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Method of the Nonlinear Monotonic Tangent in the Solution of Transcendental Equations

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Abstract—The method of a curvilinear monotonic tangent in solving transcendental equations is proposed. In the denominator of the nonlinear term of the expression for the mentioned tangent, a regulating relation, which is a straight line with a control parameter, is used. An algorithm for solving the problem is described. Three examples of solving transcendental equations are performed. The efficiency of using the proposed method is shown.

Keywords: nonlinear tangent, transcendental equation, monotonicity, numerical solution, control parameter

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When solving transcendental equations, if the distance from the initial approximation to the desired root is sufficiently large, the M.K. Gavurin method of continuation with respect to the parameter [1] or the author's method of continuation in the space of parameters [2] is normally used. This is relevant in a situation where the function defining the equation has singularities.

We consider the transcendental equation

$$\Phi(x) = 0, \tag{1}$$

where x is a real variable. Let us assume that we know the neighborhood of the root of Eq. (1), which is embedded in the interval [0, 1]. In this case, Newton's tangent method or the secant method is usually used [3]. Together with these methods, since the 1970s [4, 5], the well-established secant method in combination with the inverse quadratic interpolation method (zeroin method) has been used in the practice of calculations. When using it, the solution is found as a result of the convergence of the corresponding iterative process.

In this paper, we propose a method for solving similar transcendental equations that does not use iteration when solving the problem. To analyze its effectiveness, we compare its complexity with calculations using the Newton tangent and zeroin methods.

When solving the transcendental equation (1), we assume that in the neighborhood $x = x_0$ there is a simple real root $x = x_*$. We assume that the function $\Phi(x)$ is monotonic on the interval $[x_0, x_*] \subset [0,1]$ of change of variable x. If this is not the case, then either we choose a different interval or we use the methods mentioned above [1, 2].

Following Newton's tangent method, for $x = x_0$, we calculate the value of the function $\Phi(x) = \Phi(x_0) = \Phi_0$. For definiteness, we first consider the case when $\Phi_0 > 0$. We find the value of the derivative $d\Phi/dx\big|_{x=x_0} = \Phi_0'$, as well as the value and sign of the second derivative $d^2\Phi(x)/dx^2\big|_{x=x_0} = \Phi_0''$.

We start with a special case where $\Phi_0' < 0$ and $\Phi_0'' > 0$. Next, we write the equation of the tangent to the curve $\Phi(x)$ at the point $x = x_0$. We have

$$y = \Phi_0 + \Phi_0'(x - x_0). \tag{2}$$

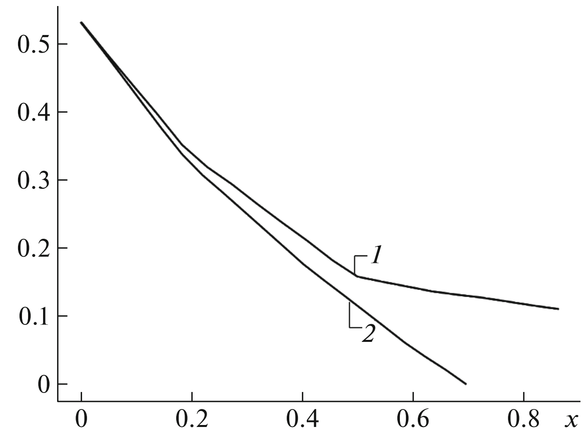

Fig. 1. Curve *2* of dependence $\Phi(x)$ and parabolas (curve *1*).

Tangent (2) is a straight line and it intersects the x-axis at the point $x = x_1$. We find the value of coordinate $x = x_1$ using the equation

$$\Phi_0 + \Phi_0'(x_1 - x_0) = 0. \tag{3}$$

After that, at this point, we calculate the value of the function $\Phi(x) = \Phi(x_1) = \Phi_1$. This function, just as Φ_0, will be positive and, most often, still far from the root $x = x_*$ of function $\Phi(x)$. Therefore, it is necessary to again use the equation of the tangent at the point $x = x_1$, etc. The value of the root of the function $\Phi(x)$ can be found more quickly if, instead of a straight-line tangent, we use a monotonic curvilinear tangent containing in its expression the ratio $(x - x_0)^2$ and the straight line $1 + k(x - x_0)$. In other words,

$$z = \Phi_0 + \Phi_0'(x - x_0) + 0.5\Phi_0'' \frac{(x - x_0)^2}{1 + k(x - x_0)}. \tag{4}$$

This function depends on parameter k. If $k = 0$, then expression (4) turns into a parabola; in this case, into to its lower branch. At $x = x_0$, it touches the curve $\Phi(x)$ (Fig. 1, upper curve) and, due to the presence of a positive term proportional to $(x - x_0)^2$, departs from the curve $\Phi(x)$, gradually (monotonically) moving away from it.

When $k > 0$, the denominator $1 + k(x - x_0)$ becomes greater than unity, increasing along with the growth of coordinate x. As a result, we can specify such a value $k > 0$, at which the curve $z(x)$, touching the curve $\Phi(x)$ at the point $x = x_0$, is then in the immediate vicinity of the curve $\Phi(x)$. With the growth of x, it can gradually move away from the curve $\Phi(x)$ or can intersect the curve $\Phi(x)$. When crossing the x-axis ($z = 0$), it turns out to be in a sufficiently close neighborhood from the point $x = x_*$, where $\Phi(x) = 0$.

Expression (4) will be used as monotonic on segments having a length within the interval $[x_0, x_*]$. To satisfy this condition, when $x = x_1$ we find the value of the function $\Phi(x) = \Phi(x_1)$ from interval $[x_0, x_*]$ and assume variable $z = z(x_1) = \Phi(x_1) + \delta$. Here, δ is a small number. Then, from equality (4) at $x = x_1$ and $z = \Phi(x_1) + \delta$, we find

$$k = (A - 1)/(x_1 - x_0). \tag{5}$$

Here,

$$A = 0.5\Phi_0'' \frac{(x_1 - x_0)^2}{\Phi(x_1) + \delta - \Phi_0 - \Phi_0'(x_1 - x_0)}. \tag{6}$$

For the given value k, equating $z = 0$, from expression (4) we obtain

$$a_1(x_2 - x_0)^2 + a_2(x_2 - x_0) + \Phi_0 = 0, \tag{7}$$

where $a_1 = k\Phi_0' + 0.5\Phi_0''$; $a_2 = k\Phi_0 + \Phi_0'$.

If $a_1 < 0$, then, solving Eq. (7), we will have

$$x_2 = \begin{cases} x_0 - a_2/(2a_1) + \sqrt{a_2^2/(4a_1^2) - \Phi_0/a_1} & \text{at} \quad a_2 < 0, \\ x_0 - a_2/(2a_1) - \sqrt{a_2^2/(4a_1^2) - \Phi_0/a_1} & \text{at} \quad a_2 > 0. \end{cases} \tag{8}$$

If $a_1 = 0$, then instead of Eq. (7) we have a linear equation

$$a_2(x_2 - x_0) + \Phi_0 = 0. \tag{9}$$

We assume that $a_2 < 0$ at $a_1 = 0$. If in Eq. (7) $a_2 = 0$, then

$$x_2 = x_0 + \sqrt{-\Phi_0/a_1} \quad \text{and} \quad a_1 < 0. \tag{10}$$

We consider that the found value x_2 will be close to the root of Eq. (1) $x = x_*$. At $x = x_2$, we find the value of the function $\Phi(\tilde{x}_1)$. Through the three points $(x_0, \Phi(x_0))$, $(x_1, \Phi(x_1))$, and $(x_2, \Phi(x_2))$, we draw a parabola

$$\tilde{z} = \Phi_0 + \alpha(x - x_0) + \beta(x - x_0)^2, \tag{11}$$

where

$$\alpha = 2\frac{\Phi(x_1) - \Phi(x_0)}{x_1 - x_0} - \frac{\Phi(x_2) - \Phi(x_0)}{x_2 - x_0}; \tag{12}$$

$$\beta = \frac{1}{x_2 - x_1}\left[\frac{\Phi(x_2) - \Phi(x_0)}{x_2 - x_0} - \frac{\Phi(x_1) - \Phi(x_0)}{x_1 - x_0}\right]. \tag{13}$$

At $\tilde{z} = 0$ we get a quadratic equation

$$\beta(x_* - x_0)^2 + \alpha(x_* - x_0) + \Phi_0 = 0 \tag{14}$$

as a result of quadratic interpolation if $\Phi(x_2) < 0$ or quadratic extrapolation if $\Phi(x_2) > 0$.

Solving Eq. (14) with $\alpha < 0$ and $\beta > 0$, we get

$$x_* = x_0 - \alpha/(2\beta) - \sqrt{(\alpha/(2\beta))^2 - \Phi_0/\beta}. \tag{15}$$

We consider an example of the solution of the problem. To do this, we use the transcendental function

$$\Phi(x) = a/(1 + x) - (1 - \exp(-bx)) \tag{16}$$

when $a = b = 0.5$. At $x = x_0 = 0$, the function $\Phi(0) = 0.5$; i.e., it is positive. Its first term $a/(1 + x)$ only decreases when x increases. Its second term $1 - \exp(-bx)$ only grows when x increases. Hence, the considered function $\Phi(x)$, depending on x, has a single root. The dependence of $\Phi(x)$ on x is shown in Fig. 1 (curve *2*). We see that this is a monotonic function in x, vanishing at $x = x_* = 0.69773$. Its first derivative is expressed as

$$d\Phi(x)/dx = -a/(1 + x)^2 - b\exp(-bx). \tag{17}$$

Derivative (17) is a function decreasing in absolute value and having negative values (Fig. 2, lower curve), and $\Phi_0' = -1$ at $x = 0$.

The second derivative of the function $\Phi(x)$ in x has the expression

$$d^2\Phi(x)/dx^2 = 2a/(1 + x)^3 + b^2\exp(-bx) \tag{18}$$

and is a modulo decreasing function of coordinate x. Its graph (upper curve) is shown in Fig. 2. We see that, on the interval of the change in coordinate x from $x = 0$ to $x = x_*$, it more than doubles. At $x = 0$, $\Phi_0'' = 1.25$.

Under the conditions of the example under consideration, the equation for the curvilinear tangent can be written as

$$z = 0.5 - x + 0.625x^2/(1 + kx). \tag{19}$$

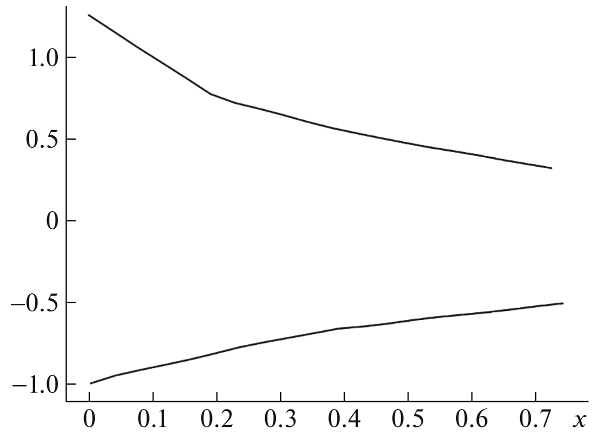

Fig. 2. The first (lower curve) and second derivatives for the function $\Phi(x)$.

If parameter $k = 0$, then expression (19) for the curvilinear tangent as a function of coordinate x (Fig. 1, curve *1*) turns into the lower branch of the parabola, which, when $x = 0$ touches the curve $\Phi(x)$ (Fig. 1, curve *2*). However, at $x > 0$, it deviates from this curve and as x increases, it turns out to be higher than the curve $\Phi(x)$, gradually moving away from it. At $x = 0.8$, this parabola passes through a minimum at a distance of 0.1 above the x axis and then only grows.

At $x = x_1 = 0.6$, $\Phi(x) = 0.0533$. We take the value $\delta = 0.001$. Then, using expressions (6) and (5), $k = 0.763664$ and curve $z(x)$ turns out to be higher than the curve $\Phi(x)$, almost merging with it. Table 1 shows the results of calculating the values of the function $\Phi(x)$ and curvilinear tangent $z(x)$. Table 1 shows that the curve $z(x)$ with the growth of x gradually moves upward from the curve $\Phi(x)$. At $x = x_1 = 0.6$, value z turns out to be at a distance of 0.001 above the curve $\Phi(x)$. As a result, the curve $z(x)$ intersects the x axis at $x = x_2 = 0.699181$, where the function $\Phi(x) = \Phi(x_2) = -0.000763316$. Using quadratic interpolation, by formula (15) with coefficients $\alpha = -0.915612$, $\beta = 0.285187$, we find the value of the root $x = x_* = 0.69773$ of the function $\Phi(x)$.

Now we find the root of the function $\Phi(x)$ by the Newton tangent method using the tangent Eq. (2). The calculation results are shown in Table 2.

We see that after the first approximation the solution is still far from the root of the function $\Phi(x)$. The second approximation is also not sufficient to obtain the desired result. However, the third approximation allows us to get a result that differs from the root in the 4th significant figure, or by 0.011%. Thus, Newton's method in this case allows successfully solving the problem, requiring, as in this example, three approximations.

We now solve the same problem using the zeroin algorithm. To do this, we draw a secant, that is, a straight line

$$y = \Phi_0 + (\Phi_1 - \Phi_0)x/x_1 \tag{20}$$

Table 1. Changes in function $\Phi(x)$ and curvilinear tangent $z(x)$ at $k = 0.763664$ depending on x

x	0	0.2	0.4	0.6	0.69773/0.69918
$\Phi(x)$	0.5	0.3215	0.17590	0.0533	0
$z(x)$	0.5	0.3217	0.17660	0.0536	0

Table 2. Finding the root of the function $\Phi(x)$ by Newton's tangent method

x	0.5	0.6833	0.69765
$\Phi(x)$	0.1121	0.007612	0

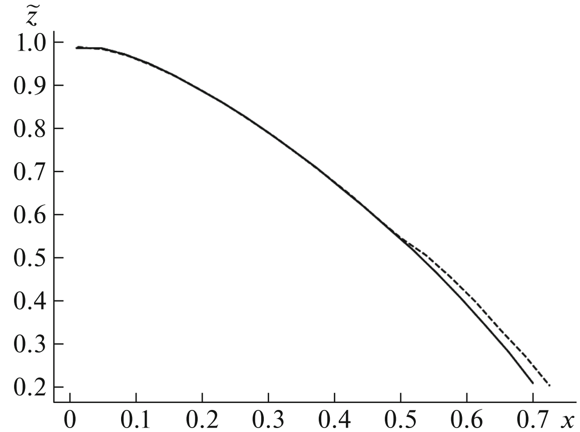

Fig. 3. Dependency curve $\tilde{\Phi}(x)$ (dotted line) and parabola as functions of x.

through the points $x = 0$ and $x = 0.6$. This line will intersect the x axis at $x = x_2 = 0.6716$ and $\Phi(x) = \Phi(x_2) = 0.01388$. Using Eq. (11), we obtain the expression

$$x_3 = x_0 - \alpha/(2\beta) - \sqrt{(\alpha/(2\beta))^2 - \Phi_0/\beta} \tag{21}$$

to determine the value of the coordinate $x = x_3$ of the point of intersection of parabola (11) with the abscissa axis. Under the conditions of the example under consideration, coefficient $\alpha = -0.91783$ and coefficient $\beta = 0.288785$. Then the value of the coordinate of point $x = x_3$ turns out to be 0.69810. This point differs by just 0.00037 or 0.053% from the value of the root $x = x_* = 0.69773$ of the function $\Phi(x)$, but it is not known to the program. Therefore, the zeroin program will perform one more approximation and, having provided the required accuracy of the convergence of the iterative process, will give the desired solution.

As a result, the considered model equation is solved by Newton's tangent method using three approximations; by the zeroin method, by using two approximations; and by the proposed method, by using one approximation.

Now consider the case when, for a positive function $\tilde{\Phi}(x)$ (Fig. 3, dotted line), its first and second derivatives are negative (Fig. 4, curves *1* and *2*, respectively). For a curvilinear tangent, we again use expression (4); and for linear tangents, (2). We also use equality (6) to determine the point of intersection $\tilde{x}_1$ of the curvilinear tangent with the abscissa axis and equality (9) to determine the value of the simple root of Eq. (14).

We consider the transcendental equation

$$\tilde{\Phi}(x) = 0, \tag{22}$$

which is valid for $x = x_*$. Furthermore,

$$d\tilde{\Phi}(x)/dx\big|_{x=x_0} = \tilde{\Phi}'_0 < 0 \quad \text{and} \quad d^2\tilde{\Phi}(x)/dx^2\big|_{x=x_0} = \tilde{\Phi}''_0 < 0.$$

As an example, we take the transcendental function

$$\tilde{\Phi}(x) = a\cos((\pi/2)x) - (\exp(bx) - 1), \tag{23}$$

where $a = 1$ and $b = 0.45$. The first term in (23) on the interval $[0, -1]$ of the change in variable x only decreases from the value of multiplier a at $x = 0$ and to zero at $x = 1$. Its second term on the same interval of the change in variable x only grows starting from zero at $x = 0$. This means that on the interval $[0, -1]$ they can only intersect once.

The first and second derivative functions $\tilde{\Phi}(x)$ have the following expressions:

$$d\tilde{\Phi}(x)/dx = \tilde{\Phi}'(x) = -a(\pi/2)\sin((\pi/2)x) - b\exp(bx); \tag{24}$$

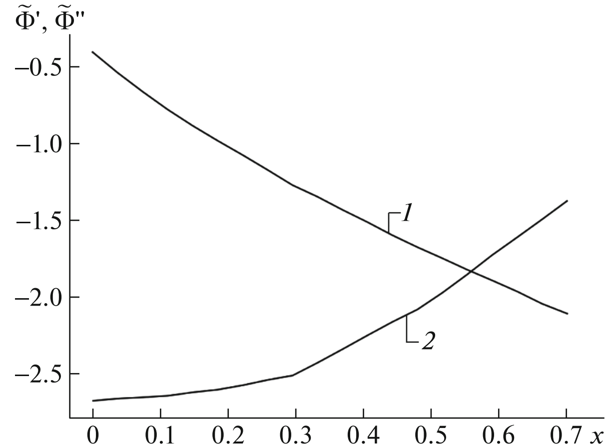

Fig. 4. First and second derivatives of $\tilde{\Phi}(x)$ depending on x.

$$d^2\tilde{\Phi}(x)\big/dx^2 = \tilde{\Phi}''(x) = -a(\pi/2)^2\cos((\pi/2)x) - b^2\exp(bx). \tag{25}$$

The curve of the change in the function $\tilde{\Phi}(x)$ when using the above-mentioned values of constants a and b is shown in Fig. 3; the upper curve. We see that the curve $\tilde{\Phi}(x)$ has a fairly rapidly increasing slope. This is consistent with the noticeable increase in the modulus of the first derivative $\tilde{\Phi}'(x)$ (Fig. 4, curve *1*). The curve of the second derivative $\tilde{\Phi}''(x)$ is shown in Fig. 4 (curve *2*).

Taking into account the named quantities for constants a and b, the curvilinear tangent expression $\tilde{z}(x)$ will take the form

$$\tilde{z}(x) = 1 - 0.45x - 1.334951x^2/(1 + kx). \tag{26}$$

Figure 3 (lower curve) shows a plot of the curvilinear tangent $\tilde{z}(x)$ to function $\tilde{\Phi}(x)$ when parameter $k = 0$. This is the upper branch of the parabola. It passes below the curve $\tilde{\Phi}'(x)$, but due to the steep decrease in the curve $\tilde{\Phi}'(x)$ it doesn't deviate much from it. As a result, the variable $\tilde{z}(x)$ at $k = 0$ intersects the x axis at the point $x = x_1 = 0.713213$, and it corresponds to the value of the function $\tilde{\Phi}(x) = \tilde{\Phi}(x_1) = 0.0569703$. We see that $\tilde{\Phi}(x)$ is sufficiently small and its root $x = x_* = 0.740966$ can be found without using parameter $k > 0$. Indeed, let us draw the secant

$$\tilde{y} = \tilde{\Phi}(0) + [\tilde{\Phi}(x_1) - \tilde{\Phi}(0)]x/x_1 \tag{27}$$

through points $x = 0$ and $x = x_1$. It will cross the x axis at point $x = x_2 = 0.756299$, where the function $\tilde{\Phi}(x_2) = -0.0318936$. As the third point, we take the arithmetic mean of the numbers x_1 and x_2. We get $x_2 = 0.734756$. It matches $\tilde{\Phi}(x_2) = 0.0128352$. Then the former x_2 will be $x_3 = 0.756299$ and $\tilde{\Phi}(x_3) = -0.0318936$.

We draw parabola (11) with coefficients α and β determined by expressions (12) and (13) through points $(x_1, \tilde{\Phi}(x_1))$, $(x_2, \tilde{\Phi}(x_2))$, and $(x_3, \tilde{\Phi}(x_3))$. At $\tilde{z}(x) = 0$ we get the quadratic equation (14) to find the root $x = x_*$. Solving Eq. (14) with $\alpha = -2.0348976$ and $\beta = -0.6397686$, and using the plus sign in front of the square root, we get $x = x_* = 0.740966$.

Thus, for a large modulus of the first derivative, the problem can be successfully solved for the parameter value $k = 0$.

If, however, we use $k > 0$, its value should be close to zero. Indeed, from (5), (6) with $x_1 = 0.6$ and $\delta = 0.01$, we get $k = 0.1447465$. In this case, the curve $\tilde{z}(x)$ is above the curve $\tilde{\Phi}(x)$; see Table 3.

We see that at $x = 0.2$, the difference between $\tilde{\Phi}(x)$ and $\tilde{z}(x)$ is one unit in the 3rd significant digit. With the growth of x, it also increases, and with $x = 0.6$ it is 0.01. At $x > 0.6$ it decreases, but remains above the

Table 3. Changed in $\tilde{\Phi}(x)$ and curvilinear tangent $\tilde{z}(x)$ at $k = 0.1447465$ depending on x

x	0	0.2	0.4	0.6	0.7	0.740966/0.743117
$\tilde{\Phi}(x)$	1	0.85688	0.61180	0.27782	0.0837312	0
$\tilde{z}(x)$	1	0.85810	0.61810	0.28782	0.0910541	0

curve $\tilde{\Phi}(x)$ until it intersects the x axis. Function $\tilde{z}(x)$ at $k = 0.1447465$ crosses the x axis at $x = x_2 = 0.743117$. For this value of coordinate x, the function $\tilde{\Phi}(x) = -0.0044559$. We draw a parabola

$$\tilde{\tilde{y}} = 1 + ax + bx^2 \tag{28}$$

to find the root of the function $\tilde{\Phi}(x)$ through points 0, x_1, and x_2 on the curve $\tilde{\Phi}(x)$. Its coefficients are $a = -0.5829692$ and $b = -1.0344401$. Then at $\tilde{\tilde{y}} = 0$, from the quadratic equation

$$bx_3^2 + ax_3 + 1 = 0, \tag{29}$$

we find the value of the coordinate $x = x_3 = 0.7410134$, which differs from the root only in the 5th significant digit $x = x_* = 0.740966$ of the function $\tilde{\Phi}(x)$.

We solve this problem using the zeroin program. We draw secant (27), in which $\tilde{\Phi}(0) = 1$ and $\tilde{\Phi}(x_1) = 0.27782$, through the points on the curve $\tilde{\Phi}(x)$ corresponding to coordinates $x = 0$ and $x = x_1 = 0.6$. This secant intersects the x axis at $x = x_2 = 0.8308178$. In this case, $x = x_2$ of the function $\tilde{\Phi}(x_2) = -0.1907115$. We draw parabola (28) through the points $(0, \tilde{\Phi}(0))$, $(x_1, \tilde{\Phi}(x_1))$, and $(x_2, \tilde{\Phi}(x_2))$. Its coefficients are $a = -0.6069373$ and $b = -0.9944930$. At $\tilde{\tilde{y}} = 0$ we get the quadratic equation (29). Solving (29), we find $x = x_3 = 0.7430177$. This value corresponds to the function $\tilde{\Phi}(x_3) = -0.0042501$. We draw the secant

$$\bar{y} = \tilde{\Phi}(x_1) + [\tilde{\Phi}(x_3) - \tilde{\Phi}(x_1)](x - x_1)/(x_3 - x_1) \tag{30}$$

through the points $(x_1, \tilde{\Phi}(x_1))$ and $(x_3, \tilde{\Phi}(x_3))$. At $\bar{y} = 0$ we get $x = x_4 = 0.7408628$. We determine the value $\tilde{\Phi}(x_4) = 0.0002146$ for this coordinate $x = x_4$. We draw the parabola

$$\tilde{\tilde{y}} = \tilde{\Phi}(x_1) + \bar{a}(x - x_1) + \bar{b}(x - x_1)^2 \tag{31}$$

through points $(x_1, \tilde{\Phi}(x_1))$, $(x_3, \tilde{\Phi}(x_3))$, and $(x_4, \tilde{\Phi}(x_4))$. Its coefficients are $\bar{a} = -1.87116948$ and $\bar{b} = -0.70693962$. At $\tilde{\tilde{y}} = 0$ we get the quadratic equation

$$\bar{b}(x_5 - x_1)^2 + \bar{a}(x_5 - x_1) + \tilde{\Phi}(x_1) = 0 \tag{32}$$

to determine the value of coordinate $x = x_5$. Solving Eq. (32), we obtain coordinate $x = x_5 = 0.740966$. The value of the function $\tilde{\Phi}(x_5)$ is practically zero. The difference between coordinates $x = x_4$ and $x = x_5$ is 0.0139%. If the given accuracy of convergence in terms of the value of the determined root is sufficient, the problem will be considered solved. If not, then another approximation is required.

As we can see, the problem is successfully solved by the zeroin method and the computational process converges in two approximations.

Using the Newton tangent method, it is inconvenient to solve this problem, starting from $x = 0$ since the tangent crosses the x axis at $x > 2.2$, i.e., too far from $x = x_* = 0.740966$, when the monotonicity condition for $\cos(\pi/2)x$ will be violated. To use Newton's tangent method in this case, it is necessary to solve the problem starting approximately from $x = x_0 = 0.6$ using the appropriate $x = x_0$ values of $\tilde{\Phi}(x_0)$ and $\tilde{\Phi}'(x_0)$. Therefore, we will not solve this problem using the Newton tangent method. This, in this case, we solve the problem using the zeroin method using two approximations; and using the proposed method, using one approximation.

However, in addition to the two considered cases, there are two others, in which the left parts of the equations being solved are negative, and their first derivatives are positive (Fig. 5, curves *1* and *2*). In the former (the second derivative is negative), the algorithm of the first task is realized, and in the second one, the algorithm of the second task is realized. Therefore, the remaining two cases can not be specially ana-

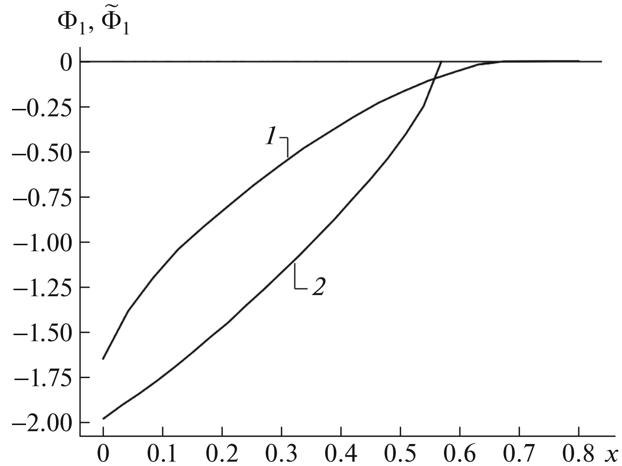

Fig. 5. Left parts of $\Phi_1(x)$ and $\tilde{\Phi}_1(x)$ of transcendental equations if $\Phi_1(x) < 0$ and $\tilde{\Phi}_1(x) < 0$.

lyzed. Note that when solving the second problem, the growth rate of the function $\tilde{\Phi}_1(x)$ (curve *2* in Fig. 5) can be so large that curve (4) at $k = 0$ (parabola) will lag behind it. In this situation, the effect of the nonlinear term in equality (4) should not be reduced, as was the case in the examples considered, but, on the contrary, it should be increased. Such an increase in the effect of the nonlinear term in equality (4) can be carried out using parameter $k < 0$.

As an example, we consider the equation

$$\Phi_2(x) = \exp(bx) - 2 = 0, \tag{33}$$

which we will solve as a transcendental equation on the interval [0, 1]. We take $b = 1$ in Eq. (19). $\Phi_2(0) = -1$ at $x = 0$. Its first and second derivatives are

$$d\Phi_2(x)/dx = b\exp(bx) \quad \text{and} \quad d^2\Phi_2(x)/dx^2 = b^2\exp(bx) \tag{34}$$

and, under the conditions of the considered example, they will be equal to each other at $x = 0$:

$$d\Phi(0)/dx = d^2\Phi_2(0)/dx^2 = 1.$$

The nonlinear monotonic tangent to the function $\Phi_2(x)$ at the point $x = 0$ will be written in the form

$$z_2(x) = -1 + x + 0.5x^2/(1 + kx). \tag{35}$$

The dependences of functions $\Phi_2(x)$ and $z_2(x)$ at $k = 0$ (lower curve) are shown in Fig. 6. We see that the curve for $z_2(x)$ at $k = 0$ lags the curve for $\Phi_2(x)$. To increase the intensity of its increase, we decrease the modulus of the value of the function $\Phi_2(0.6)$ at $x = x_1 = 0.6$, equal to –0.1779, by 0.002 and equate it to the obtained value $z_2(0.6) = -0.1759$. Using equality (5), we find the value $k = k_1 = -0.3279786$. The dependence $z_2(x)$ with this value $k = k_1$ is given in Table 4. The values of the function $\Phi_2(x)$ are also given there for comparison.

We see that the curve $z_2(x)$ at $k = k_1$ runs slightly above the curve $\Phi_2(x)$ and intersects the x axis by a value of 0.002 more to the left than the function $\Phi_2(x)$, at the point $x_2 = 0.69115$. The value of the function $\Phi_2(x_2) = -0.0039968$. After that, we draw the parabola

$$\tilde{\Phi}_2 = \Phi_2(0) + \tilde{a}x + \tilde{b}x^2, \tag{36}$$

Table 4. Change in function $\Phi_2(x)$ and nonlinear tangent $z_2(x)$ at $k_1 = -0.3005464$ depending on variable x

x	0	0.2	0.4	0.6	0.69315/0.69115
$\Phi_2(x)$	−1	−0.7786	−0.5082	−0.1779	0
$z_2(x)$	−1	−0.7786	−0.5079	−0.1759	0

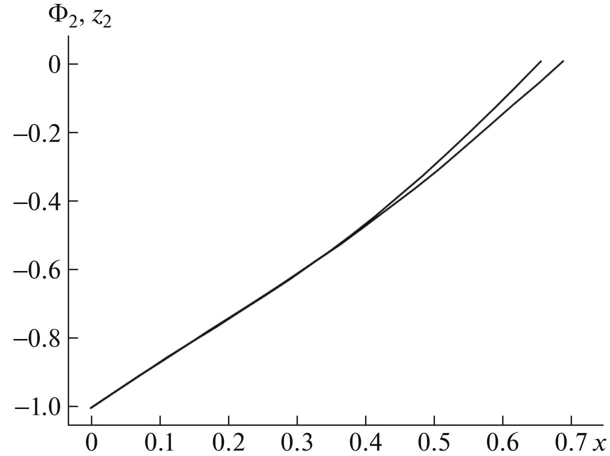

Fig. 6. Dependences $\Phi_2(x)$ and $z_2(x)$ at $k = 0$ (lower curve) on x.

where $\tilde{a} = 0.903606$ and $\tilde{b} = 0.7776533$, through the points $(0, \Phi_2(0))$, $(x_1, \Phi_2(x_1))$, and $(x_2, \Phi_2(x_2))$. At $\tilde{\Phi}_2(x_3) = 0$ we get a quadratic equation

$$\tilde{b}x_3^2 + \tilde{a}x_3 - 1 = 0. \quad (37)$$

From (37) it follows that $x_3 = 0.69315$; i.e., x_3 coincides with the root of Eq. (33).

CONCLUSIONS

In this study, it is proposed to use a curvilinear monotonic curve as a tangent to solve the transcendental equation. To control the impact on the value of the nonlinear term in the expression for the nonlinear tangent, parameter k (positive or negative) is used. The examples show that the curvilinear monotonic tangent method is equally effective for both positive and negative second derivatives of the left-hand sides of the transcendental equations being solved. For completeness of the analysis, the proposed method is compared with Newton's tangent method and the zeroin method. The comparison shows that the proposed method is at least as efficient as the latter two methods.

CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.

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