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Inverse-square-type potential from the interaction of the magnetic quadrupole moment of a neutral particle with a magnetic field

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Abstract. We analyse the attractive inverse-square-type potential that arises from the interaction of the magnetic quadrupole moment of a neutral particle with the magnetic field. Then, we search for bound state solutions to the Schrödinger equation. Besides, we analyse the repulsive inverse-square-type potential that arises from the interaction of the magnetic quadrupole moment of a neutral particle with the magnetic field. Thus, we discuss the influence of this repulsive inverse-square-type potential on the neutral particle subject to two cylindrical surfaces and a cylindrical surface.

1 Introduction

In recent decades, the magnetic quadrupole moment has drawn attention in quantum physics due to the studies of geometric quantum phases [1,2], noncommutative quantum mechanics [3] and under the effects of rotation [4]. The main interest in the magnetic quadrupole moment is in systems with molecules [5,6] and atoms [7,8], but it goes further by exploring the chiral anomaly [9], the violation of the time-reversal symmetry in molecules [10] and the violation of CP-symmetry [11]. Recently, the appearance of effective uniform magnetic fields [2,12] and Coulomb-type interactions [13,14] from the interaction of the magnetic quadrupole moment with external fields has been analysed in search of bound states. In chemical physics, it is worth citing studies of nuclear quadrupole moments [15–20] and molecular quadrupole moments [21,22].

Based on the appearance of analogues of the Landau quantization [2,12] and the Coulomb interaction [13] in the magnetic quadrupole moment system, other interactions can rise and show bound states. For instance, a singular potential like the attractive inverse-square potential is an interesting topic for discussion. From the analysis of singular potentials, Case [23] made one of the first studies that involves attractive inverse-square potentials. Landau and Lifshitz [24] also investigated the attractive inverse-square potential and showed its unusual behaviour, which they called as the fall of the particle to the centre. This unusual behaviour is characterized by having energy levels that go to $-\infty$ when the limit $r_0 \to 0$ is taken on the short-distance cut-off r_0. At present days, studies of the attractive inverse-square potential can be found in systems of atoms that interact with the magnetic field of a ferromagnetic wire [25] and with the electric field produced by a long charged wire [26–28]. Other studies have dealt with the Efimov effect [29], the generalized uncertainty principle [30] and the Aharonov–Bohm effect [31]. In the interface of quantum mechanics and general relativity, the inverse-square potential has been investigated from nonrelativistic effects on a scalar field in the Reissner–Nordström black hole spacetime [32] and from the AdS/CFT correspondence [33–35].

Thereby, in this work, we raise a discussion about the attractive inverse-square potential that can rise in a magnetic quadrupole moment system. The appearance of this singular potential is due to the interaction of the magnetic quadrupole moment of a neutral particle with the magnetic field produced by a long cylindrical wire. We show that this interaction gives rise to an attractive inverse-square-type potential; thus, we search for bound state solutions to the Schrödinger equation. Further, we show another perspective where this interaction can give rise to a repulsive inverse-square-type potential. We thus analyse the quantum effects of the repulsive inverse-square-type potential on the neutral particle subject to two cylindrical surfaces and a cylindrical surface.

The structure of this paper is: in Sect. 2, we show that an attractive inverse-square potential can arise from the interaction of the magnetic quadrupole moment of a neutral particle with an external magnetic field and obtain the bound state solutions to the Schrödinger equation; in Sect. 2.2, we discuss the case where a repulsive inverse-square potential arises from the interaction

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between the magnetic quadrupole moment of a neutral particle and an external magnetic field. Thus, we analyse the confinement of this neutral particle to two cylindrical surfaces and to a cylindrical surface; in Sect. 3, we present our conclusions.

2 Interaction with an azimuthal magnetic field

In the rest frame of the neutral particle, such as an atom or a molecule, the potential energy that describes the interaction of the magnetic quadrupole moment of the (spinless) neutral particle with a magnetic field is given by $V_m = -\sum_i \sum_j M_{ij}\, \partial_i\, B_j$ [36,37], where M_{ij} is the magnetic quadrupole moment tensor and $\vec{B}$ is the magnetic field. In addition, the tensor M_{ij} is a symmetric and traceless tensor. On the other hand, when the neutral particle moves with velocity $v \ll c$ (c is the velocity of light), the quantum description of interaction of the magnetic quadrupole moment of the (spinless) neutral particle with external fields is given by the time-independent Schrödinger equation (with the units $\hbar = 1$ and $c = 1$) [2,12,13]:

$$\mathcal{E}\phi = \frac{1}{2m}\left[\hat{p} - \vec{M}\times\vec{E}\right]^2 \phi - \vec{M}\cdot\vec{B}\,\phi. \tag{1}$$

Note that the components of the vector $\vec{M}$ are determined by $M_i = \sum_j M_{ij}\, \partial_j$. The fields $\vec{B}$ and $\vec{E}$ are the electric and magnetic fields in the laboratory frame, respectively. Moreover, m corresponds to the mass of the neutral particle, $\hat{p} = -i\vec{\nabla}$ is the momentum operator, and $\mathcal{E}$ corresponds to the energy eigenvalue.

In this work, we assume that the non-null components of the tensor M_{ij} are given by

$$M_{r\,\varphi} = M_{\varphi\, r} = M, \tag{2}$$

where M is a constant ($M > 0$). From now on, our focus is on the arising of an inverse-square-type potential [23,28,38–40] from the interaction of magnetic quadrupole moment (21) with a magnetic field. We shall discuss the arising of attractive and repulsive inverse-square potentials and the possibility of obtaining bound state solutions to Schrödinger equation (1). Recently, one of us has studied the arising of inverse-square-type potentials in a spin-dependent potential [41], from Lorentz symmetry breaking effects [42] and in a system of a neutral particle with an induced electric dipole moment [43].

2.1 Attractive inverse-square-type potential

Let us consider the magnetic field produced by a long cylindrical wire of radius R_0, where the electric current i_0 is uniformly distributed inside the wire. Then, the magnetic field at $r > R_0$ is given by

$$\vec{B} = -\frac{I}{r}\,\hat{\varphi}, \tag{3}$$

where $I = \frac{\mu_0\, i_0}{2\pi} > 0$ and $\hat{\varphi}$ is a unit vector that indicates the azimuthal direction. With the magnetic quadrupole tensor given in Eq. (2) and azimuthal magnetic field (3), hence, the last term of the right-hand side of Eq. (1) becomes

$$V_{\text{eff}}\,(r) = -\vec{M}\cdot\vec{B} = -\frac{M\,I}{r^2}. \tag{4}$$

Thereby, Eq. (4) shows us that the interaction of the magnetic quadrupole moment with the azimuthal magnetic field gives rise to an attractive inverse-square potential [23,28,38–42].

Hence, Schrödinger equation (1) becomes

$$\mathcal{E}\phi = -\frac{1}{2m}\left[\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2}\right] - \frac{M\,I}{r^2}\,\phi. \tag{5}$$

With the purpose of solving Eq. (5), let us write $\phi\,(r,\varphi,z) = G\,(\varphi)\; Z\,(z)\; u\,(r)$. After substituting ψ into Eq. (5), we obtain $G\,(\varphi) = e^{i\,\ell\,\varphi}$ and $Z\,(z) = e^{ip_z z}$, where $\ell = 0, \pm1, \pm2, \ldots$ and p_z is a constant. In addition, the function $u\,(r)$ is the solution to the following second-order differential equation:

$$u'' + \frac{1}{r}\,u' - \frac{\left(\ell^2 - 2m\,MI\right)}{r^2}\,u + \left(2m\mathcal{E} - p_z^2\right)\,u = 0. \tag{6}$$

Our focus is on the bidimensional system; therefore, let us take $p_z = 0$. Furthermore, let us deal with s-waves from now on. The s-waves are defined when we consider $\ell = 0$. From this perspective, radial equation (6) becomes

$$u'' + \frac{1}{r}\,u' + \frac{2mMI}{r^2}\,u + 2m\mathcal{E}\,u = 0. \tag{7}$$

In the search for bound state solutions, let us consider $\mathcal{E} < 0$ [44–47] and define the parameter:

$$\zeta = \sqrt{-2m\mathcal{E}}. \tag{8}$$

In this way, radial equation (7) is rewritten in the form:

$$u'' + \frac{1}{r}\,u' + \frac{2mMI}{r^2}u - \zeta^2\,u = 0. \tag{9}$$

Let us define $\xi = \zeta\, r$, thus, radial equation (9) becomes

$$u'' + \frac{1}{\xi}\,u' + \frac{2mMI}{\xi^2}\,u - u = 0. \tag{10}$$

Second-order differential equation (10) is known in the literature as the Bessel differential equation [28,38–40].

Let us impose that $u(\xi) \to 0$ when $\xi \to \infty$, then, the solution to Eq. (10) is given in terms of the modified Bessel function of third kind of imaginary order [28,38,40,44–46]:

$$u(\xi) = c_1\, K_{i\sqrt{2mMI}}(\xi)\,, \tag{11}$$

where c_1 is a constant.

Next, we assume that the radius R_0 of the cylindrical wire is very small. This permits us to impose that the wave function vanishes at a short-distance cut-off $r_0 = R_0$ in agreement with Refs. [44–47]. Thereby, we can write $\xi_n = \zeta_n\, r_0 = \zeta_n\, R_0$, and thus, we have the boundary condition:

$$u(\xi_n) = c_1\, K_{i\sqrt{2mMI}}(\xi_n) = 0. \tag{12}$$

Obverse that, with this short-distance cut-off $r_0 = R$, we can assume that $\xi_n \ll 1$. As a consequence, the function $K_{i\sqrt{2mMI}}(\xi_n)$ can be written in the form [23,28,38,40]:

$$K_{i\sqrt{2mMI}}(\xi_n) \approx \sqrt{\frac{\pi}{\sqrt{2mMI}\,\sinh\left(\pi\sqrt{2mMI}\right)}} \times \sin\left(\sqrt{2mMI}\,\ln(\xi_n/2) + \delta\right), \tag{13}$$

where δ is a constant [23,28,38,40]. Hence, by substituting (13) into Eq. (12), we obtain

$$\xi_n = \frac{2}{e^{\delta/\sqrt{2mMI}}}\, e^{\nu\pi/\sqrt{2mMI}}, \tag{14}$$

where $\nu = 0, \pm1, \pm2, \pm3, \ldots$. With the aim of having the condition $\xi_n \ll 1$ satisfied, the possible values of the parameter ν are given by $\nu = -n < 0$, where $n = 1, 2, 3, 4, \ldots$ [47]. Thereby, after substituting $\xi_n = \zeta_n\, R_0$ and Eq. (8) into Eq. (14), the energy eigenvalues, for s-waves, are given by

$$\mathcal{E}_n = -\frac{2}{m\,R_0^2\, e^{2\delta/\sqrt{2mMI}}}\, e^{-2n\pi/\sqrt{2mMI}}, \tag{15}$$

where $n = 1, 2, 3, \ldots$ is the radial quantum number.

The eigenvalues of energy (15) stem from the interaction of the magnetic quadrupole moment of neutral particle (2) with azimuthal magnetic field (3). These eigenvalues of energy are obtained for s-waves when we impose that the radial wave function is well-behaved at $r \to \infty$ and vanishes at a short-distance cut-off $r_0 = R_0$. An interesting aspect of energy levels (15) is that $\mathcal{E}_n \to -\infty$ when $R_0 \to 0$. This aspect of the energy levels means that no ground state exists and corresponds to what Landau and Lifshitz [24] called as the fall of the particle to the centre. Hence, the energy levels are finite due to the presence of the short-distance cut-off $r_0 = R_0$. In other words, the short-distance cut-off $r_0 = R_0$ is responsible for the renormalization of the energy levels [47].

Another aspect of the spectrum of energy (15) is that it decreases exponentially with the radial quantum number n. Then, when $n \to \infty$ we have that $\mathcal{E}_{n\to\infty} \to 0$. With the increase in the quantum number n, the energy eigenvalues become closer to the energy level $\mathcal{E}_n = 0$. This yields a point of accumulation of energy levels in the energy level $\mathcal{E}_n = 0$. The ground state, by contrast, is determined by $n = 1$; thus, its energy is given by $\mathcal{E}_1 = -\frac{2}{m\,R_0^2\,e^{2\delta/\sqrt{2mMI}}}\, e^{-\frac{2\pi}{\sqrt{2mMI}}}$. Thereby, the energy eigenvalues for s-waves are defined in the range:

$$-\frac{2}{m\,R_0^2\, e^{2\delta/\sqrt{2mMI}}}\, e^{-\frac{2\pi}{\sqrt{2mMI}}} \leq \mathcal{E}_n \leq 0. \tag{16}$$

Observe that for other values of the quantum number ℓ, i.e. for $\ell \neq 0$, bound states associated with an inverse-square potential can be achieved if $2mMI > \ell^2$. In this case, we can define in radial equation (6) the parameter:

$$\alpha^2 = 2mMI - \ell^2, \tag{17}$$

and thus, with $p_z = 0$, radial equation (6) would be written in the form:

$$u'' + \frac{1}{r}u' + \frac{\alpha^2}{r^2}\,u + 2m\mathcal{E}\,u = 0. \tag{18}$$

Then, after following the steps from Eq. (8) to Eq. (14), we obtain the energy levels:

$$\mathcal{E}_{n,\,\ell} = -\frac{2}{m\,R_0^2\, e^{2\delta/\alpha}}\, e^{-2n\pi/\alpha}, \tag{19}$$

which are determined in terms of the parameter α. Observe that Eq. (15) is a particular case of Eq. (19). Besides, the eigenvalues of energy $\mathcal{E}_{n,\,\ell}$ are also defined in range (16).

Finally, a point to be observed with respect to the interaction of the magnetic quadrupole moment determined in Eq. (2) and the azimuthal magnetic field is that the eigenvalues of energy given in Eqs. (15) and (19) are no longer obtained if we have $\ell^2 > 2mMI$ or if the direction of the magnetic field is changed. We shall discuss this point in the following.

2.2 Repulsive inverse-square-type potential

Let us begin by observing that if $\ell^2 > 2mMI$, then, the parameter α in Eq. (17) becomes an imaginary number. Therefore, the solution to radial equation (6) or (18) that we have obtained previously is no longer valid. Another perspective is given if we change the direction of the azimuthal magnetic field:

$$\vec{B} = +\frac{I}{r}\,\hat{\varphi}, \tag{20}$$

where $I > 0$ has been defined in Eq. (3). In this case, with the magnetic quadrupole moment tensor defined in Eq. (2), the last term of the right-hand side of Eq. (1) becomes

$$V_{\text{eff}}\left(r\right) = -\vec{M}\cdot\vec{B} = +\frac{M\,I}{r^2}. \tag{21}$$

In this case, the interaction of the magnetic quadrupole moment with azimuthal magnetic field (20) gives rise to a repulsive inverse-square potential [41]. Thereby, with $p_z = 0$, radial equation (6) becomes

$$u'' + \frac{1}{r}\,u' - \frac{\left(\ell^2 + 2mMI\right)}{r^2}\,u + 2m\mathcal{E}\,u = 0. \tag{22}$$

Henceforth, we assume that $\mathcal{E} > 0$ and define the parameters:

$$\begin{aligned}\lambda^2 &= \ell^2 + 2mMI;\\ \beta^2 &= 2m\mathcal{E}.\end{aligned} \tag{23}$$

In this way, radial equation (22) becomes

$$u'' + \frac{1}{r}\,u' - \frac{\lambda^2}{r^2}\,u + \beta^2\,u = 0. \tag{24}$$

Hence, Eq. (24) is the Bessel differential equation [48], where its general solution is given by

$$u\left(r\right) = a_1\,J_{|\lambda|}\left(\beta\,r\right) + a_2\,N_{|\lambda|}\left(\beta\,r\right), \tag{25}$$

where a_1 and a_2 are constants, and $J_{|\lambda|}\left(\beta r\right)$ and $N_{|\lambda|}\left(\beta r\right)$ are the Bessel functions of first and second kinds, respectively [48,49].

Let us assume that the neutral particle is confined to a region between two cylindrical surfaces $r = R_0$ and $r = R_1$, where $R_1 > R_0$ are fixed [50]. This confinement gives two boundary conditions:

$$u\left(R_0\right) = 0;\;\; u\left(R_1\right) = 0. \tag{26}$$

Observe that the boundaries of the region $R_0 < r < R_1$ are impenetrable, which means the radial wave function vanishes at the boundaries and outside them. By substituting (25) in boundary conditions (26), we obtain:

$$J_{|\lambda|}\left(\beta\,R_0\right)\,N_{|\lambda|}\left(\beta\,R_1\right) - J_{|\lambda|}\left(\beta\,R_1\right)\,N_{|\lambda|}\left(\beta\,R_0\right) = 0. \tag{27}$$

Next, we analyse a particular case where we consider $\beta\,R_0 \gg 1$ and $\beta\,R_1 \gg 1$. For a fixed ζ, then, we can use Hankel's asymptotic expansion [48,50]:

$$\begin{aligned}J_{|\lambda|}\left(y_i\right) &\approx \sqrt{\frac{2}{\pi\,y_i}}\left[\cos\left(y_i - \frac{\lambda\,\pi}{2} - \frac{\pi}{4}\right) - \frac{4\,\lambda^2 - 1}{8y_i}\,\sin\left(y_i - \frac{\lambda\,\pi}{2} - \frac{\pi}{4}\right)\right];\\ N_{|\lambda|}\left(y_i\right) &\approx \sqrt{\frac{2}{\pi\,y_i}}\left[\sin\left(y_i - \frac{\lambda\,\pi}{2} - \frac{\pi}{4}\right) + \frac{4\,\lambda^2 - 1}{8y_i}\,\cos\left(y_i - \frac{\lambda\,\pi}{2} - \frac{\pi}{4}\right)\right],\end{aligned} \tag{28}$$

where we have labelled $y_i = \beta R_0,\, \beta R_1$. By substituting the functions given in Eq. (28) into Eq. (27), we can write [50]:

$$\beta^2 \approx \frac{\bar{n}^2\pi^2}{\left(R_1 - R_0\right)^2} + \frac{4\,\lambda^2 - 1}{4\,R_0 R_1}, \tag{29}$$

where $\bar{n} = 0, 1, 2, 3, \ldots$ describes the radial quantum number. Thus, by using Eq. (23) we obtain the energy levels $\mathcal{E}_{n,\,\ell}$:

$$\mathcal{E}_{\bar{n},\,\ell} \approx \frac{\bar{n}^2\pi^2}{2m\left(R_1 - R_0\right)^2} + \frac{4\,\lambda^2 - 1}{8m\,R_1 R_0}. \tag{30}$$

Hence, we have obtained energy levels (30) when the neutral particle is confined to a region between two cylindrical surfaces under the influence of the repulsive inverse-square potential. It is worth emphasizing that this repulsive inverse-square potential stems from the interaction of magnetic quadrupole moment (2) and azimuthal magnetic field (20). The repulsive inverse-square potential influences the energy levels through the presence of the parameter λ.

Let us proceed our discussion with the confinement of the neutral particle to a cylindrical surface. This kind of confinement is given by the limit $R_1 \to R_0$. However, by taking $R_1 \to R_0$ in Eq. (30), we have that $\mathcal{E}_{\bar{n},\,\ell} \to \infty$. According to Refs. [50–52], we can obtain a finite spectrum of energy by introducing an attractive scalar potential:

$$V_{\bar{n}} = -\frac{\bar{n}^2\pi^2}{2m\left(R_1 - R_0\right)^2}. \tag{31}$$

This attractive scalar potential is introduced in the region between the cylindrical surfaces. Therefore, with the introduction of this attractive scalar potential, the divergence that stems from the radial modes is removed when we take the limit $R_1 \to R_0$. Hence, from Eq. (30), we obtain the energy levels:

$$\mathcal{E}_\ell \approx \frac{\lambda^2}{2m\,R_0^2} - \frac{1}{8m\,R_0^2}. \tag{32}$$

Energy levels (32) stem from the confinement of the neutral particle to a cylindrical surface under the influence of the repulsive inverse-square potential. The influence of the repulsive inverse-square potential is also observed through the presence of the parameter λ in energy levels (32). Note that the last term of Eq. (32) corresponds to the Costa term [53]. It arises from the dynamics of a particle in a two-dimensional surface inside a three-dimensional space.

An interesting point to be observed is that if we have considered magnetic field (3); thus, the repulsive inverse-square potential is achieved when $\ell^2 > 2mMI$. In this case, the parameter α defined in Eq. (17) becomes an imaginary number, i.e. $\alpha = i\,\alpha'$, where $\alpha' = \ell^2 - 2mMI$. Thereby, energy levels (30) and (32) would be given in terms of the parameter α'. However, these energy levels are valid for $\ell \neq 0$. In the case $\ell = 0$, the parameter α' gives rise to the attractive inverse-square potential as well as the parameter α that we have seen in the previous section; thus, Eqs. (30) and (32) are no longer valid.

3 Conclusions

We have seen that the interaction of the magnetic quadrupole moment of a neutral particle with an azimuthal magnetic field can yield an effective scalar potential that plays the role of attractive or repulsive inverse-square potentials. In the case of the attractive inverse-square-type potential, we have obtained bound state solutions to the Schrödinger equation by imposing that the radial wave function is well-behaved at $r \to \infty$ and vanishes at a short-distance cut-off $r_0 = R_0$. Then, we have seen that the spectrum of energy is discrete and characterized by decreasing exponentially with the radial quantum number n. Besides, the spectrum of energy has a point of accumulation in the energy level $\mathcal{E}_{n,\,\ell} = 0$ as $n \to \infty$. Another aspect of the spectrum of energy is $\mathcal{E}_{n,\,\ell} \to -\infty$ when $R_0 \to 0$, which is called by Landau and Lifshitz [24] as the fall of the particle to the centre.

With respect to the repulsive inverse-square-type potential, we have analysed its influence on the confinement of the neutral particle to two cylindrical surfaces and also to a cylindrical surface. We have seen that the spectra of energy are discrete, but they are valid for $\ell \neq 0$. Otherwise, we do not have the repulsive inverse-square-type potential and the energy levels are no longer valid.

Recently, the Rényi entropy [54,55] has been studied from the bound states obtained in the non-central Kratzer potential [56]. Therefore, from the bound states obtained in this work, an interesting perspective of exploring the attractive inverse-square-type potential is in studies of the Rényi entropy [54,55]. Another perspective is in studies of the thermodynamics properties of quantum systems. In recent years, the thermodynamics properties have been studied from the bound states of a radial scalar power potential [57], a quantum ring [58], the harmonic oscillator in the presence of global monopole [59], the Landau-Aharonov-Casher quantization under the influence of a disclination [60] and the Dirac oscillator [61]. Thereby, from the discrete spectrum of energy obtained in Eqs. (15) and (19), the study of the thermodynamics properties of the attractive inverse-square-type potential proposed in the present work is an interesting topic for discussion.

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Author contributions

S.L.R.V. and K.B. conceived the mathematical model, interpreted the results and wrote the paper. S.L.R.V. made most of the calculations in consultation with K.B. All authors gave final approval for publication.

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