^2R}{\mathrm{d}\rho^2}+\frac{1}{\rho}\frac{\mathrm{d}R}{\mathrm{d}\rho}-\frac{l^2}{\rho^2}R-gv_\varphi\vartheta k\rho R\\ &\quad-\frac{g^2v_\varphi^2\vartheta^2}{4}\rho^2R+(\mathcal{E}^2-m^2-k^2)R=0. \end{aligned} \tag{12}$$

Differential equation (12) describes the relativistic quantum dynamics of a massive scalar field subjected to an electric field that linearly varies with axial distance which is induced by LSV governed by a vector field. We can note that, in Eq. (12), we have a liner potential term (fourth term) and a harmonic-type potential term (fifth term). The presence this two terms, both induced by LSV, gives us two cases to be analyzed: the quantum system in the plane, by taking $k=0$, and the general case for $k\neq0$. From now on, we will analyze these two cases.

3.1 Particular case: $k=0$

3.1.1 Harmonic-type potential

Let us consider the case $k=0$. In this particular case, the relativistic quantum system is restricted in xy-plane and Eq. (12) becomes

$$\frac{\mathrm{d}^2R}{\mathrm{d}\rho^2}+\frac{1}{\rho}\frac{\mathrm{d}R}{\mathrm{d}\rho}-\frac{l^2}{\rho^2}R-\frac{g^2v_\varphi^2\vartheta^2}{4}\rho^2R+(\mathcal{E}^2-m^2)R=0. \tag{13}$$

Equation (13) is radial wave equation that describes the quantum motion of massive scalar field in xy-plane under effects of an electric field that linearly varies with axial coordinate in an anisotropic space-time governed by a constant vector field. In this case, we have a harmonic-type potential in Eq. (13) which is induced by LSV. This type of potential is extremely important, as it is an ideal model for the description of systems characterized by vibration around a reference point. This feature has wide applicability in solids, condensed matter and molecular atomic physics [44]. In addition, the harmonic-type potential has been extensively investigated in several scenarios, for example, thermodynamic properties [45], in a global monopole space-time [46], in the cosmic string space-time [47] and in the presence of a cosmic screw dislocation [48].

Now, we proceed with a change of variables given by $\xi=\frac{gv_\varphi\vartheta}{2}\rho^2$, and thus, we rewrite Eq. (13) in the form

$$\frac{\mathrm{d}^2R}{\mathrm{d}\xi^2}+\frac{1}{\xi}\frac{\mathrm{d}R}{\mathrm{d}\xi}-\frac{l^2}{4\xi^2}R+\frac{\alpha}{\xi}R-\frac{1}{4}R=0, \tag{14}$$

where we define the parameter

$$\alpha=\frac{\mathcal{E}^2-m^2}{2gv_\varphi\vartheta}. \tag{15}$$

By imposing that radial wave function $R(\xi)\to0$ when $\xi\to0$ and $\xi\to\infty$, we have that $R(\xi)$ can be written as follows:

$$R(\xi)=\xi^{\frac{|l|}{2}}e^{-\frac{\xi}{2}}f(\xi). \tag{16}$$

Then, by substituting Eq. (16) into Eq. (14), we obtain

$$\xi\frac{\mathrm{d}^2f}{\mathrm{d}\xi^2}+(|l|+1-\xi)\frac{df}{\mathrm{d}\xi}+\left(\alpha-\frac{|l|}{2}-\frac{1}{2}\right)f=0, \tag{17}$$

which is known as confluent hypergeometric equation [49] and $f(\xi)$ is the confluent hypergeometric function:

$f(\xi) = {}_1F_1(A, B; \xi)$, with

$$A = \frac{1}{2} + \frac{|l|}{2} - \alpha; \quad B = |l| + 1. \tag{18}$$

It is well known that the confluent hypergeometric series becomes a polynomial of degree $n = 0, 1, 2, \ldots$, when the condition $A = -n$ is satisfied [49]. Then, this equality we obtain the expression

$$\mathcal{E}_{l,n} = \pm\sqrt{m^2 + 2m\omega\left(n + \frac{|l|}{2} + \frac{1}{2}\right)}, \tag{19}$$

where

$$\omega = \frac{gv_\varphi\vartheta}{m}. \tag{20}$$

Equation (19) represents the relativistic energy levels of massive scalar field restricted in xy-plane and subjected to an electric field that linearly varies with axial coordinate in a LSV background governed by a constant vector field. We can note the relativistic energy profile (19) is produced by the harmonic-type interaction, which is induced by a possible LSV scenario, which is governed by a constant background vector field. In addition, we can also note that relativistic energy spectrum is analogous to the energy levels calculated in Ref. [30]. However, there is a difference between these results, as our result comes from the CPT-odd coupling inspired by the SME gauge sector, while in Ref. [30] is used the non-minimum CPT-even coupling inspired by the SME gauge sector.

It is important to note that the relativistic energy profile determined in Eq. (19) was only possible to determine due to the consideration of quadratic terms in Eq. (8). Since we are investigating a particular case ($k = 0$) of the relativistic quantum dynamics of a scalar particle in the xy-plane, the disregard of the quadratic terms in Eq. (8), as it occurs in Ref. [37], would result in the self-functions in terms of the Bessel cylindrical harmonics [50].

3.1.2 Harmonic-type potential plus a hard-wall confining potential

Now, let us consider the presence of a rigid wall potential in the scenario described in the previous section, that is, a scalar field in the xy-plane subject to an LSV-induced harmonic potential plus a potential described by the following boundary condition:

$$R(\rho_0) = 0, \tag{21}$$

where $\rho_0 = \text{const}$. From the mathematical point of view, the boundary condition given in Eq. (21) is the Dirichlet boundary condition; from the physical point of view, Eq. (21) represents the presence of a hard-wall potential in the system, that is, the radial wave function vanishes at a fixed radius ρ_0. This type of potential has been analyzed on some relativistic quantum systems, for example, on a scalar field under effects of the relativistic Landau quantization and of the Klein–Gordon oscillator in the space-time with torsion [51], on the Dirac and Klein–Gordon oscillators in the global monopole space-time [52] and in LSV scenarios [50,53]. The hard-wall confining potential is important because it is a very good approximation to consider when discussing the quantum properties of a gas molecule system and other particles, which are necessarily confined in a box of certain dimensions. In addition to the fixed radius ρ_0, let us consider $\alpha \gg 1$ and fixed angular momentum eigenvalues l. In this conditions, the confluent hypergeometric function ${}_1F_1(A, B; \xi_0)$ becomes [54]

$${}_1F_1(A, B; \xi_0) \approx \cos\left(\sqrt{2B\xi_0 - 4A\xi_0} - \frac{B}{2}\pi + \frac{\pi}{4}\right). \tag{22}$$

Thereby, by substituting Eqs. (16) and (22) into Eq. (21), we obtain the expression

$$\mathcal{E}_{l,n} = \pm\sqrt{m^2 + \frac{\pi^2}{\rho_0^2}\left(n + \frac{|l|}{2} + \frac{1}{2}\right)^2}, \tag{23}$$

which represents the relativistic energy profile of a massive scalar field under effects of a harmonic-type potential induced by the LSV plus a hard-wall potential in xy-plane. We can note that, in this case, the scalar field isn't influenced by the harmonic-type central potential induced by the LSV scenario, that is, the scalar field is only influenced by the hard-wall confining potential; the hard-wall potential inhibits the anisotropic effects of LSV. By taking $\rho_0 \to \infty$, we recover the rest energy of scalar field.

3.2 General case: $k \neq 0$

In this section, we analyze the general case characterized by the definition $k \neq 0$. In this case, in addition to the presence of the harmonic-type potential, there is also the presence of a central linear potential into Eq. (12), both induced by LSV. It is worth mentioning that the linear potential has been investigated in several systems of relativistic quantum mechanics, for example, in quark–antiquark interaction [55], on a scalar field subjected to the relativistic Landau quantization in space-time with torsion [51] and in cosmic string space-time [56], on a scalar field in Kaluza–Klein theory [57–60], on a scalar field in Gödel-type space-time [61], in spin-0 relativistic quantum particles in (1 + 2)-dimensions Gürses space-time [62] and on the Klein–Gordon oscillator in Minkowski space-time [63,64].

Let us consider the variable change $\varrho = \sqrt{\frac{g v_\varphi \vartheta}{2}}\rho$, such that Eq. (12) becomes

$$\frac{\mathrm{d}^2R}{\mathrm{d}\varrho^2} + \frac{1}{\varrho}\frac{\mathrm{d}R}{\mathrm{d}\varrho} - \frac{l^2}{\varrho^2}R - \beta\varrho R - \varrho^2 R + \gamma R = 0, \tag{24}$$

where we define the follows parameters

$$\beta = 2k\sqrt{\frac{2}{g v_\varphi \vartheta}}; \quad \gamma = \frac{2(\mathcal{E}^2 - m^2 - k^2)}{g v_\varphi \vartheta}. \tag{25}$$

The solution to Eq. (24) can be written in the form [56]

$$R(\varrho) = \varrho^{|l|} e^{-\frac{1}{2}\varrho(\varrho+\beta)} h(\varrho), \tag{26}$$

where $h(\varrho)$ is an unknown function. Then, by substituting Eq. (26) into Eq. (24), we have the differential equation

$$\frac{\mathrm{d}^2h}{\mathrm{d}\varrho^2} + \left(\frac{2|l|+1}{\varrho} - 2\varrho - \beta\right)\frac{\mathrm{d}h}{\mathrm{d}\varrho} + \left(\delta - \frac{\epsilon}{\varrho}\right)h = 0, \tag{27}$$

with

$$\delta = \gamma + \frac{\beta^2}{4} - 2(1+|l|); \quad \epsilon = \frac{\beta}{2}(2|l|+1). \tag{28}$$

Eq. (27) is known as the biconfluent Heun equation [56,65] and $h(\varrho)$ is the biconfluent Heun function:

$$h(\varrho) = H_b\left(2|l|, \beta, \gamma + \frac{\beta^2}{4}, 0; \varrho\right). \tag{29}$$

Eq. (27) contains two singular points: the origin and the infinite, a regular singular point and an irregular singular point, respectively [56]. Since the origin is a regular singular point, Eq. (27) has at least one solution around this point given by the power series [49,56]:

$$h(\varrho) = \sum_{0}^{\infty} c_j \varrho^j. \tag{30}$$

By substituting Eq. (30) into Eq. (27), we obtain the recurrence relation

$$c_{j+2} = \frac{[\beta(j+1)+\epsilon]c_{j+1} - (\delta - 2j)c_j}{(j+2)(j+2+2|l|)}, \tag{31}$$

with the coefficients

$$c_1 = \frac{\epsilon}{1+2|l|}c_0 = \frac{\beta}{2}c_0;$$
$$c_2 = \frac{(\beta+\epsilon)c_1 - \delta c_0}{2(2+2|l|)} = \frac{c_0}{4(1+|l|)}\left[\frac{(\beta+\epsilon)\epsilon}{(1+2|l|)} - \delta\right]. \tag{32}$$

Our aim is to determine solutions of bound states. Therefore, we must truncate the biconfluent Heun power series (30) and this procedure is possible through the following conditions [56]:

$$c_{\bar{n}+1} = 0; \quad \delta = 2\bar{n}, \tag{33}$$

where $\bar{n} = 1, 2, 3, \ldots$ are the radial modes. From now on, we must analyze these two conditions by imposing values of $\bar{n}$ separately. In this case, let us consider the radial mode $\bar{n} = 1$, which represents the lowest energy state of the relativistic quantum system, that is, for $\bar{n} = 1$ in the condition $c_{\bar{n}+1} = 0$ we obtain $c_2 = 0$. Then, imposing this condition in Eq. (32), we obtain the following expression

$$\vartheta_{k,l,1} = \frac{k^2}{g v_\varphi}(2|l|+3). \tag{34}$$

Eq. (34) gives us the allowed values for the parameter associated to the volumetric distribution of electric charges corresponding to the radial mode $\bar{n} = 1$. For the lowest energy state of the relativistic quantum system, this permit us to construct a first degree polynomial to the biconfluent Heun series (30). Since the allowed values of the parameter associated to the volumetric distribution of electric charges are determined by the quantum numbers $\{k, l, \bar{n}\}$, we have labeled $\vartheta = \vartheta_{k,l,\bar{n}}$ with $k \neq 0$, where we have assumed assume that the parameter ϑ can be adjusted in such a way that the condition $c_{\bar{n}+1} = 0$ is satisfied for any value of $\bar{n}$.

From the condition $\delta = 2\bar{n}$, for $\bar{n} = 1$, we have

$$\mathcal{E}^2_{k,l,1} = m^2 + g v_\varphi \vartheta_{k,l,1}(2+|l|). \tag{35}$$

Then, by substituting Eq. (34) into Eq. (35), we obtain the expression

$$\mathcal{E}_{k,l,1} = \pm\sqrt{m^2 + k^2(2|l|+3)(|l|+3)}, \tag{36}$$

which represents the allowed values of relativistic energy for the lowest energy state of the quantum system. By comparing Eqs. (19) and (36), we can see that the system relativistic energy levels are drastically modified. This modification results from the condition $k \neq 0$ which induces a linear central potential in the system. In addition, the lower energy state is determined by the radial mode $\bar{n} = 1$, in contrast to the quantum number $n = 0$ as obtained in Eq. (19). We can also note that, in contrast to Eq. (19), the lowest energy state of the quantum system (36) doesn't depend of the parameters associated to LSV, that is, the lowest energy state of the relativistic quantum system isn't influenced by the anisotropic background.

4 Conclusion

We have analyzed the relativistic quantum dynamics of a scalar field in a possible CPT-odd LSV scenario. For this, through a non-minimal coupling in the Klein–Gordon equation, we modified it in order to investigate the possible anisotropic effects governed by a vector field which violates the CPT symmetry and, consequently, violates the Lorentz symmetry. Differently from Ref. [37], we consider all terms from the non-minimum coupling, taking into account the quadratic terms, previously neglected.

Having made the complete modification in the Klein–Gordon equation, we propose a Lorentz break scenario, which, from a theoretical point of view, is possible and is characterized by a simple azimuth vector and electromagnetic field configuration defined by the absence of a magnetic field and by the presence of an electric field that varies linearly with the axial coordinate. Through this particular scenario, we can notice the induction of a central harmonic-type potential plus a linear potential on the relativistic quantum system. So, we analyze our system in two parts: in the plane and in the whole space, that is, for $k = 0$ and $k = 1$, respectively.

We can notice that, in the plane, the radial wave equation is reduced in a quantum system of a massive scalar field in interaction with a harmonic-type potential, where it is possible to obtain a relativistic energy profile of a relativistic oscillator, where the frequency of the oscillator is determined by the parameters associated with the LSV. In addition, we can also note that this harmonic-type potential induced by LSV in the presence of a rigid wall potential does not influence the scalar field, that is, the hard-wall confinement potential inhibits the anisotropic effects on the scalar field.

In the general case characterized by $k \neq 1$, we can note that the relativistic energy levels of the system are drastically modified. This occurs due to the presence of the linear potential in the system competing with a harmonic-type potential, both induced by LSV. This modification is perceived by the following effects: unlike the system previously analyzed, the lowest energy state of the system is determined by the radial mode $\bar{n} = 1$ and not by $n = 0$; it is not possible to obtain a closed expression for the relativistic energy levels, but to determine the allowed energy values for the radial modes separately; in addition to relativistic energy, the permitted values of the parameter associated with the volumetric distribution of electrical charges is determined by the quantum numbers of the system; the lowest energy state of the relativistic quantum system isn't influenced by the anisotropic background.

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Author contributions

R. L. L. Vitória is responsible for preparing all stages of article construction, literature review, definition of methodology, discussion of results and conclusion. H. Belich is responsible for guiding and correcting all steps performed by the first author.

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