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Structural Break Detection in Autoregressional Conditional Heteroskedasticity Model: Case of Student's Distribution

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Abstract—Two methods of structural break detection in a piecewise generalized model of autoregressive conditional heteroscedasticity are considered. The first method is based on Kolmogorov–Smirnov statistics and is called the KS method. The second one is based on the cumulative sums and is called the KL method. In this paper, the KS and KL methods are compared under the assumption of Student's conditional distribution of random errors. The results of our Monte Carlo experiments are as follows: the KL method is inferior to the KS method both in terms of the average probability of errors of the first type and in terms of the average power of detecting a structural break.

Keywords: GARCH-t, t distribution, Student's distribution, volatility, structural breaks, structural shifts, ICSS, CUSUM

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1. INTRODUCTION

The authors of [1] proposed a new method for detecting a structural break in the generalized autoregressive conditional heteroscedasticity model or GARCH(1,1) model. The method is based on the Kolmogorov–Smirnov statistics and is called the KS method. This method is compared with the well-known method based on cumulative sums (hereinafter, the KL method) [2] under the assumption of conditionally normally distributed random errors. In contrast to [1], this paper considers the situation when random errors in the GARCH(1,1) model have Student's conditional distribution rather than a normal conditional distribution. For the first time, a generalized model of autoregressive conditional heteroscedasticity with Student's conditional distribution of random errors (without the presence of structural breaks) was proposed in [3] and called the GARCH(1,1)-t model. This model is interesting from a practical point of view, since the distribution of the returns of many financial instruments has more *heavy tails* and *higher peaks* than the usual GARCH(1,1) model with a normal conditional distribution of random errors can provide [3].

In this study, we consider the problem of detecting one structural break for a piecewise given GARCH(1,1)-*t* model, which is described as follows. We assume $\tau \in \{1, \dots, T\}$ is a possible moment of a structural break in the time series $Y = (Y_t)_{t=1}^T$, which separates the time series Y into two identical segments:

$$\begin{cases} Y_t = \varepsilon_t, \quad \varepsilon_t = \sigma_t\xi_t, \quad \sigma_t^2 = \omega_1 + \delta_1\sigma_{t-1}^2 + \gamma_1\varepsilon_{t-1}^2, \quad t \in [1; \tau - 1], \\ Y_t = \varepsilon_t, \quad \varepsilon_t = \sigma_t\xi_t, \quad \sigma_t^2 = \omega_2 + \delta_2\sigma_{t-1}^2 + \gamma_2\varepsilon_{t-1}^2, \quad t \in [\tau; T], \end{cases} \tag{1}$$

where $\tau \in \{1, \dots, T\}$ and $\theta_j := (\omega_j, \delta_j, \gamma_j, \nu_j)$, $j = 1, 2$ are unknown model parameters. It is assumed that vectors θ_1 and θ_2 belong to the set of allowed values

$$\Theta = \{(\omega, \delta, \gamma, \nu) : \omega > 0,\ \delta \ge 0,\ \gamma \ge 0,\ \delta + \gamma < 1,\ \nu > 2\},$$

$\sigma_0, \xi_0, \xi_1, \ldots, \xi_T$ are independent random variables, and σ_0 is a nonnegative random variable with $E[\sigma_0^2] = \omega_1/(1-\delta_1-\gamma_1)$, $\varepsilon_0 = \sigma_0\xi_0$. Values $\xi_0, \xi_1, \ldots, \xi_T$ have Student's standardized distribution with ν degrees of freedom, i.e.,

$$\xi_t = \eta_t \Big/ \sqrt{\nu/(\nu-2)}, \quad t = 1, \ldots, T,$$

where $\eta_1, \ldots, \eta_T$ are independent random variables having Student's distribution with ν degrees of freedom ($\nu > 2$, where ν is not necessarily an integer). The distribution density of the random variables ξ_t is given by the formula

$$f_{\xi_t}(x) = \frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)\sqrt{(\nu-2)\pi}}\left(1 + \frac{x^2}{\nu-2}\right)^{(\nu+1)/2}, \quad x \in \mathbb{R},$$

where $\Gamma(a) := \int_0^\infty t^{a-1}e^{-t}dt$ and $a > 0$ is Euler's gamma function. Note that the random variables ξ_t are normalized to $\sqrt{\nu/(\nu-2)}$ so that the values ξ_t have unit variance.

The task is to test the hypothesis $H_0 : \theta_1 = \theta_2$ on the absence of a structural break against the hypothesis $H_1 : \theta_1 \neq \theta_2$ on the presence of a structural break at the moment of time $t = \tau$. Some approaches to solve this problem can be found, for example, in [2, 4–11]. A detailed review of many methods for detecting structural breaks for piecewise GARCH models is contained in [12].

In the next section, the situations of the occurrence of structural breaks in GARCH-t models are simulated using the Monte Carlo statistical test method. In order for these tests to be closer to reality when generating GARCH-t processes, the parameters of GARCH-t models were used, which were previously estimated from the time series of the returns from 28 international indices. We carried out a series of four Monte Carlo experiments, each of which consisted of 28 calculations, and each calculation consisted of 5000 simulations. Using these experiments, we compared the average probabilities of errors of the first kind and the average powers of detecting structural breaks that the KS and KL methods have in the case of Student's conditional distribution of random errors.

2. NUMERICAL EXPERIMENTS

This section describes Monte Carlo experiments in which structural volatility shifts were modeled using piecewise GARCH(1,1)-*t* models. In order for the conditions of the experiments to be sufficiently close to the real conditions, the prices are taken from $N = 28$ global financial indices with the following tickers: GSPC, DJI, IXIC, NYA, XAX, BUK100P, RUT, VIX, GDAXI, FCHI, STOXX50E, N100, BFX, IMOEX.ME, N225, HSI, AXJO, AORD, BSESN, JKSE, KS11, TWII, GSPTSE, BVSP, MXX, IPSA, MERV, and TA125.TA. The data are taken from Yahoo! Finance (https://finance.yahoo.com/world-indices) for the period from January 1, 2011 to December 31, 2013. This time period was chosen due to its relative stability for financial markets. The selected index prices have been converted to log returns. According to these data, using GARCH(1,1)-t model for each index, the vector of parameters $\theta = (\omega, \delta, \gamma, \nu)$ was estimated. Thus, 28 parameter vectors were obtained $\theta^{(j)} = (\omega^{(j)}, \delta^{(j)}, \gamma^{(j)}, \nu^{(j)})$, $j = 1, \ldots, N$, each of which corresponds to one of these financial indices. For the KL method, the calculations were performed at a significance level of 1% with the function $q(N) = \sqrt{N}$. The parameters of the KS method were chosen as follows: $\Delta_1 = 4$, $\Delta_2 = 400$, and $\alpha_{KS} = 0.00001$.

It should be noted that in some previous works, for example, in [14] or in [6], when performing the numerical experiments, the parameter vectors were set without reference to the real data. From the point of view of parameter interpretation, such an approach can lead to ambiguous results. Thus, in [14] the values were chosen as one of the parameter vectors $\theta = (\omega, \delta, \gamma) = (0.05, 0.4, 0.3)$, which corresponds to the expected annual volatility of over 600% and is extremely rare in practice. In addition, the sampling error of the first kind and the power strongly depend on the values of the parameters.

Numerical experiment 1. The case of the absence of a structural shift. Calculation of the average probabilities of errors of the first kind.

• For each security and parameter vector $\theta = (\omega, \delta, \gamma, \nu)$, according to model (1) provided there is no structural break ($\omega_1 = \omega_2 = \omega$, $\delta_1 = \delta_2 = \delta$, $\gamma_1 = \gamma_2 = \gamma$, and $\nu_1 = \nu_2 = \nu$), a series of $S = 5000$ random process generations $\left(Y_t^{(j)}\right)_{t=1}^T$ were performed for $T = 2000$.

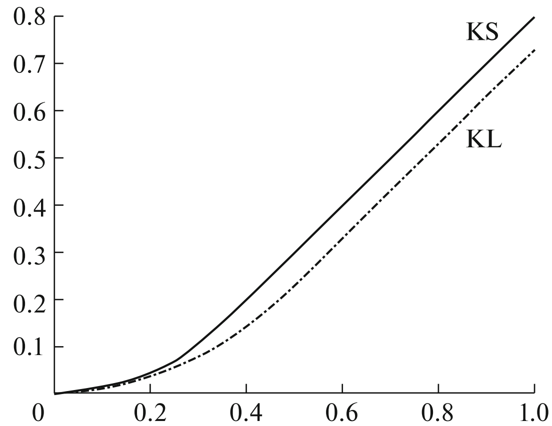

Fig. 1. Sample cumulative functions of distributions of empirical error of the first kind.

• Using the results of these simulations, the average probabilities of errors of the first kind $\alpha_{KS}^{(j)}$ and $\alpha_{KL}^{(j)}$ are calculated as the proportion of cases in which the KS and KL methods erroneously indicated the presence of a structural shift.

• Based on values $\alpha_{KS}^{(j)}$ and $\alpha_{KL}^{(j)}$, the average probabilities of errors of the first kind of the KS and KL methods are calculated:

$$\bar{\alpha}_{KS} = \frac{1}{N}\sum_{j=1}^{N}\alpha_{KS}^{(j)}, \quad \bar{\alpha}_{KL} = \frac{1}{N}\sum_{j=1}^{N}\alpha_{KL}^{(j)}.$$

• The calculation results are shown in Table 1.

• Sample integral distribution functions of errors of the first kind are calculated and their graphs are shown in Fig. 1.

As can be seen from Table 1, the KL method is somewhat inferior to the KS method, since the KL method has a higher average probability of a type I error. In addition, according to Fig. 1, they turned out to be in relation to the stochastic dominance of the second order $F_{KL} \succ_2 F_{KS}$.

Table 1. Average probabilities of type I error

$\bar{\alpha}_{KS}$	$\bar{\alpha}_{KL}$
0.21	0.27

Numerical experiments 2–4 used three types of structural breaks. Within their framework, the average powers of the methods were calculated. In each experiment, a structural break was modeled in model (1) in one of parameters ω, δ, or γ.

• For each security and parameter vector $\theta = (\omega, \delta, \gamma, \nu)$ according to model (1) at $T = 2000$ and in the presence of a structural break at time $\tau = 1001$, a series of $S = 5000$ generations of the random process $\left(Y_t^{(j)}\right)_{t=1}^{T}$ was performed.

—In experiment 2, a spike in parameter ω upward by a factor of five was simulated, i.e., $\omega_1 = \omega$, $\omega_2 = 5\omega$, $\delta_1 = \delta_2 = \delta$ and $\gamma_1 = \gamma_2 = \gamma$, $\nu_1 = \nu_2 = \nu$;

—In experiment 3, a spike in parameter δ downward by a factor of 0.1 was simulated, i.e., $\omega_1 = \omega_2 = \omega$, $\delta_1 = \delta$, $\delta_2 = \delta - 0.1$ and $\gamma_1 = \gamma_2 = \gamma$, $\nu_1 = \nu_2 = \nu$;

—In experiment 4, a spike of parameter γ downward by a factor of 0.04 was simulated, i.e., $\omega_1 = \omega_2 = \omega$, $\delta_1 = \delta_2 = \delta$, $\gamma_1 = \gamma$ and $\gamma_2 = \gamma - 0.04$, $\nu_1 = \nu_2 = \nu$.

Table 2. Average power for different types of spikes

Type of jump	$\bar{W}_{\mathrm{KS}}$	$\bar{W}_{\mathrm{KL}}$
Experiment 2: Five-fold increase in ω	0.50	0.51
Experiment 3: A 0.1 decrease in δ	0.92	0.49
Experiment 4: A 0.04 decrease in γ	0.82	0.45

• Using the results of these simulations, the powers $W_{\mathrm{KS}}^{(j)}$ and $W_{\mathrm{KL}}^{(j)}$ are calculated as the proportion of cases in which the KS and KL methods correctly indicated the presence of a structural break. Based on values $W_{\mathrm{KS}}^{(j)}$ and $W_{\mathrm{KL}}^{(j)}$, the average powers of the KS and KL methods were calculated:

$$\bar{W}_{\mathrm{KS}} = \frac{1}{N}\sum_{j=1}^{N} W_{\mathrm{KS}}^{(j)}, \quad \bar{W}_{\mathrm{KL}} = \frac{1}{N}\sum_{j=1}^{N} W_{\mathrm{KL}}^{(j)}.$$

• The calculation results are given in Table 2.

• Selective integral functions of power distributions are calculated, their graphs are presented in Figs. 2–4.

Table 2 shows that in experiments 3 and 4, the KL method is significantly inferior to the KS method, since in these experiments the KL method has significantly lower average structural break detection pow-

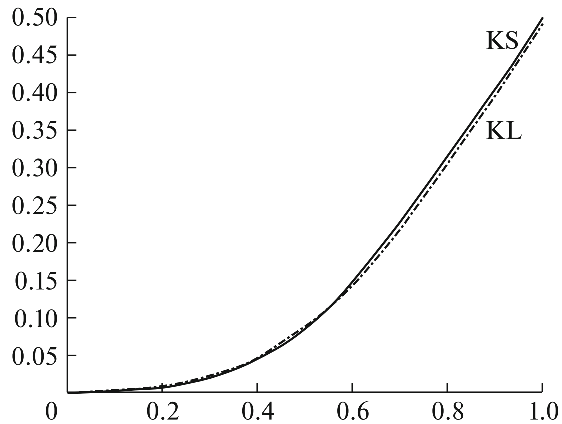

Fig. 2. Sample integral distribution functions of empirical power for experiment 2.

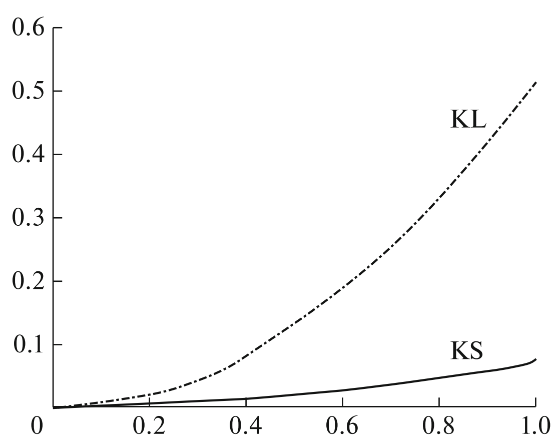

Fig. 3. Sample integral distribution functions of empirical power for experiment 3.

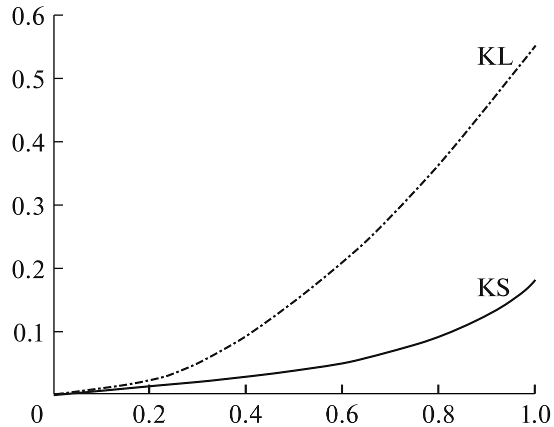

Fig. 4. Sample integral distribution functions of empirical power for experiment 4.

ers. In the second experiment, the KS and KL methods demonstrated comparable average powers of detecting a structural break.

In addition, according to Figs. 3 and 4, the methods turned out to be in relation to the stochastic dominance of the second order: $F_{KS} \succ_2 F_{KL}$. No benefit was found in the second experiment.

The technical condition under which the securities were selected was that the “shifted” parameter vectors $\theta_2 = (\omega_2, \delta_2, \gamma_2, \nu_2)$ in model (1) for each of the selected securities must fall into the set of acceptable parameter values Θ in each of experiments 2–4. In particular, the conditions $\delta_2 + \gamma_2 < 1$ and $\gamma_2 \geq 0$ must be met. For this reason, in experiments 3 and 4, parameters δ and γ move downward and not upward. In this case, the shift in parameter γ is insignificant (only by a factor of 0.04). Otherwise, for some calculations, the shifted parameter vectors θ_2 are not included in the admissible set Θ.

3. CONCLUSIONS

This paper compared the KL and KS methods for detecting a structural shift in GARCH(1,1)-t models in the case of Student’s conditional distribution of random errors. The comparison was carried out using the Monte Carlo statistical test method in which the situations of the occurrence of structural breaks in the GARCH-t models were simulated. In order for these tests to be more realistic when generating the GARCH-t processes, we used the parameters of GARCH-t models that were preestimated in the time series of the returns of 28 global financial indices. A series of four Monte Carlo experiments were carried out, each of which consisted of 28 calculations, and each calculation consisted of 5000 simulations. These experiments were used to compare the average probabilities of errors of the first kind and the average powers of detection of structural shifts of the KS and KL methods.

The results of the tests were as follows: the KL method was inferior to the KS method in terms of the average probability of errors of the first kind (27% vs 21%); for two of the three types of structural breaks, the average power of structural break detection of the KS method was significantly superior to the KL method (92% vs 49% and 82% vs 45%, respectively). For the third type of structural shift, both the KL and KS methods showed approximately the same power (51% and 50%, respectively). Thus, based on the results of the experiments, we conclude that the KS method has a significant advantage over the KL method in terms of the power of detecting a structural break in the case of Student’s conditional distribution of random errors.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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