Simulation of a Thermoporoelastic Medium Taking the Damage into Consideration

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Abstract—The issues of the mathematical modeling of a thermoporoelastic medium taking its damage into account are considered. The employed model generalizes the classical Biot model simulating the behavior of a poroelastic medium taking the thermoelastic effects into account. In order to describe the damage of the medium, the approach of continual damage mechanics is used, in which the state of the medium is described by the scalar damage parameter, which affects the elastic and poroperm characteristics of the medium. The system of the governing equations for the model consists of the fundamental mass, momentum, and energy conservation laws and is closed by thermodynamically consistent constitutive relations. Moreover, the medium's energy expression takes into account its changes due to the formation of damaged zones. The computational algorithm is based on the finite-element method. A monolithic approach is used, which assumes that all groups of equations (mechanics, heat transfer, and flow) are solved simultaneously without being split into physical processes and/or iterations between groups of equations. The system of equations of thermoporoelasticity is approximated by a fully implicit scheme. The damage parameter's evolution depending on the stress-strain state of the medium can be described in terms of both instant and finite-time kinetics. This paper briefly describes the mathematical model and presents a detailed description of the computational algorithm and its implementation. A significant part of the study is devoted to the application of the developed approaches for solving several model and realistic three-dimensional problems. The analysis of the geomechanical problems of thermal enhanced oil recovery methods, which require a consistent description of the elastic, filtration, and thermal fields' dynamics taking into account the evolution of the medium's fracture, is considered to be the main field of the application of the model and algorithm.

1. INTRODUCTION

This work is devoted to the mathematical modeling of the dynamics of a thermoporoelastic medium that takes the damage into account. The employed mathematical model is a generalization of M. Biot's classical model of poroelastic medium. The model is thermodynamically consistent in the sense of the fulfillment of the basic laws of mass, energy, and momentum conservation, as well as entropy inequality in the required form, see [1].

Within the Biot model, the medium is described as a set of two interpenetrating continua (phases): the elastic skeleton (solid phase) and the fluid that saturates it (mobile phase). Under the influence of external loads applied to it, the dynamics of the medium is determined by the joint evolution of the stress-strain state of a porous permeable deformed solid and the pressure fields of the fluid filtering into it. The equations of the model represent a strongly connected system of equations of thermomechanics and filtration. Typical applications of the Biot model include problems of hydrogeology [2], analysis of processes in bone and soft biological tissues [3, 4], and hydrogels.

It should be noted that the Biot model is not the only known model for describing the dynamics of a poroelastic medium. In particular, there is a notable model proposed in the works of V.N. Dorovskii and his colleagues. The model is thermodynamically consistent and is obtained within the theory of multive-

locity continuum developed by the authors [5–7]. The specific feature of this model is that the poroelastic medium is described by three elastic moduli, which are uniquely determined by the propagation velocities of elastic waves [5]. This model is used to describe the motion of Stoneley waves [6], in problems of seismic exploration [7], and in other areas.

One of the important applications for the models of the considered class is the problem of petroleum geomechanics, especially in relation to the analysis of modern methods for enhanced oil recovery, in particular, the so-called thermal methods, which supply heat to the reservoir by pumping a thermophore or using a conductive heat exchange [8]. On applying the thermal treatment methods, the development process involves manifested effects such as the change in the properties of the formation fluid, the course of chemical reactions, and the change in the poroperm properties of the formation due to the change in the mobility rate of the fluid filling the pores and the formation of microcracking zones. The understanding of the complex interrelated processes accompanying the thermal effect on an oil-bearing stratum can only be enhanced by providing a comprehensive description of the main processes occurring in the stratum, including the geomechanical effects and the formation's damage.

There are several models that enable taking these phenomena into account. In most of these models, the damageability of the medium is described within the theory of continuous damage [9, 10], when the so-called damage parameter is introduced in the set of thermodynamic parameters describing the state of the medium. The evolution of this parameter is determined by a kinetic equation that connects the value of the parameter (or the rate of its change) with the stress-strain state of the medium. In turn, the same parameter is included in the main constitutive relations and affects the poroperm, thermodynamic, and geomechanical parameters of the oil reservoir.

In this study, the thermodynamically consistent mathematical model proposed in [1] is used to describe the evolution of a thermoporoelastic medium taking into account the continual damage.

The distinctive feature of the model is that, on the one hand, it is obtained within a strict thermodynamic formalism, which guarantees its correctness, and on the other hand, it can be practically applied to solve problems in realistic settings.

In the considered approximation, the system of poroelasticity equations is a coupled problem for an elliptic equation (describing the stress-strain state of a saturated medium) and parabolic equations (describing the laws of conservation of the fluid mass and energy of the system). To solve this system of equations, various methods can be used, including the finite-volume method, the finite-element method, and the method of boundary integral equations.

For approximations in time, completely implicit schemes are most often used; they will also be applied in this work. In this case, the complete system of equations can be approximated as a whole, or the problem is "decoupled" and its solution is obtained in the course of certain iterations between the groups of filtration and elasticity equations. The latter approach is most often employed when two different programs are used to solve the problem for calculating the filtration part (for example, using an industrial filtration simulator) and for solving the problem of elasticity theory. The construction of effective iterative methods for solving such problems is a separate problem; see, for example, [11].

A characteristic property of the system of poroelasticity equations in the considered approximation is that, after approximation in time, the system of equations has the form of a saddle point problem [12]. In this case, for the stability of the problem's solution both in the continuous and in the discrete setting, it is necessary to satisfy the so-called inf-sup conditions (the Ladyzhenskaya–Babushka–Brezzi conditions) [12]. Violations of these conditions in poroelasticity problems leads to numerical instabilities and effects of the locking of the finite-dimensional solution, especially in the incompressible limit (see [13] and references there). The simplest example of a finite element satisfying these conditions is the Taylor–Hood element [14], in which second-order finite elements are used to approximate displacements, and first-order finite elements are used to approximate pressure. Moreover, both systems of functions are specified on the same partition of the domain into finite elements. This finite-element type guarantees the optimal rate of convergence, including in poroelasticity problems [15]. This approach will be used in this study as well.

An alternative approach is such a regularization of the original problem (see, for example, [16–18]), for which the inf-sup conditions are satisfied using the standard pairs of spaces (finite elements of the same, first, order for both the displacement field and the pressure and temperature).

In this study, the finite-element method is used. Its advantages include a fairly formal way of constructing a finite-dimensional problem and efficiency when using unstructured computational grids. Unlike a number of other works (see, for example, [19, 20]), in this paper the problem is considered in a completely

linked (monolithic) statement, when all the equations of the model are solved simultaneously, within a single system of equations. Thus, there is no iterative connection between the groups of equations of the theory of elasticity, filtration, and energy balance, which is especially important in relation to the multidisciplinary nature of the problem and taking into account the additional effects (direct damage and the dependence of the model parameters on it).

The main aim of the study is to describe the model, a set of computational algorithms, and their software implementation, which give an idea of their capabilities and show the applicability of the considered approaches for solving problems in statements that are close to real ones.

2. MATHEMATICAL MODEL

Let us consider a thermoporoelastic medium consisting of two continuums: a porous deformable skeleton and a single-phase weakly compressible fluid. The system of equations describing the behavior of such a medium has the following form [1]:

$$\frac{\partial m_f}{\partial t} + \mathrm{div}(\rho_f \mathbf{w}) = 0,$$
$$\mathrm{div}\,\boldsymbol{\sigma} = 0, \tag{1}$$
$$\frac{\partial(m_s e_s + m_f e_f)}{\partial t} + \mathrm{div}(\rho_f e_f \mathbf{w}) = \mathrm{div}(-p\mathbf{w}) - \mathrm{div}(\mathbf{q}),$$

where m_α is the mass of phase α in some elementary volume, ρ_f is the fluid density, $\mathbf{w} = -(\mathbf{k}/\mu)\,\mathrm{grad}(p)$ stands for the filtration rate, $\boldsymbol{\sigma}$ denotes the total stress tensor, e_α is the specific internal energy, p is the pore pressure, $\mathbf{q} = -\boldsymbol{\kappa}\,\mathrm{grad}(T)$ stands for the heat flux vector, $\mathbf{k}$ is the permeability tensor, μ is the fluid viscosity, $\boldsymbol{\kappa}$ stands for the coefficient of thermal conductivity, and T is the temperature.

The system of equations (1) is closed by the following set of defining relations [1]:

$$\Delta\boldsymbol{\sigma} = \mathbf{C}(1 - D) : \Delta\boldsymbol{\varepsilon} - \mathbf{b}\Delta p - \mathbf{C} : \alpha_T \Delta T - \mathbf{C} : \boldsymbol{\varepsilon}^0 : \Delta D,$$
$$\Delta m_f = \rho_f \mathbf{b} : \Delta\boldsymbol{\varepsilon} + \rho_f \frac{\Delta p}{M} - \alpha_m \rho_f \Delta T,$$
$$\Delta S_s = \mathbf{C} : \alpha_T \Delta\boldsymbol{\varepsilon} - \alpha_\phi \Delta p + \frac{C_{ps}}{T^0} \Delta T,$$
$$\Delta\mathbf{Y} = \mathbf{C}\left(\boldsymbol{\varepsilon}^0 + \frac{1}{2}\Delta\boldsymbol{\varepsilon}\right)\Delta\boldsymbol{\varepsilon}, \tag{2}$$
$$\Delta\frac{1}{\rho_f} = \frac{1}{K_f}\Delta p - \alpha_f \Delta T,$$
$$\Delta S_f = -\alpha_f \Delta p + \frac{C_{pf}}{T^0}\Delta T,$$

where $\mathbf{C}$ is the elastic coefficients tensor, $\mathbf{b}$ is the Biot coefficient, $\mathbf{C} : \alpha_T$ is the tensor of the thermoelastic coefficients, $\boldsymbol{\varepsilon} = (\nabla \otimes \boldsymbol{\xi} + (\nabla \otimes \boldsymbol{\xi})^T)/2$ is the strain tensor, $\boldsymbol{\xi}$ stands for the skeleton displacement vector, M denotes the Biot module, α_ϕ and α_f are the coefficients of thermal expansion of the skeleton and fluid, respectively, C_{ps} and C_{pf} is the heat capacity of the skeleton and fluid, $\mathbf{Y}$ is the generalized thermodynamic force related to damageability, K_f is the bulk fluid compression module, S_α is the phase entropy α, D is the damageability parameter, $\Delta f = f - f^0$, and f^0 is the reference parameter value f.

The expression for the internal energy of the skeleton is

$$\Delta E_s = \Delta E_{s\varepsilon} + \Delta E_{sp} + \Delta E_{sT} + \Delta E_{sD},$$

where

$$\Delta E_{s\varepsilon} = \left[\boldsymbol{\sigma}^0 + \frac{1}{2}\mathbf{C} : \Delta\boldsymbol{\varepsilon} + p^0\mathbf{b} + T^0\mathbf{C} : \alpha_T \right]\Delta\boldsymbol{\varepsilon},$$

$$\Delta E_{sp} = \left[\frac{1}{N}\left(p - \frac{1}{2}\Delta p \right) - \alpha_\varphi\left(T - \frac{1}{2}\Delta T \right) \right]\Delta p,$$

$$\Delta E_{sT} = \left[-\alpha_\varphi\left(p - \frac{1}{2}\Delta p \right) + \frac{C_{ps}}{T^0}\left(T - \frac{1}{2}\Delta T \right) \right]\Delta T, \tag{3}$$

$$\Delta E_{sD} = -\left[\mathbf{Y}^0 + \mathbf{C}\left(\boldsymbol{\varepsilon}^0 + \frac{1}{2}\Delta\boldsymbol{\varepsilon} \right)\Delta\boldsymbol{\varepsilon} \right]\Delta\mathbf{D}.$$

The expression for the internal energy of the fluid has the form

$$\Delta E_f = \left[\frac{\phi}{K_f}\left(p - \frac{1}{2}\Delta p \right) - \phi\alpha_f\left(T - \frac{1}{2}\Delta T \right) \right]\Delta p - \left[\phi\alpha_f\left(p - \frac{1}{2}\Delta p \right) - \frac{C_{pf}}{T^0}\left(T - \frac{1}{2}\Delta T \right) \right]\Delta T, \tag{4}$$

where $\phi = m_f/\rho_f$ is the porosity.

The coefficients in the defining constitutive can be functions of the parameters of the medium's state $\chi = \{\varepsilon, p, T, D\}$. In relation to this, the model was supplemented in order to take into account the dependence of the formation's permeability and fluid viscosity on the current state parameters $\mathbf{k} = \mathbf{k}(\varepsilon, p, T, D)$ and $\mu = \mu(p, T)$.

The primary unknowns of the system of equations (1) are displacements ξ, pressure p, and temperature T. The problem is considered in a three-dimensional domain Ω with boundary $\partial\Omega$ on which the Dirichlet ($\partial\Omega_D$) or Neumann ($\partial\Omega_N$) boundary conditions are defined for each parameter $\partial\Omega = \partial\Omega_N \cup \Omega_D$. The Dirichlet condition $x|_{\partial\Omega_D} = 0$ is the rigidly fixed boundary of the domain, $p|_{\partial\Omega_D} = \tilde{p}$ is the pressure on the domain boundary, and $T|_{\partial\Omega_D} = \tilde{T}$ is the temperature on the domain boundary. The Neumann condition $\boldsymbol{\sigma} : \mathbf{n}|_{\partial\Omega_N} = \tilde{\mathbf{t}}$ is the normal stress vector on the domain boundary, $\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega_N} = \tilde{q}$ is the fluid flow across the domain boundary, and $\mathbf{q}_T \cdot \mathbf{n}|_{\partial\Omega_N} = \tilde{q}_T$ is the heat flux across the domain boundary.

Note that for each variable ξ, p, and T, domains $\partial\Omega_N$ and $\partial\Omega_D$ are specific, that is, for variable $\alpha : \partial\Omega = \partial\Omega_N^{(\alpha)} \cup \partial\Omega_D^{(\alpha)}$. However, to simplify the further calculations, we will assume that they coincide.

3. WEAK PROBLEM STATEMENT

Let the variables of the problem (ξ, p, T, D) be elements of spaces of sufficiently smooth functions that satisfy the Dirichlet boundary condition. In order to construct a weak statement of the problem, we multiply each equation in system (1) by the corresponding test function and integrate over domain Ω. Then, in accordance with Green's formula, taking into account the defining relations, the system of equations (1) will take the form

$$\int_\Omega (\mathbf{Lv})^T : \mathbf{C}(1 - D) : \boldsymbol{\varepsilon}(\xi)d\Omega - \int_\Omega (\mathbf{Lv})^T \mathbf{b}pd\Omega - \int_\Omega (\mathbf{Lv})^T : \mathbf{C} : \alpha_T Td\Omega = \int_{\partial\Omega_N} \mathbf{v}^T\tilde{\mathbf{t}}dS,$$

$$\int_\Omega v\mathbf{b} : \mathbf{L}\frac{\partial\xi}{\partial t}d\Omega + \int_\Omega v\frac{1}{M}\frac{\partial p}{\partial t}d\Omega - \int_\Omega v\alpha_m\frac{\partial T}{\partial t}d\Omega + \int_\Omega (\text{grad }v)^T\frac{\mathbf{k}}{\mu}\text{grad}(p)d\Omega = -\int_{\partial\Omega_N} v\tilde{q}d\Omega,$$

$$\int_\Omega v\left[\frac{\partial E_s}{\partial t} + \frac{\partial E_f}{\partial t} \right]d\Omega - \int_\Omega (\text{grad }v)^T\frac{1}{\phi}E_f\mathbf{w}d\Omega - \int_\Omega (\text{grad }v)^T p\mathbf{w}d\Omega$$

$$- \int_\Omega (\text{grad }v)^T\mathbf{q}_T d\Omega = -\int_{\partial\Omega_N} v\rho_f e_f\tilde{q}dS - \int_{\partial\Omega_N} vp\tilde{q}dS - \int_{\partial\Omega_N} v\tilde{q}_T dS, \tag{5}$$

where $\mathbf{v} \in V_\alpha^0$ are the trial functions defined in domain Ω such that $\mathbf{v}|_{\partial\Omega_D} = 0$.

We represent the symmetric strain and stress tensors using the Voigt vector notation [21], i.e.,

$$\varepsilon = \left[\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{xz}\right]^T, \quad \gamma_{ij} = 2\varepsilon_{ij}, \quad \sigma = \left[\sigma_x, \sigma_y, \sigma_z, \sigma_{xy}, \sigma_{yz}, \sigma_{xz}\right]^T.$$

In this case, the relationship between stress and strain for an isotropic medium can be written as $\sigma = \mathbf{C} : \varepsilon$, $\varepsilon = \mathbf{L}\xi$, where $\xi = [\xi_x, \xi_y, \xi_z]^T$,

$$\mathbf{C} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2 \end{bmatrix},$$

$$\mathbf{L} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 & 0 & \dfrac{\partial}{\partial y} & 0 & \dfrac{\partial}{\partial z} \\ 0 & \dfrac{\partial}{\partial y} & 0 & \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} & 0 \\ 0 & 0 & \dfrac{\partial}{\partial z} & 0 & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}^T.$$

Using expressions (3) for the internal energy of the skeleton, we find its time derivative

$$\frac{\partial E_s}{\partial t} = \left[\mathbf{C}(1-\Delta D) : \Delta\varepsilon + p^0\mathbf{b} + T^0\mathbf{C} : \alpha_T\right]\frac{\partial \varepsilon}{\partial t}$$
$$+ \left[\frac{1}{N}p - \alpha_\varphi T\right]\frac{\partial p}{\partial t} + \left[-\alpha_\varphi p + \frac{C_{ps}}{T^0}T\right]\frac{\partial T}{\partial t} - \left[\frac{1}{2}\frac{dD}{dt}\mathbf{C} : \varepsilon\right]\varepsilon.$$

In the last formula, it is assumed that $\varepsilon^0 = 0$, $D^0 = 0$ and $\mathbf{Y}^0 = 0$. Similarly, for the internal energy of the fluid (4), we obtain

$$\frac{\partial E_f}{\partial t} = \left[\frac{\phi}{K_f}p - \phi\alpha_f T\right]\frac{\partial p}{\partial t} + \left[-\phi\alpha_f p + \frac{C_{pf}}{T^0}T\right]\frac{\partial T}{\partial t}.$$

Thus, the time derivative of the total internal energy is written as $\partial E/\partial t = \partial E_s/\partial t + \partial E_f/\partial t$ where the form of the terms is defined above.

4. FINITE-DIMENSIONAL APPROXIMATIONS

Let us consider a spatial approximation of the system of equations (5). We introduce finite-dimensional spaces $V_\alpha^h \subset V_\alpha$ and $V_\alpha^{0,h} \subset V_\alpha^0$, while $V_\alpha^h = \mathrm{span}(\phi_i^{(\alpha)})$, where $\phi_i^{(\alpha)}$ are the basic functions. Then for an arbitrary function f we have

$$f = \sum_{i=1}^{N_\alpha}\phi_i^{(\alpha)}f_i, \quad f = \xi, p, T.$$

Accordingly, the approximation of system (5) in matrix form is written as

$$\begin{bmatrix} -\mathbf{A}_{\xi\xi} & \mathbf{A}_{\xi p} & -\mathbf{A}_{\xi T} \\ \mathbf{A}_{\xi p}^T & \mathbf{A}_{pp} & \mathbf{A}_{pT} \\ \mathbf{A}_{T\xi} & \mathbf{A}_{Tp} & \mathbf{A}_{TT} \end{bmatrix}\begin{bmatrix} \partial\xi/\partial t \\ \partial p/\partial t \\ \partial T/\partial t \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathbf{B}_{pp} & 0 \\ \mathbf{B}_{T\xi} & \mathbf{B}_{Tp} & \mathbf{B}_{TT} \end{bmatrix}\begin{bmatrix} \xi \\ p \\ T \end{bmatrix} = \begin{bmatrix} -\partial\mathbf{f}_\xi/\partial t \\ \mathbf{f}_p \\ \mathbf{f}_T \end{bmatrix}, \tag{6}$$

where the following denotations are used:

$$\left[\mathbf{A}_{\xi\xi}\right]_{ij} = \int_\Omega \left[\mathbf{L}\phi_\xi\right]_{ik} \left[\mathbf{C}(1-D)\right]_{kl} \left[\mathbf{L}\phi_\xi\right]_{lj} d\Omega,$$

$$\left[\mathbf{A}_{\xi p}\right]_{ij} = \left[\mathbf{A}_{p\xi}\right]_{ji} = \int_\Omega \left[\mathbf{L}\phi_\xi\right]_{ik} \mathbf{b}_k \phi_{p,j} d\Omega,$$

$$\left[\mathbf{A}_{\xi T}\right]_{ij} = -\int_\Omega \left[\mathbf{L}\phi_\xi\right]_{ik} \left[\mathbf{C}(1-D):\alpha_T\right]_{kl} \phi_{T,j} d\Omega,$$

$$\left[\mathbf{B}_{pp}\right]_{ij} = \int_\Omega \nabla\phi_{p,i} \frac{\mathbf{k}}{\mu} \nabla\phi_{p,j} d\Omega, \quad \left[\mathbf{A}_{pp}\right]_{ij} = \int_\Omega \phi_{p,i} \frac{1}{M} \phi_{p,j} d\Omega,$$

$$\left[\mathbf{A}_{pT}\right]_{ij} = -\int_\Omega \phi_{p,i} \alpha_m \phi_{T,j} d\Omega,$$

$$\left[\mathbf{A}_{T\xi}\right]_{ij} = \int_\Omega \phi_{T,i} \left[\sigma^0 + \mathbf{C}(1-D):\varepsilon + p^0\mathbf{b} + T^0\mathbf{C}:\alpha_T\right]_{ik} \left[\mathbf{L}\phi_\xi\right]_{kj} d\Omega,$$

$$\left[\mathbf{A}_{Tp}\right]_{ij} = \int_\Omega \phi_{T,i} \left(\frac{1}{M} p - \alpha_m T\right) \phi_{p,j} d\Omega,$$ (7)

$$\left[\mathbf{A}_{TT}\right]_{ij} = \int_\Omega \phi_{T,i} \left[-\alpha_m p + \left(C_{ps} + C_{pf}\right)\frac{T}{T^0}\right]_{ij} \phi_{T,j} d\Omega,$$

$$\left[\mathbf{B}_{T\xi}\right]_{ij} = \int_\Omega \nabla\phi_{T,i} \left[\frac{1}{2}\frac{dD}{dt}\mathbf{C}:\varepsilon\right]_{ik} \left[\mathbf{L}\phi_\xi\right]_{kj} d\Omega,$$

$$\left[\mathbf{B}_{Tp}\right]_{ij} = \int_\Omega \nabla\phi_{T,i} \left\{\frac{E_f^0}{\varphi} + \left[\frac{1}{K_f} p - \alpha_f T\right]\Delta p - \left[\alpha_f p - \frac{C_{pf}}{f}\right]\Delta T\right\}_{ij} \frac{\mathbf{k}}{\mu} \nabla\phi_{p,j} d\Omega,$$

$$\left[\mathbf{B}_{TT}\right]_{ij} = \int_\Omega \nabla\phi_{T,i}\kappa\nabla\phi_{T,j} d\Omega, \quad \mathbf{f}_{\xi,i} = \int_{\partial\Omega_N} \left[\phi_\xi\right]_{ij} \tilde{\mathbf{t}}_j dS, \quad \mathbf{f}_{p,i} = -\int_{\partial\Omega_N} \phi_{p,i}\tilde{q} d\Omega,$$

$$\mathbf{f}_{T,i} = -\int_{\partial\Omega_N} \phi_{T,i}\rho_f e_f \tilde{q} dS - \int_{\partial\Omega_N} \phi_{T,i} p\tilde{q} dS - \int_{\partial\Omega_N} \phi_{T,i}\tilde{q}_T dS.$$

The integrals in Eq. (7) are calculated using second-order Gauss quadrature formulas [22].

In order to discretize Eq. (8) in space, we used a tetrahedral grid with quadratic basis functions for displacements and linear ones for pressure and temperature (Taylor−Hood elements [14]).

In order to approximate the system of equations in time, we will use a scheme that is completely implicit with respect to displacements x, pressure p, and temperature T. The damage parameter D will be taken into account explicitly. Let $f = f(t)$ be the value of some quantity at a point in time t, then $\hat{f} = f(t + \Delta t)$ is its value at time $t + \Delta t$. Accordingly, the system of equations (6) after approximation in time takes the form

$$\begin{bmatrix} -\mathbf{A}_{\xi\xi} & \mathbf{A}_{\xi p} & -\mathbf{A}_{\xi T} \\ \mathbf{A}_{\xi p}^T & \mathbf{A}_{pp} + \Delta t\mathbf{B}_{pp} & \mathbf{A}_{pT} \\ \mathbf{A}_{T\xi} + \Delta t\mathbf{B}_{T\xi} & \mathbf{A}_{Tp} + \Delta t\mathbf{B}_{Tp} & \mathbf{A}_{TT} + \Delta t\mathbf{B}_{TT} \end{bmatrix} \cdot \begin{bmatrix} \hat{\xi} \\ \hat{p} \\ \hat{T} \end{bmatrix}$$
$$= \begin{bmatrix} -\mathbf{A}_{\xi\xi} & \mathbf{A}_{\xi p} & -\mathbf{A}_{\xi T} \\ \mathbf{A}_{\xi p}^T & \mathbf{A}_{pp} & \mathbf{A}_{pt} \\ \mathbf{A}_{T\xi} & \mathbf{A}_{Tp} & \mathbf{A}_{TT} \end{bmatrix} \cdot \begin{bmatrix} \xi \\ p \\ T \end{bmatrix} + \Delta t \begin{bmatrix} -\Delta\mathbf{f}_\xi/\Delta t \\ \mathbf{f}_p \\ \mathbf{f}_T \end{bmatrix}.$$ (8)

Due to the fact that the system of equations (8) is nonlinear, Newton's method [23] is used for its solution. The criterion for the convergence of the iterative algorithm is the simultaneous fulfillment of the following conditions:

$$\|\Delta\mathbf{x}\|_\infty < \varepsilon_x, \quad \|\mathbf{R}\|_\infty < \varepsilon_R,$$

where $\Delta\mathbf{x}$ is the increment value of unknowns at the current iteration, $\mathbf{R}$ is the value of nonlinear residual, and ε_x and ε_R are the parameters.

Taking into account the form of matrices (7), the Jacobian of system (8) is

$$\mathbf{J} = \begin{bmatrix} -\mathbf{A}_{\xi\xi} & \mathbf{A}_{\xi p} & -\mathbf{A}_{\xi T} \\ \mathbf{A}_{\xi p}^T & \mathbf{A}_{pp} + \Delta t \mathbf{B}_{pp} & \mathbf{A}_{pT} \\ \mathbf{J}_{T\xi} & \mathbf{J}_{Tp} & \mathbf{J}_{TT} \end{bmatrix},$$

where

$$\mathbf{J}_{T\xi} = \mathbf{A}_{T\xi} + \Delta t \mathbf{B}_{T\xi} + \frac{\partial \mathbf{A}_{T\xi}}{\partial \boldsymbol{\xi}} \Delta \boldsymbol{\xi} + \Delta t \mathbf{B}_{T\xi} \hat{\dot{\boldsymbol{\xi}}},$$

$$\mathbf{J}_{Tp} = \mathbf{A}_{Tp} + \Delta t \mathbf{B}_{Tp} + \frac{\partial \mathbf{A}_{Tp}}{\partial p} \Delta p + \frac{\partial \mathbf{A}_{TT}}{\partial p} \Delta T + \Delta t \frac{\partial \mathbf{B}_{Tp}}{\partial p} \hat{p},$$

$$\mathbf{J}_{TT} = \mathbf{A}_{TT} + \Delta t \mathbf{B}_{TT} + \frac{\partial \mathbf{A}_{Tp}}{\partial T} \Delta p + \frac{\partial \mathbf{A}_{TT}}{\partial T} \Delta T + \Delta t \frac{\partial \mathbf{B}_{Tp}}{\partial T} \hat{p}$$

and

$$\frac{\partial \mathbf{A}_{T\xi}}{\partial \boldsymbol{\xi}} = \int_{\Omega} \phi_T^T \left[\mathbf{C}(1-D) : \mathbf{I} \right] (\mathbf{L}\phi_\xi) d\Omega, \quad \frac{\partial \mathbf{A}_{Tp}}{\partial p} = \int_{\Omega} \phi_T^T \frac{1}{M} \phi_p d\Omega,$$

$$\frac{\partial \mathbf{A}_{Tp}}{\partial T} = -\int_{\Omega} \phi_T^T \alpha_m \phi_p d\Omega, \quad \frac{\partial \mathbf{A}_{TT}}{\partial p} = -\int_{\Omega} \phi_T^T \alpha_m \phi_T d\Omega,$$

$$\frac{\partial \mathbf{A}_{TT}}{\partial T} = \int_{\Omega} \phi_T^T \frac{C_{ps} + C_{pf}}{T^0} \phi_T d\Omega, \quad \frac{\partial \mathbf{B}_{T\xi}}{\partial \boldsymbol{\xi}} = \int_{\Omega} \nabla \phi_T^T \left[\frac{\partial}{\partial \boldsymbol{\xi}} \left(\frac{1}{2} \frac{dD}{dt} \mathbf{C} : \boldsymbol{\varepsilon} \right) \right] (\mathbf{L}\phi_\xi) d\Omega,$$

$$\frac{\partial \mathbf{B}_{Tp}}{\partial p} = \int_{\Omega} (\mathbf{L}\phi_\xi)^T \left[\frac{1}{K_f} (2p - p^0) - \alpha_f (2T - T^0) \right] \frac{\mathbf{k}}{\mu} \nabla \phi_p d\Omega,$$

$$\frac{\partial \mathbf{B}_{Tp}}{\partial T} = \int_{\Omega} (\mathbf{L}\phi_\xi)^T \left[\frac{C_{pf}}{\phi} - \alpha_f (2p - p^0) \right] \frac{\mathbf{k}}{\mu} \nabla \phi_p d\Omega.$$

The system of linear equations (8) is solved at each Newtonian iteration using the biconjugate gradient stabilized method (BiCGStab) [24]. Incomplete LU decomposition with single-level filling (ILU (1)) was used as a preconditioner.

In order to ensure the stability of the finite-dimensional problem, several approaches were used, in accordance with which the matrix of system (8) was transformed. For mass matrices $\mathbf{A}_{pp}$, $\mathbf{A}_{pT}$, $\mathbf{A}_{Tp}$, and $\mathbf{A}_{TT}$, the method of mass matrices' diagonalization (mass lumping, [25]) was used. In order to reduce the bandwidth of the sparse matrix in the system, the Cuthill–McKee algorithm [26] was used.

5. MODELING RESULTS

This section presents the results of simulation using a software package based on the mathematical model and computational algorithm described earlier. The algorithm was validated using several tests (Terzaghi's problem, Mandel's test, and the test for one-dimensional nonisothermal expansion), for which an analytical solution is known [27]. Further we present the simulation results for several problems in realistic settings.

In the calculations, the following settings for the convergence of Newtonian iterations were used: $\varepsilon_x = 0.001$ and $\varepsilon_R = 0.001$. If the number of iterations in one time step exceeds 5, the time step is halved and the iterative algorithm is repeated from the beginning. In the case of the successful convergence of the algorithm, the next time period is calculated with a gradual increase in the time step. The criterion for stopping the linear iterative process is the achievement of the residual norm of quantity 10^{-5}.

In order to simulate the dependence of the permeability on the parameters of the medium state, we used the dependence [28], i.e.,

$$k = k_0 \exp\left[-\beta (\tilde{\sigma} - \alpha p) \right], \tag{9}$$

Table 1. Values of the input parameters for the thermal stimulation model

Parameter	Value
Young's modulus, E	20 GPa
Poisson's ratio, ν	0.3
Bio module, N	10 GPa
Module of volumetric fluid compression, K_f	3.3 GPa
Biot coefficient, b	0.79
Permeability, k	1×10^{-16} m^2
Porosity, φ	0.1
Viscosity, μ	1 mPa s
Skeleton density, ρ_s	2100 kg/m^3
Fluid density, ρ_f	1000 kg/m^3
Volumetric thermal expansion coefficient of the skeleton, α_s	1×10^{-6} 1/K
Volumetric thermal expansion coefficient of fluid, α_f	1×10^{-4} 1/K
Specific heat capacity of the skeleton, c_{ps}	1000 J/(kg K)
Specific heat capacity of the fluid, c_{pf}	4200 J/(kg K)
Effective thermal conductivity, κ	2 W/(m K)

where k_0 is the value of the initial permeability and $\tilde{\sigma} = 1/3\,(\sigma_x + \sigma_y + \sigma_z)$ is the average stress. For the further calculations, the values from [28] $\alpha = 0.1$ and $\beta = 0.05$ Pa^{-1} were taken as the values of coefficients α and β.

In order to describe the dependence of viscosity on temperature, the Baggs–Robinson correlation is used [29]:

$$\mu = 10^X - 1, \quad X = 10^Z T^{-1.163}, \quad Z = 3.0324 - 0.02023\gamma_0, \tag{10}$$

where μ is the pore fluid's viscosity in cP (1 cP = 0.001 Pa s) and γ_0 is the fluid's density in $^\circ API$.

In order to describe the evolution of the damageability parameter, an explicit dependence is used [30]:

$$D = \begin{cases} 0, & \tilde{\varepsilon} < \tilde{\varepsilon}_c, \\ \dfrac{D_{\text{off}}}{\tilde{\varepsilon}_{\text{off}} - \tilde{\varepsilon}_c}\tilde{\varepsilon} - D_{\text{off}}\dfrac{\varepsilon_c}{\tilde{\varepsilon}_{\text{off}} - \tilde{\varepsilon}_c}, & \tilde{\varepsilon}_c \le \tilde{\varepsilon} \le \tilde{\varepsilon}_{\text{off}}, \\ D_{\text{lim}} - (D_{\text{lim}} - D_{\text{off}})\dfrac{\tilde{\varepsilon}_{\text{off}}}{\tilde{\varepsilon}}, & \tilde{\varepsilon} > \tilde{\varepsilon}_{\text{off}}, \end{cases} \tag{11}$$

where ε_i are the principal strains and $\tilde{\varepsilon} = \sqrt{\sum_{i=1}^{3}\langle\varepsilon_i\rangle^2}, \langle\varepsilon_i\rangle = (\varepsilon_i + |\varepsilon_i|)/2$. The parameters of dependence (11) were taken equal to $\tilde{\varepsilon}_c = 0.0002$, $\tilde{\varepsilon}_{\text{off}} = 0.003$, $D_{\text{lim}} = 1$, and $D_{\text{off}} = 0.9$.

The values of the other input parameters for all the presented models are given in Table 1.

5.1. Modeling Thermal Action on a Formation with Production and Injection Wells

This calculation simulates the response of the formation when a system for maintaining reservoir pressure is used, in which an injection well pumps fluid at a temperature of 400°C with an injectivity of 0.2 m^3/day. The production well operates at a constant bottomhole pressure of 100 bar. Initial reservoir pressure is 200 bar, initial temperature is 100°C; the stresses along the axes $x, y,$ and z are 300, 330, and 550 bar, respectively; and the initial strains are equal to zero.

An isotropic formation of $100 \times 100 \times 1$ m is considered. The wells are located at opposite ends of the formation's diagonal. For a more correct account of the effects arising in the near-wellbore zone, the
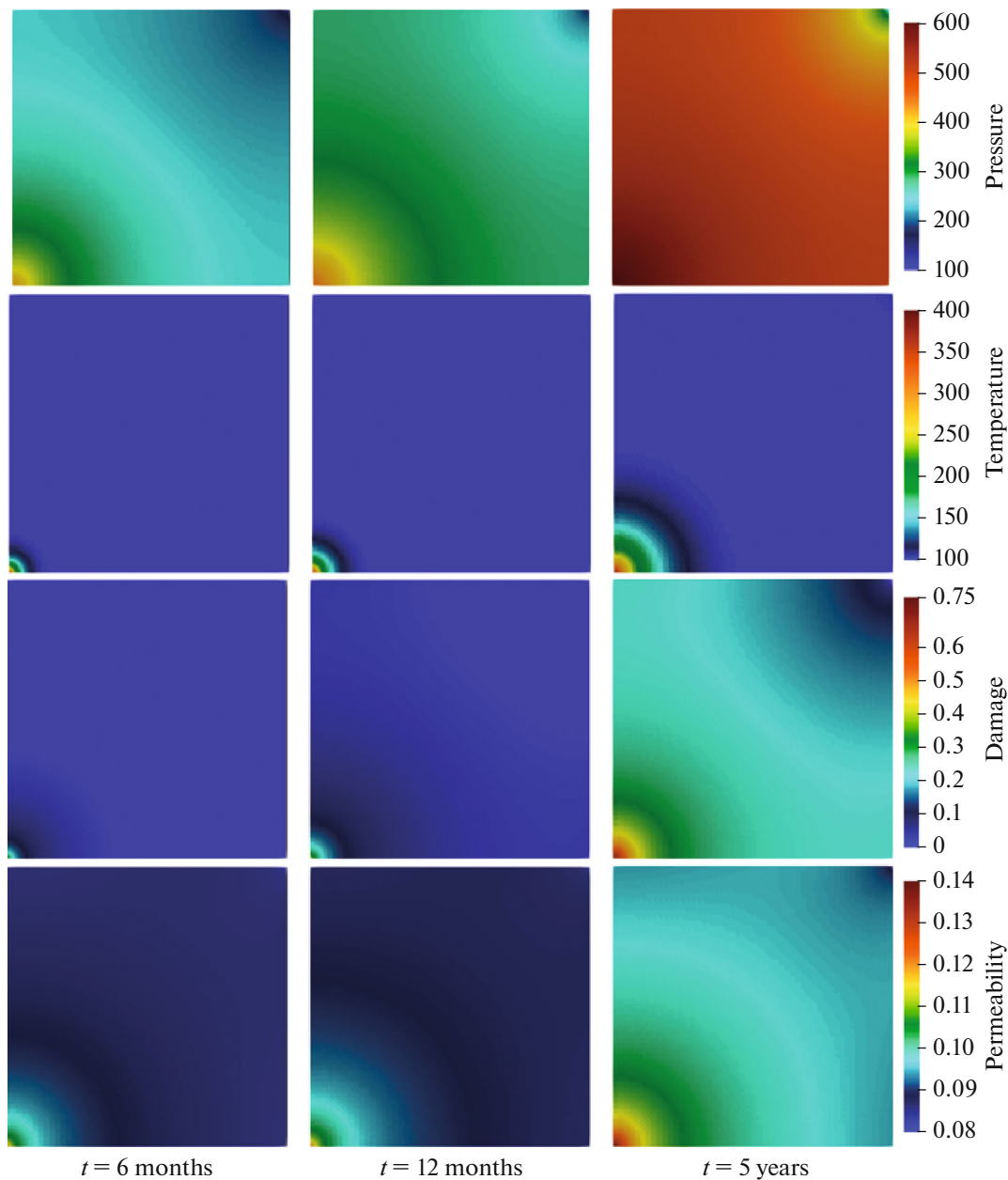
Fig. 1. Distribution of pressure p (row 1), temperature T (row 2), damage parameter D (row 3), permeability k (row 4) after 6 months (left), 12 months (center), and 5 years (right).

computational grid is locally refined along the model boundary. The mechanical boundary conditions are $\xi_x(x=0)=0$, $\xi_y(y=0)=0$, $\xi_z(z=0)=0$, $\mathbf{t}_x(x=100)=300$ bar, $\mathbf{t}_y(y=100)=330$ bar, and $\mathbf{t}_z(z=1)=550$ bar. The filtration and thermal boundary conditions include a thermally insulated impermeable boundary.

The development of the reservoir was simulated for five years. Based on the results of the calculations, the change in the pressure, temperature, damageability parameter, and permeability with time was estimated. The field distributions of the corresponding values in the lateral section after 6, 12, and 60 months are shown in Fig. 1. The distribution of the values in the vertical section is not given due to the constancy of the properties in the vertical direction.

Also, this model was calculated without taking into account the damage. Comparison of the distributions of the strain tensor component ε_{xx} at the time point of five years both in the presence and in the
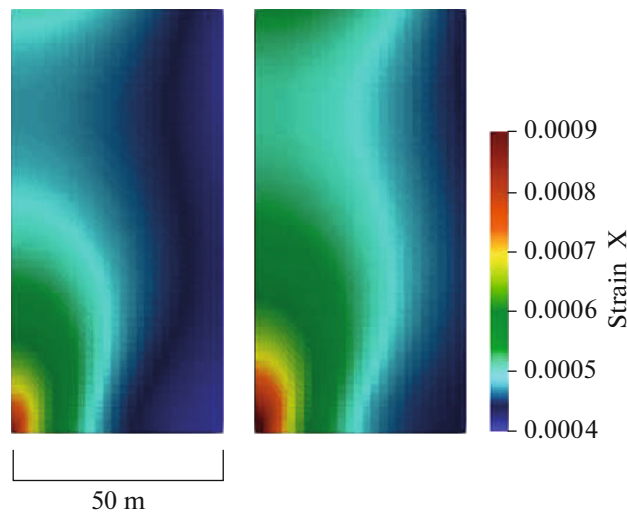
Fig. 2. The component ε_{xx} of the stress tensor at the moment of time 5 years when the damage was taken into account (left) and without the damage being taken into account (right).

absence of damage is shown in Fig. 2. It can be seen from this figure that taking into account the destruction of the rock significantly affects the calculation of the stress-strain state (for this test, the difference in the values of the strain tensor reaches 20%).

5.2. Modeling the Development of Damageability Near an Injection Well

The calculation presented below considers the damage to the bottomhole zone of an injection well when a heat transfer fluid is pumped into the formation at a high pressure at the initial moments of time. In order to assess the emerging effects, the numerical simulation was carried out for a model sized $0.5 \times 0.5 \times 0.2$ m, in the corner of which a well with a radius of 0.1 m was located, injecting fluid at a temperature of 300°C at a constant injectivity of 1 m/day. The initial and boundary conditions are similar to those used in the previous case. The permeability and Young's modulus are specified in the model in accordance with the Gaussian distribution. The computational grid was constructed taking into account the well geometry, the total number of finite elements of the grid was 59 508, and the number of nodes was 120 183.

Figure 3 shows the fields of the dynamic parameters at the time of 12 h. The calculation results show almost complete destruction of the reservoir within a radius of 0.2 m from the well. At the same time, the damage is largely caused by the increased reservoir pressure (over 800 bar), since the temperature increase in this zone is insignificant. Also, it was found that the permeability was twice as high as the initial one.

For this model, the influence of permeability on the distribution of the pressure field was assessed. In Fig. 4 the comparison of the distribution of the pressure field is presented for the case when the change in permeability of the state of the medium (on the left) is taken into account and in the case of constant permeability (on the right). As a result of the calculations, a difference in pressure of more than 50 bar was obtained.

6. CONCLUSIONS

In this paper, a mathematical model and a computational algorithm for modeling the evolution of a thermoporoelastic medium taking damage into account are presented. The model is a generalization of the Biot model and takes into account deformation, filtration, and nonisothermal effects. The damage of the medium is modeled within the continual damage theory. The system of equations consists of fundamental conservation laws and is closed by thermodynamically consistent constitutive relations.
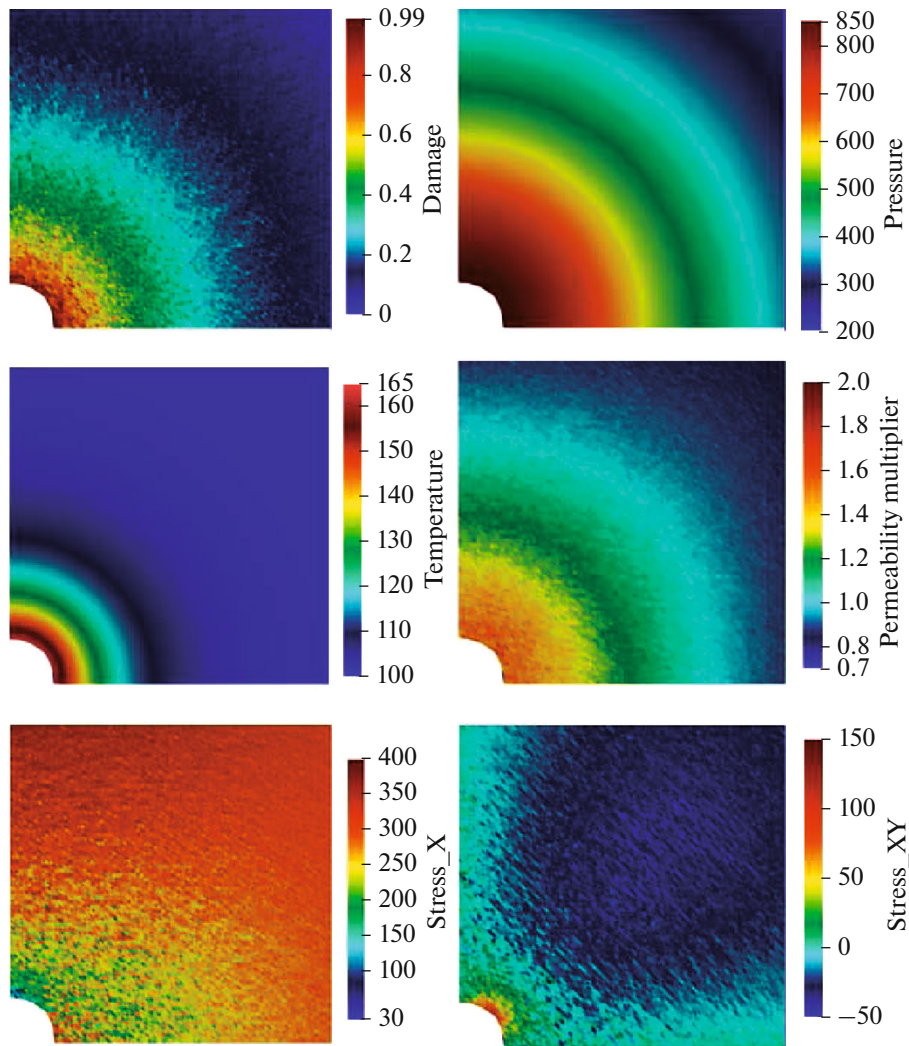
Fig. 3. The distribution of the damageability parameter D, pressure p, temperature T, ratio of current and initial permeability k/k^0, and the lateral components of the stress tensor σ_{xx} and σ_{xy} after 12 h.
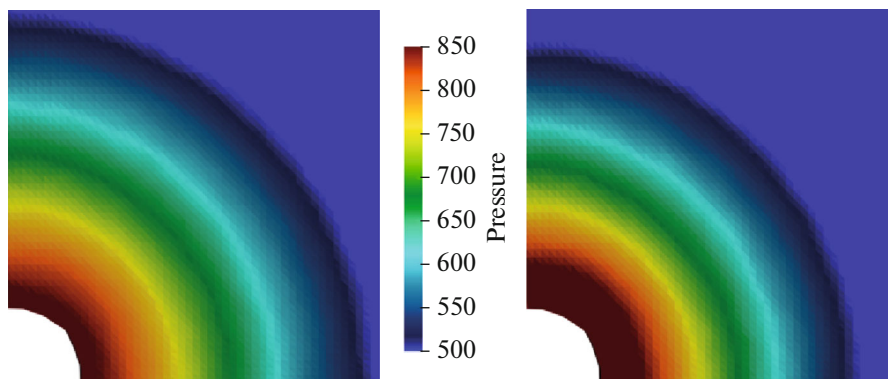

Fig. 4. Comparison of the pressure field after 12 h taking into account the change in permeability (left) and with constant permeability (right).

The system of equations is solved in a fully coupled statement and the computational algorithm is based on the finite element method. For the approximation in time, a completely implicit scheme is used, and the Taylor−Hood tetrahedral elements are used as finite elements

The operation of the algorithm is exemplified by problems of modeling thermal effects on an oil reservoir. Based on the results of the calculations, the impact of the damage was evaluated for various parameters of the wells and the formation. It was found that the damage noticeably affected the value of the elastic moduli and, as a consequence, the stress-strain state of the medium.

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