ISSN 2070-0482, Mathematical Models and Computer Simulations, 2020, Vol. 12, No. 6, pp. 883–896. © Pleiades Publishing, Ltd., 2020.
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Tensor Expansions of the Angular Particle Distribution

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Received April 8, 2019; revised April 8, 2019; accepted November 25, 2019

Abstract—We establish a relation between the class of symmetric spherical tensors and even/odd polynomials. The expansions of the scattering operator of photons or neutrons in a series of symmetric spherical tensors are obtained. Among them are expansions that have a higher rate of uniform convergence in comparison with expansions in spherical functions and Legendre polynomials. It is shown that, in problems of radiation transport in a substance with predominant forward or backward scattering, it is advisable to use expansions in the system of Chebyshev polynomials and tensors.

Keywords: photon or neutron transport equation, spherical tensors, expansions of the scattering operator, reduction of the expansion orders

DOI: 10.1134/S2070048220060150

INTRODUCTION

Radiation transport equation. The numerical simulation of nuclear reactors, radiation protection, and problems of remote diagnostics of objects in biology and technology requires the solution of the linear integro-differential equation of photon or neutron transport

$$\left[\frac{1}{v}\frac{\partial}{\partial t} + \Omega^i \frac{\partial}{\partial r^i} + \Sigma(E,\mathbf{r},t)\right]\varphi(E,\mathbf{\Omega},\mathbf{r},t) = q^s(\varphi) + q^{\text{ext}}(E,\mathbf{\Omega},\mathbf{r},t), \tag{1}$$

$$q^s(\varphi) = \int_0^\infty \frac{\nu^s(E',\mathbf{r},t)}{2\pi}\Sigma^s(E',\mathbf{r},t)\int_{4\pi} w(E' \to E,\mathbf{\Omega\Omega}')\varphi(E',\mathbf{\Omega}',\mathbf{r},t)d\mathbf{\Omega}'dE', \tag{2}$$

$$\begin{gathered}\mathbf{\Omega} = (\sqrt{1-\mu^2}\cos\alpha, \sqrt{1-\mu^2}\sin\alpha, \mu), \quad \mu = \cos\theta, \\ d\mathbf{\Omega}' = d\mu' d\alpha', \quad \int_0^\infty\int_{-1}^1 w(E' \to E,\eta,\mathbf{r},t)d\eta dE = 1.\end{gathered} \tag{3}$$

Here $\varphi(E,\mathbf{\Omega},\mathbf{r},t)$ is the particle distribution function depending on the coordinates $\mathbf{r}$, time t, energy E, and the direction $\mathbf{\Omega}$ of flight of the particles ($|\mathbf{\Omega}| = 1$); θ and α are the polar and azimuthal angles; μ is the directional cosine of vector $\mathbf{\Omega}$; $\Sigma(E,\mathbf{r},t)$ is the total cross section for the interaction of particles with a substance; q^{ext} is an external source of particles independent of the distribution function; $q^s(\varphi)$ is the source of secondary particles arising in scattering processes and reactions of particle multiplication; $\Sigma^s(E',\mathbf{r},t)$ is the total cross section of reactions; $\nu^s(E',\mathbf{r},t)$ is the number of particles at the exit of reactions; $w(E' \to E,\mathbf{\Omega\Omega}',\mathbf{r},t)$ is the indicatrix of reactions; and$\eta = \mathbf{\Omega\Omega}'$ is the cosine of the angle between the flight directions of the primary and secondary particles. It is often possible in photon transport problems to restrict oneself to allowing the processes of the conservative scattering of photons by particles of the substance. With this approach, we have

$$\nu^s(E,\mathbf{r},t) \approx 1, \quad w(E' \to E,\mathbf{\Omega\Omega}',\mathbf{r},t) \approx w(\mathbf{\Omega\Omega}',\mathbf{r},t)\delta(E'-E),$$

$$q^s(\varphi) = \frac{1}{2\pi}\Sigma^s(E,\mathbf{r},t)\int_{4\pi} w(\mathbf{\Omega\Omega}',\mathbf{r},t)\varphi(E,\mathbf{\Omega}',\mathbf{r},t)d\mathbf{\Omega}'.$$

In what follows, we will sometimes omit part of the arguments of functions if this does not contradict the understanding of formulas.

The numerical solution of Eq. (1) by direct methods, for example, by the S_n-method or by the method of characteristics is performed by iterations over the values of the source of secondary particles $q^s(\varphi)$. It is convenient to expand the source in series according to some basis functions that depend on the direction $\mathbf{\Omega}$ of the flight of the particles. The expansion coefficients are called the moments of the distribution function. Then the problem of refining the source is reduced to the problem of clarifying the moments. In many cases, this way allows us to reduce the number of arithmetic operations and the amount of stored data during the iterations. Note that the moments of the distribution function still need to be calculated. They are usually included in the final results of numerical modeling of an object.

We expand the indicatrix $w(\eta)$ in a series of the Legendre polynomials $P_n(\eta)$ as

$$\begin{gathered} w(E' \to E, \eta, \mathbf{r}, t) \approx \sum_{n=0}^{N} \frac{2n+1}{2} \omega_n(E' \to E, \mathbf{r}, t) P_n(\eta), \\ \omega_n(E' \to E, \mathbf{r}, t) = \int_{-1}^{1} w(E' \to E, \eta, \mathbf{r}, t) P_n(\eta) d\eta, \end{gathered} \tag{4}$$

where $\eta = \mathbf{\Omega\Omega}' = \mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\alpha - \alpha')$ is the cosine of the reaction angle, ω_n are the coefficients of this expansion, and N is its order. Further, using the addition theorem for spherical functions $Y_n^l(\mathbf{\Omega})$ (see [1, p. 386 of the Russian translation] and [2, p. 184 of the Russian translation]),

$$P_n(\mathbf{\Omega\Omega}') = \sum_{l=0}^{n} \frac{(n-l)!}{(n+l)!} \frac{2P_n^l(\mu)P_n^l(\mu')}{1+\delta_{0l}} \cos(l[\alpha - \alpha']) = \sum_{l=-n}^{n} \frac{2Y_n^l(\mathbf{\Omega})Y_n^l(\mathbf{\Omega}')}{1+\delta_{0l}}, \tag{5}$$

$$Y_n^l(\mathbf{\Omega}) = \sqrt{\frac{(n-|l|)!}{(n+|l|)!}} P_n^{|l|}(\mu) \begin{Bmatrix} \cos l\alpha, & 0 \le l \le n \\ \sin|l|\alpha, & -n \le l \le -1 \end{Bmatrix},$$

$$\int_{4\pi} Y_m^p(\mathbf{\Omega}) Y_n^l(\mathbf{\Omega}) d\mathbf{\Omega} = 2\pi \frac{1+\delta_{0l}}{2n+1} \delta_{mn}\delta_{pl},$$

we transform the one-dimensional series to the two-dimensional series

$$w(\mathbf{\Omega\Omega}') = \sum_{n=0}^{N} \frac{2n+1}{2} \omega_n \sum_{l=-n}^{n} \frac{2Y_n^l(\mathbf{\Omega})Y_n^l(\mathbf{\Omega}')}{1+\delta_{0l}}. \tag{6}$$

Substituting (6) into (2) gives the expansion of the source in spherical functions:

$$\begin{gathered} q^s(\varphi) \approx \sum_{n=0}^{N} \frac{2n+1}{4\pi} \sum_{l=-n}^{n} \frac{2Y_n^l(\mathbf{\Omega})}{1+\delta_{0l}} \int_0^{\infty} \nu^s(E')\Sigma^s(E')\omega_n(E' \to E) Z_n^l(E', \mathbf{r}, t) dE', \\ Z_n^l(E', \mathbf{r}, t) = \int_{4\pi} Y_n^l(\mathbf{\Omega}')\varphi(E', \mathbf{\Omega}', \mathbf{r}, t) d\mathbf{\Omega}'. \end{gathered} \tag{7}$$

The solution of Eqs. (1) and (7) can be found by iterations over the values of the moments Z_n^l.

Series (7) is the product of two expansions, namely, the expansion of the indicatrix in Legendre polynomials and the expansion of the particle distribution function in spherical functions. The series provides far from the best approximation to the source q^s as the expansion order N increases. This is especially noticeable when we solve problems with the predominant scattering of particles forward or backward, in which it is necessary to greatly increase N and therefore the complexity of the numerical algorithm to achieve the required accuracy.

The goal of the work is to construct expansions for the source $q^s(\varphi)$ of secondary particles in systems of symmetric spherical tensors. We will not restrict ourselves to expansions of the indicatrix only in Legendre polynomials and of the source only in spherical functions. The tensor expansions constructed do not depend on the choice of the coordinate system. They are convenient for analytical transformations and can be used to numerically solve the transport equation. Among the expansions, there are expansions having a high rate of uniform convergence, in particular, expansions in tensors generated by Chebyshev polynomials. It is advisable to use these expansions in problems with the predominant scattering of particles forward or backward. (It is known that the expansion of a continuous function in series in Chebyshev polynomials has a uniform convergence rate of the finite truncated series that is not slower by a factor of more than $\ln N$ than the maximum possible convergence rate in the class of polynomials; see [3, pp. 95, 448], [4, p. 111 of the Russian translation], and [5].) We will also find the source expansion in spherical functions of the usual form (7), which is the product of the expansion of the particle distribution function in spherical functions by the expansion of the indicatrix in Chebyshev polynomials.

The power moments of the distribution are the integrals

$$\Phi_n^{ijk\dots}(E,\mathbf{r},t) = \int_{4\pi} \underbrace{\Omega^i\Omega^j\Omega^k\dots}_{n} \varphi(E,\mathbf{\Omega},\mathbf{r},t)d\mathbf{\Omega}, \tag{8}$$

where $\Omega^i\Omega^j\Omega^k\dots$ is the product of the coordinates of the unit vector and n is the degree of the moment. In order not to be tied to one predetermined coordinate system, it is convenient to combine the set of moments of the same degree into the power moment tensor $\Phi_n^{ijk\dots}$, where n is both the degree and the rank of the tensor, i.e., the number of coordinate indices. The power moment tensor is a completely symmetric tensor that does not change under the permutation of any two indices: $\Phi_n^{ijk\dots} = \Phi_n^{ikj\dots} = \Phi_n^{kij\dots} = \dots.$

Tensors of form (8) composed of moments of the same degree n are not the only possible tensor characteristics of the particle distribution. To expand the concept of power moments (8), symmetric spherical tensors are introduced in Section 1. A connection between tensors and even/odd polynomials of one variable is established. In Section 2, a general expansion of the source $q^s(\varphi)$ in spherical tensors is constructed. In Section 3, we study the properties of tensor expansions generated by the standard systems of Gegenbauer, Legendre, and Chebyshev's orthogonal polynomials. Section 4 is addressed to transformations and economization (order reduction) of expansions. Section 5 gives a tensor formulation of the system of equations for the moments of the distribution function.

1. SYMMETRIC SPHERICAL TENSORS

An object whose components are linear combinations of the products of coordinates $r^i, r^j, \dots$ and Kronecker symbols δ^{kl} is a tensor over the group of linear transformations of coordinates in a three-dimensional Euclidean space. The object is not a tensor under general nonlinear coordinate transformations. If we substitute $\mathbf{r} = r\mathbf{\Omega}$, $r = |\mathbf{r}|$, $|\mathbf{\Omega}| = 1$ and consider the group of rotation transformations of the system of coordinates $r = \text{const}$, then the object's components still change according to the tensor law. A spherical tensor $M_n^{ijk\dots}(v,\mathbf{\Omega})$ is the following tensor constituted by the products of the coordinates of the unit vector $\mathbf{\Omega}$ and the Kronecker tensors δ^{ij}:

$$M_n^{\overbrace{ij\dots qkl\dots sh}^{v}}(v,\mathbf{\Omega}) = \left[a_n^{(n)} \underbrace{\Omega^i\Omega^j\dots\Omega^q}_{n} + a_{n-2}^{(n)} \underbrace{\Omega^j\dots\Omega^p}_{n-2} \delta^{iq} + \dots + a_0^{(n)}\right]\delta^{kl}\dots\delta^{sh}.$$

Subscript n denotes the maximum degree of the product of coordinates and index v among the arguments denotes the length of the set of coordinate indices $i, j, \dots, q, k, \dots, s, h$ or the rank of this tensor. We identify covariant and contravariant tensors with lower and upper coordinate indices, respectively. To multiply and contract tensors, we adhere to the convention of summation over repeated coordinate indices.

We restrict ourselves to considering completely symmetric spherical tensors with real coefficients that do not change under permutation of any two indices. For example, $A_2^{ij}(2,\mathbf{\Omega}) = 2\Omega^i\Omega^j + 3\delta^{ij}$ is a symmetric spherical tensor $(2,2)$ of the second degree and second rank; and $B_1^{ijk}(3,\mathbf{\Omega}) = \Omega^i\delta^{jk} + \Omega^j\delta^{ik} + \Omega^k\delta^{ij}$ is a symmetric spherical tensor $(1,3)$ of the first degree and third rank. Symmetric spherical tensors depending on the degree (or rank) are even/odd functions of the unit vector: $M_n^{ijk\dots}(v,\mathbf{\Omega}) = [\pm 1]^n M_n^{ijk\dots}(v,\pm\mathbf{\Omega})$. The tensor degree can take only even values $n = 0, 2, \dots, 2k$ for an even rank $v = 2k$ and odd values $n = 1, 3, \dots, 2k+1$ for an odd rank $v = 2k+1$.

An arbitrary tensor of rank v has 3^v components (the number of words of length v composed of the letters of the three-letter alphabet x, y, z). A symmetric tensor has at most $[v+2][v+1]/2$ components distinct from each other (the number of unordered sets of coordinate indices). A symmetric spherical tensor of degree n, $n \le v$, has at most $[n+2][n+1]/2$ different components. However, even among them there are dependent components, since the modulus of the argument $\mathbf{\Omega}$ is 1. If the connection $|\mathbf{\Omega}| = 1$ is allowed, then the set of components of symmetric spherical tensors becomes equivalent to the set of spherical Wigner functions (generalized spherical functions) [6].

Normalized tensors. A basis of normalized tensors can be introduced in the class of symmetric spherical tensors. A homogeneous spherical tensor is a tensor that satisfies the similarity relation with changes in the scale of coordinates:

$$M_n^{ijk\cdots}(v, r\mathbf{\Omega}) = r^n M_n^{ijk\cdots}(v, \mathbf{\Omega}). \tag{9}$$

In the given examples, $B_1^{ijk}(3, \mathbf{\Omega})$ is a homogeneous tensor and $A_2^{ij}(2, \mathbf{\Omega})$ is an inhomogeneous tensor. A normalized tensor $\Omega_n^{ijk\cdots}(v)$ is a homogeneous symmetric spherical tensor whose components do not exceed unity in modulus. Moreover, there exists a vector $\mathbf{\Omega}$ for which at least one of the components is equal to unity:

$$\left|\Omega_n^{ijk\cdots}(v)\right| \le 1, \quad \max_{\mathbf{\Omega}\in 4\pi}\left|\Omega_n^{ijk\cdots}(v)\right| = 1, \quad |\mathbf{\Omega}| = 1, \quad \mathbf{\Omega} \in 4\pi. \tag{10}$$

It can be shown that there exists only one normalized tensor of degree n and rank v. The simplest normalized tensors are the identity element $\Omega_0(0) = 1$ and the unit vector $\Omega_1^i(1) = \Omega^i$. The other tensors can be obtained from the simplest ones by multiplying them by the unit vector or the Kronecker symbol and performing the symmetrization operation:

$$\begin{aligned}\Omega_{n+1}^{ijkl\ldots q}(v+1) &= \Omega^i \frac{\Omega_n^{jkl\ldots q}(v)}{v+1} + \Omega^j \frac{\Omega_n^{ikl\ldots q}(v)}{v+1} + \ldots + \Omega^q \frac{\Omega_n^{ijkl\ldots}(v)}{v+1},\\ \Omega_0^{ijkl\ldots q}(2u+2) &= \delta^{ij} \frac{\Omega_0^{kl\ldots q}(2u)}{2u+1} + \delta^{ik} \frac{\Omega_0^{jl\ldots q}(2u)}{2u+1} + \ldots + \delta^{iq} \frac{\Omega_0^{jkl\ldots}(2u)}{2u+1}.\end{aligned} \tag{11}$$

We give several normalized low-rank tensors,

$$\begin{gathered}\Omega_0(0) = 1, \quad \Omega_1^i(1) = \Omega^i, \quad \Omega_2^{ij}(2) = \Omega^i\Omega^j, \quad \Omega_0^{ij}(2) = \delta^{ij}, \quad \Omega_3^{ijk}(3) = \Omega^i\Omega^j\Omega^k,\\ \Omega_1^{ijk}(3) = [\Omega^i\delta^{jk} + \Omega^j\delta^{ik} + \Omega^k\delta^{ij}]/3, \quad \Omega_4^{ijkl}(4) = \Omega^i\Omega^j\Omega^k\Omega^l,\\ \Omega_0^{ijkl}(4) = [\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}]/3,\\ \Omega_2^{ijkl}(4) = [\Omega^i\Omega^j\delta^{kl} + \Omega^i\Omega^k\delta^{jl} + \Omega^i\Omega^l\delta^{jk} + \Omega^j\Omega^k\delta^{il} + \Omega^j\Omega^l\delta^{ik} + \Omega^k\Omega^l\delta^{ij}]/6,\\ \Omega_1^{ijklp}(5) = [\Omega^i\Omega_0^{jklp} + \Omega^j\Omega_0^{iklp} + \Omega^k\Omega_0^{ijlp} + \Omega^l\Omega_0^{ijkp} + \Omega^p\Omega_0^{ijkl}]/5,\\ \Omega_3^{ijklp}(5) = [\Omega_3^{ijk}\delta^{lp} + \Omega_3^{ijl}\delta^{kp} + \Omega_3^{ijp}\delta^{kl} + \Omega_3^{ikl}\delta^{jp} + \Omega_3^{ikp}\delta^{jl} + \Omega_3^{ilp}\delta^{jk}]/10\\ + [\Omega_3^{jkl}\delta^{ip} + \Omega_3^{jkp}\delta^{il} + \Omega_3^{jlp}\delta^{ik} + \Omega_3^{klp}\delta^{ij}]/10.\end{gathered} \tag{12}$$

The normalized tensors satisfy the contraction rules

$$\begin{aligned}\Omega^k\Omega_n^{ijkl\cdots}(v) &= \frac{v-n}{v}\Omega_{n+1}^{ijl\ldots}(v-1) + \frac{n}{v}\Omega_{n-1}^{ijl\ldots}(v-1),\\ \delta^{ij}\Omega_n^{ijkl\cdots}(v) &= \frac{[v-n][v+n+1]}{v[v-1]}\Omega_n^{kl\ldots}(v-2) + \frac{n[n-1]}{v[v-1]}\Omega_{n-2}^{kl\ldots}(v-2),\end{aligned} \tag{13}$$

and the binary contraction rules for two tensors of different arguments $\mathbf{\Omega}, \mathbf{\Theta} \in 4\pi$:

$$\begin{aligned}(\mathbf{\Omega\Theta})^n &= \underbrace{\Omega^i\Omega^j\ldots}_{n}\underbrace{\Theta^i\Theta^j\ldots}_{n} = \Omega_n^{ijk\cdots}(n+2u)\Theta_{n+2u}^{ijk\ldots}(n+2u)\\ &= \Omega_{n+2u}^{ijk\ldots}(n+2u)\Theta_n^{ijk\cdots}(n+2u) \quad n, u = 0, 1, 2, \ldots\end{aligned} \tag{14}$$

Here are a few rules for binary contraction in expanded form:

$$1 = \Omega^i\Omega^j\delta^{ij} = \delta^{ij}\Theta^i\Theta^j = \frac{\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}}{3}\Theta^i\Theta^j\Theta^k\Theta^l = \ldots,$$
$$(\mathbf{\Omega\Theta}) = \Omega^i\Theta^i = \frac{\Omega^i\delta^{jk} + \Omega^j\delta^{ik} + \Omega^k\delta^{ij}}{3}\Theta^i\Theta^j\Theta^k = \ldots. \quad (15)$$

Conformity with the class of even/odd polynomials. An even or odd polynomial of a variable η is a finite sum of even or odd powers of the variable:

$$C_n(\eta) = \sum_{s=0}^{\backslash n/2\backslash} a_{n-2s}^{(n)}\eta^{n-2s}, \quad a_n^{(n)} \neq 0, \quad C_n(\eta) = [-1]^n C_n(-\eta). \quad (16)$$

Here $n = 0,1,2,\ldots$ is the degree of the polynomial, $a_{n-2s}^{(n)}$ are numerical coefficients, and $\backslash b\backslash$ is the integer part of a number$b$. We associate an even or odd polynomial of degree n with a series of symmetric spherical tensors with the same coefficients with the terms of the same degree:

$$C_n^{ijk\ldots}(v,\mathbf{\Omega}) = \sum_{s=0}^{\backslash n/2\backslash} a_{n-2s}^{(n)}\Omega_{n-2s}^{ijk\ldots}(v), \quad v = n, n+2, n+4, \ldots. \quad (17)$$

The tensors $\Omega_{n-2s}^{ijk\ldots}(v)$ play the role of the powers η^{n-2s}. The symmetric spherical tensor $C_n^{ijk\ldots}(n,\mathbf{\Omega})$ of rank equal to the degree $v = n$ is called the parent tensor for descendant tensors $v > n$.

The presence of coefficient conformity with the class of even/odd polynomials allows us to make the following statement: any symmetric spherical tensor of degree n and rank $v \geq n$ can be uniquely expanded into a finite sum of normalized tensors (17) of the same parity, rank, and degree at most n.

The addition theorem (an analog of the addition theorem for spherical functions (5)) follows from the contraction rules (14). Let $C_n(\eta)$ be the even/odd polynomial (16). Making the substitution $\eta = \mathbf{\Omega\Theta}$, we obtain the equalities

$$C_n(\mathbf{\Omega\Theta}) = \sum_{s=0}^{[n/2]} a_{n-2s}^{(n)}(\mathbf{\Omega\Theta})^{n-2s} = \sum_{s=0}^{[n/2]} a_{n-2s}^{(n)}\Omega_{n-2s}^{ijk\ldots}(n-2s)\Theta_{n-2s}^{ijk\ldots}(n-2s)$$
$$= \left[\sum_{s=0}^{[n/2]} a_{n-2s}^{(n)}\Omega_{n-2s}^{ijk\ldots}(n)\right]\Theta_n^{ijk\ldots}(n) = C_n^{ijk\ldots}(n,\mathbf{\Omega})\Theta_n^{ijk\ldots}(n) \quad (18)$$
$$= \Omega_n^{ijk\ldots}(n)\left[\sum_{s=0}^{[n/2]} a_{n-2s}^{(n)}\Theta_{n-2s}^{ijk\ldots}(n)\right] = \Omega_n^{ijk\ldots}(n)C_n^{ijk\ldots}(n,\mathbf{\Theta}).$$

Now we can extend the definition of moments.

The power moments of the distribution. These are the components of the tensor of degree n and rank $v = n, n+2,\ldots$ obtained by integrating the distribution with the weight of the normalized tensor:

$$\Phi_n^{ijk\ldots}(v,E,\mathbf{r},t) = \int_{4\pi}\Omega_n^{ijk\ldots}(v)\varphi(E,\mathbf{\Omega},\mathbf{r},t)d\mathbf{\Omega}, \quad v = n, n+2,\ldots \quad (19)$$

The components of the parent tensor $\Phi_n^{ijk\ldots}(n,E,\mathbf{r},t)$ whose rank is equal to its degree are the power moments of the earlier narrower definition (8).

Distribution moments. Let the system of independent even/odd polynomials $C_n(\eta)$, $n = 0,1,2,\ldots$, be given. According to (16) and (17), the polynomials generate the system of independent symmetric spherical tensors $C_n^{ijk\ldots}(v,\mathbf{\Omega})$. The distribution moments are the components of a symmetric tensor of degree n and rank $v = n, n+2,\ldots$, obtained by integrating the distribution with the weight of the spherical tensor:

$$\Lambda_n^{ijk\ldots}(v,E,\mathbf{r},t) = \int_{4\pi} C_n^{ijk\ldots}(v,\mathbf{\Omega})\varphi(E,\mathbf{\Omega},\mathbf{r},t)d\mathbf{\Omega}, \quad v = n, n+2,\ldots. \quad (20)$$

The expansion of moment (20) into power moments (19) follows from (17):

$$\Lambda_n^{ijk\ldots}(v,E,\mathbf{r},t) = \sum_{s=0}^{\backslash n/2\backslash} a_{n-2s}^{(n)}\Phi_{n-2s}^{ijk\ldots}(v,E,\mathbf{r},t). \quad (21)$$

2. TENSOR EXPANSIONS OF THE PARTICLE SOURCE

Power expansion. Let the indicatrix $w(\eta)$ be a continuous function on the interval $-1 < \eta < 1$. We approximate it by a polynomial as follows:

$$w(E' \to E, \eta, \mathbf{r}, t) \approx \sum_{n=0}^{N} f_n(E' \to E, \mathbf{r}, t)\eta^n, \quad f_N \neq 0. \tag{22}$$

Substituting (22) into (2), we obtain the expansion of the particle source in terms of the normalized tensors:

$$\begin{aligned} q^s(\varphi) &= \frac{1}{2\pi}\sum_{n=0}^{N}\int_0^{\infty} \nu^s(E')\Sigma^s(E')\int_{4\pi} f_n(E' \to E)(\mathbf{\Omega\Omega}')^n \varphi(E', \mathbf{\Omega}')d\mathbf{\Omega}'dE' \\ &= \frac{1}{2\pi}\sum_{n=0}^{N}\Omega_n^{ijk\dots}(n)\int_0^{\infty} \nu^s(E')\Sigma^s(E') f_n(E' \to E)\Phi_n^{ijk\dots}(n, E')dE'. \end{aligned} \tag{23}$$

As an approximating polynomial, we can try to use the expansion of the function in a Taylor series or the Lagrange interpolation polynomial. However, in the general case, these polynomials approximate the indicatrix nonuniformly on the interval $-1 < \eta < 1$. They also often violate additional properties of the indicatrix, for example, its positivity. A good solution is to use as (22) the polynomial of the best uniform approximation of the function $w(\eta)$ [3–5]. Unfortunately, the problem of finding such polynomials is difficult to solve.

Expansions based on orthogonal polynomials. In practice, as (22), we use finite segments of the expansion of the indicatrix in a series with respect to the system of the orthogonal polynomials $C_n(\eta)$, $n = 0, 1, \dots$. These approximations have relatively good properties of uniform convergence. We restrict ourselves to considering systems of even/odd polynomials that are generated by the even weight functions $h(\eta) = h(-\eta)$ [2–6]:

$$w(E' \to E, \eta, \mathbf{r}, t) \approx \sum_{n=0}^{N} \frac{\omega_n(E' \to E, \mathbf{r}, t)}{h_n} C_n(\eta), \tag{24}$$

$$\omega_n(E' \to E, \mathbf{r}, t) = \int_{-1}^{1} h(\eta)C_n(\eta)w(E' \to E, \eta, \mathbf{r}, t)d\eta,$$

$$C_n(\eta) = [-1]^n C_n(-\eta), \quad \int_{-1}^{1} h(\eta)C_m(\eta)C_n(\eta)d\mu = h_n\delta_{mn}.$$

Here ω_n are the expansion coefficients and h_n are the normalization factors.

Based on (24), we construct the expansions of the particle source $q^s(\varphi)$ in spherical tensors. We apply the addition theorem (18) to (24):

$$\begin{aligned} w(\mathbf{\Omega\Theta}) &\approx \sum_{n=0}^{N} \frac{\omega_n}{h_n} C_n(\mathbf{\Omega\Theta}) = \sum_{n=0}^{N} \frac{\omega_n}{h_n} \sum_{s=0}^{\backslash n/2 \backslash} a_{n-2s}^{(n)} [\mathbf{\Omega\Theta}]^{n-2s} \\ &= \sum_{n=0}^{N} \frac{\omega_n}{h_n} \left[\sum_{s=0}^{\backslash n/2 \backslash} a_{n-2s}^{(n)} \Omega_{n-2s}^{ijk\dots}(n)\right] \Theta_{ijk\dots}^{(n)}(n) = \sum_{n=0}^{N} \frac{\omega_n}{h_n} \Omega_n^{ijk\dots}(n) \left[\sum_{s=0}^{\backslash n/2 \backslash} a_{n-2s}^{(n)} \Theta_{n-2s}^{ijk\dots}(n)\right] \\ &= \sum_{n=0}^{N} \frac{\omega_n}{h_n} C_n^{ijk\dots}(n, \mathbf{\Omega}) \Theta_n^{ijk\dots}(n) = \sum_{n=0}^{N} \frac{\omega_n}{h_n} \Omega_n^{ijk\dots}(n) C_n^{ijk\dots}(n, \mathbf{\Theta}). \end{aligned}$$

The tensors $C_n^{ijk\ldots}(n,\mathbf{\Omega})$ and $C_n^{ijk\ldots}(n,\mathbf{\Theta})$ have the same coefficients with the polynomial $C_n(\mu)$. Substituting the expansion into (2), we obtain two adjoint expansion of the particle source in the system of symmetric tensors:

$$q^s(\varphi) = \sum_{n=0}^{N} \frac{C_n^{ijk\ldots}(n,\mathbf{\Omega})}{2\pi h_n} \int_0^\infty \nu^s \Sigma^s(E',\mathbf{r},t)\omega_n(E' \to E,\mathbf{r},t)\Phi_n^{ijk\ldots}(E',\mathbf{r},t)dE',$$

$$q^s(\varphi) = \sum_{n=0}^{N} \frac{\Omega_n^{ijk\ldots}(n)}{2\pi h_n} \int_0^\infty \nu^s \Sigma^s(E',\mathbf{r},t)\omega_n(E' \to E,\mathbf{r},t)\Lambda_n^{ijk\ldots}(n,E',\mathbf{r},t)dE', \tag{25}$$

$$C_n^{ijk\ldots}(n,\mathbf{\Omega}) = \sum_{s=0}^{\backslash n/2 \backslash} a_{n-2s}^{(n)} \Omega_{n-2s}^{ijk\ldots}(n), \quad \Lambda_n^{ijk\ldots}(n,E') = \int_{4\pi} C_n^{ijk\ldots}(n,\mathbf{\Theta})\varphi(\mathbf{\Theta},E')d\mathbf{\Theta}.$$

3. STANDARD TENSOR EXPANSIONS OF THE SOURCE

Gegenbauer polynomials and tensors. Let us consider the classical system of even/odd orthogonal Gegenbauer polynomials $C_n(\lambda;\mu)$

$$h(\lambda;\eta) = \frac{1}{[1-\eta^2]^{1/2-\lambda}}, \quad \lambda > -\frac{1}{2}, \tag{26}$$

$$\int_{-1}^{1} h(\lambda;\eta)C_k(\lambda;\eta)C_n(\lambda;\eta)d\eta = h_n(\lambda)\delta_{kn}, \quad h_n(\lambda) = \frac{2^{1-2\lambda}\pi\Gamma(n+2\lambda)}{\Gamma^2(\lambda)[n+\lambda]\Gamma(n+1)}.$$

Many properties of the tensors $C_n^{ijk\ldots}(\lambda;v,\mathbf{\Omega})$ and $\Lambda_n^{ijk\ldots}(\lambda;v,\mathbf{r},t)$ can be obtained from the known polynomial properties. The Gegenbauer system is remarkable due to the fact that there are explicit formulas for the coefficients of the polynomials [2–6]:

$$C_n(\lambda;\eta) = \sum_{s=0}^{\backslash n/2 \backslash} a_{n-2s}^{(n)}(\lambda)\eta^{n-2s}, \quad a_{n-2s}^{(n)}(\lambda) = \frac{[-1]^s 2^{n-2s}\Gamma(n-s+\lambda)}{\Gamma(\lambda)\Gamma(s+1)\Gamma(n-2s+1)},$$

$$C_0(\lambda;\eta) = 1, \quad C_1(\lambda;\eta) = 2\lambda\eta, \quad C_2(\lambda;\eta) = 2\lambda[1+\lambda]\eta^2 - \lambda, \ \ldots \tag{27}$$

$$\left\{C_n^{ijk\ldots}(\lambda;v,\mathbf{\Omega}), \Lambda_n^{ijk\ldots}(\lambda;v,E,\mathbf{r})\right\} = \sum_{s=0}^{\backslash n/2 \backslash} a_{n-2s}^{(n)}(\lambda)\left\{\Omega_{n-2s}^{ijk\ldots}(v), \Phi_{n-2s}^{ijk\ldots}(v,E,\mathbf{r})\right\}.$$

The expansions of the indicatrix and the particle source are given by formulas (24) and (25).

Legendre polynomials (tensors) $P_n(\mu)$ are Gegenbauer polynomials (tensors) for $\lambda = 1/2$:

$$h(1/2;\eta) = 1, \quad \int_{-1}^{1} P_k(\eta)P_n(\eta)d\mu = \frac{2\delta_{kn}}{2n+1}, \quad h_n(1/2) = \frac{2}{2n+1},$$

$$P_n(\eta) = \sum_{s=0}^{\backslash n/2 \backslash} a_{n-2s}^{(n)}(1/2)\eta^{n-2s}, \quad a_{n-2s}^{(n)}(1/2) = \frac{[-1]^s 2^{n-2s}\Gamma(n-s+1/2)}{\Gamma(1/2)\Gamma(s+1)\Gamma(n-2s+1)}, \tag{28}$$

$$\left\{P_n^{ijk\ldots}(v,\mathbf{\Omega}), \Psi_n^{ijk\ldots}(v,\mathbf{r})\right\} = \sum_{s=0}^{\backslash n/2 \backslash} a_{n-2s}^{(n)}(1/2)\left\{\Omega_{n-2s}^{ijk\ldots}(v), \Phi_n^{ijk\ldots}(v,\mathbf{r})\right\}.$$

The properties of Legendre tensors follow from (13) and (28). Contraction of the parent tensor (n, n) with the unit vector gives the parent tensor of a lower rank:

$$\Omega^k P_n^{ijk\ldots}(n,\mathbf{\Omega}) = P_{n-1}^{ij\ldots}(n-1,\mathbf{\Omega}), \quad n \geq 1.$$

Contraction of the descendant tensor with respect to any pair of indices is proportional to the descendant tensor of two ranks lower. Contraction of the parent tensor with respect to any pair of indices is zero:

Table 1. Legendre polynomials and spherical tensors

Legendre polynomial	Legendre tensors
$P_0(\eta) = 1$	$P_0(0,\mathbf{\Omega}) = \Omega_0(0) = 1,\ P_0^{ijk\ldots}(2u,\mathbf{\Omega}) = \Omega_0^{ijk\ldots}(2u)$
$P_1(\eta) = \eta$	$P_1^i(1,\mathbf{\Omega}) = \Omega_1^i(1) = \Omega^i,\ P_1^{ijk\ldots}(2u+1,\mathbf{\Omega}) = \Omega_1^{ijk\ldots}(2u+1)$
$P_2(\eta) = \frac{3}{2}\eta^2 - \frac{1}{2}$	$P_2^{ij}(2,\mathbf{\Omega}) = \frac{3}{2}\Omega_2^{ij}(2) - \frac{1}{2}\Omega_0^{ij}(2) = \frac{3}{2}\Omega^i\Omega^j - \frac{\delta^{ij}}{2}$, $P_2^{ijk\ldots}(2u+2,\mathbf{\Omega}) = \frac{3}{2}\Omega_2^{ijk\ldots}(2u+2) - \frac{1}{2}\Omega_2^{ijk\ldots}(2u+2)$
$P_3(\eta) = \frac{5}{2}\eta^3 - \frac{3}{2}\eta$	$P_3^{ijk\ldots}(2u+3,\mathbf{\Omega}) = \frac{5}{2}\Omega_3^{ijk\ldots}(2u+3) - \frac{3}{2}\Omega_1^{ijk\ldots}(2u+3)$
$P_4(\eta) = \frac{35}{8}\eta^4 - \frac{30}{8}\eta^2 + \frac{3}{8}$	$P_4^{ijk\ldots}(v,\mathbf{\Omega}) = \frac{35}{8}\Omega_4^{ijk\ldots}(v) - \frac{30}{8}\Omega_2^{ijk\ldots}(v) + \frac{3}{8}\Omega_0^{ijk\ldots}(v)$, $v = 2u+4$

$$\begin{aligned} \delta^{jk} P_n^{ijkl\ldots}(v,\mathbf{\Omega}) &= \frac{[v-n][v+n+1]}{v[v-1]} P_n^{il\ldots}(v-2,\mathbf{\Omega}), \quad \delta^{jk} P_n^{ijkl\ldots}(n,\mathbf{\Omega}) = 0, \\ \delta^{jk}\Psi_n^{ijkl\ldots}(v,E,\mathbf{r}) &= \frac{[v-n][v+n+1]}{v[v-1]}\Psi_n^{il\ldots}(v-2,E,\mathbf{r}), \quad \delta^{jk}\Psi_n^{ijkl\ldots}(n,E,\mathbf{r}) = 0. \end{aligned} \tag{29}$$

It is important to note that the parent tensors are deviator tensors with zero trace. The expansions of the indicatrix and the source of particles in terms of Legendre polynomials and tensors are obtained by substituting (28) into (24) and (25):

$$\begin{aligned} q^s(\varphi) &\approx \sum_{n=0}^{N} \frac{2n+1}{4\pi} P_n^{ijk\ldots}(n,\mathbf{\Omega}) \int_0^\infty \nu^s \Sigma^s(E',\mathbf{r})\omega_n(E' \to E,\mathbf{r})\Phi_n^{ijk\ldots}(n,E',\mathbf{r})dE' \\ &\approx \sum_{n=0}^{N} \frac{2n+1}{4\pi} \Omega_n^{ijk\ldots}(n) \int_0^\infty \nu^s \Sigma^s(E',\mathbf{r})\omega_n(E' \to E,\mathbf{r})\Psi_n^{ijk\ldots}(n,E',\mathbf{r})dE' \\ &\approx \sum_{n=0}^{N} \frac{2n+1}{4\pi} \frac{P_n^{ijk\ldots}(\mathbf{\Omega})}{a_n^{(n)}} \int_0^\infty \nu^s \Sigma^s(E',\mathbf{r})\omega_n(E' \to E,\mathbf{r})\Psi_n^{ijk\ldots}(n,E',\mathbf{r})dE'. \end{aligned} \tag{30}$$

The third expansion in (30) follows from the vanishing of the trace of deviator moments (29). Expansions (30) are equivalent to expansions (6) and (7) of the indicatrix and the source in the system of spherical functions.

If we take $w(\eta) = \delta(\eta - 1)\delta(E' - E)$ in (2), (30), we get the expansions of the particle distribution function in Legendre tensors:

$$\begin{aligned} \varphi(E,\mathbf{\Omega},\mathbf{r},t) &\simeq \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} P_n^{ijk\ldots}(n,\mathbf{\Omega})\Phi_n^{ijk\ldots}(n,E,\mathbf{r},t) \\ &\simeq \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \Omega_n^{ijk\ldots}(n)\Psi_n^{ijk\ldots}(n,E,\mathbf{r},t) \\ &\simeq \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \frac{P_n^{ijk\ldots}(n,\mathbf{\Omega})}{a_n^{(n)}} \Psi_n^{ijk\ldots}(n,E,\mathbf{r},t). \end{aligned} \tag{31}$$

The system of Legendre polynomials and tensors is distinguished by its simplicity and by the presence of the tensor expansion (31) of the distribution function.

Chebyshev polynomials (tensors) $T_n(\mu)$ are Gegenbauer polynomials (tensors) for $\lambda = 0$ (see [2, p. 186 of the Russian translation] and [4, p. 19 of the Russian translation]):

Table 2. Chebyshev polynomials and spherical tensors

Chebyshev polynomial	Chebyshev tensors
$T_0(\eta) = 1$	$T_0(0,\mathbf{\Omega}) = \Omega_0(0) = 1,\ T_0^{ijk\ldots}(2u,\mathbf{\Omega}) = \Omega_0^{ijk\ldots}(2u)$
$T_1(\eta) = \eta$	$T_1^i(1,\mathbf{\Omega}) = \Omega_1^i(1) = \Omega^i,\ T_1^{ijk\ldots}(2u+1,\mathbf{\Omega}) = \Omega_1^{ijk\ldots}(2u+1)$
$T_2(\eta) = 2\eta^2 - 1$	$T_2^{ijk\ldots}(2u+2,\mathbf{\Omega}) = 2\Omega_2^{ijk\ldots}(2u+2) - \Omega_2^{ijk\ldots}(2u+2)$
$T_3(\eta) = 4\eta^3 - 3\eta$	$T_3^{ijk\ldots}(2u+3,\mathbf{\Omega}) = 4\Omega_3^{ijk\ldots}(2u+3) - 3\Omega_1^{ijk\ldots}(2u+3)$
$T_4(\eta) = 8\eta^4 - 8\eta^2 + 1$	$T_4^{ijk\ldots}(v,\mathbf{\Omega}) = 8\Omega_4^{ijk\ldots}(v) - 8\Omega_2^{ijk\ldots}(v) + \Omega_0^{ijk\ldots}(v),\ v = 2u+4$

$$h(0,\eta) = \frac{1}{\sqrt{1-\eta^2}}, \quad \int_{-1}^{1} \frac{T_k(\eta)T_n(\eta)}{\sqrt{1-\eta^2}} d\eta = h_n(0)\delta_{kn}, \quad h_n(0) = \frac{\pi}{2}[1+\delta_{0n}]. \tag{32}$$

$$T_n(\eta) = \lim_{\lambda\to 0} \frac{\Gamma(\lambda)[n+\lambda]}{2} C_n(\lambda,\eta) = \cos(n \arccos\eta) = \sum_{s=0}^{\backslash n/2 \backslash} a_{n-2s}^{(n)} \eta^{n-2s},$$

$$a_{n-2s}^{(n)} = \lim_{\lambda\to 0} \frac{\Gamma(\lambda)[n+\lambda]}{2} a_{n-2s}^{(n)}(\lambda) = \frac{[-1]^s 2^{n-2s-1} n\Gamma(n-s)}{\Gamma(s+1)\Gamma(n-2s+1)},$$

$$\left\{T_n^{ijk\ldots}(v,\mathbf{\Omega}), \tau_n^{ijk\ldots}(v,\mathbf{r})\right\} = \sum_{s=0}^{\backslash n/2 \backslash} a_{n-2s}^{(n)} \left\{\Omega_{n-2s}^{ijk\ldots}(v), \Phi_{n-2s}^{ijk\ldots}(v,\mathbf{r})\right\}.$$

The properties of Chebyshev spherical tensors follow from (13) and (32). Contraction of the parent tensor with the unit vector gives the parent tensor of rank lower; contraction of the tensor with respect to two indices is the Chebyshev tensor of the second kind $U_{n-2}^{ij\ldots}$ (Gegenbauer spherical tensor for $\lambda = 1$):

$$\Omega_k T_n^{ijk\ldots}(n,\mathbf{\Omega}) = T_{n-1}^{ij\ldots}(n-1,\mathbf{\Omega}), \quad n \ge 1,$$
$$\delta^{kl} T_n^{ijkl\ldots}(n,\mathbf{\Omega}) = -\frac{U_{n-2}^{ij\ldots}(n-2,\mathbf{\Omega})}{n-1}, \quad \delta^{kl}\tau_n^{ijkl\ldots}(n,\mathbf{r}) = -\frac{\Lambda_n^{ijkl\ldots}(1;n-2,\mathbf{r})}{n-1}, \tag{33}$$

The expansions of the particle source in Chebyshev polynomials and tensors are obtained by substituting (32) into (24) and (25):

$$q^s(\varphi) \approx \sum_{n=0}^{N} \frac{T_n^{ijk\ldots}(n,\mathbf{\Omega})}{\pi^2[1+\delta_{0n}]} \int_0^\infty \nu^s \Sigma^s(E',\mathbf{r},t)\omega_n(E' \to E,\mathbf{r},t)\Phi_n^{ijk\ldots}(n,E',\mathbf{r},t)dE'$$
$$\approx \sum_{n=0}^{N} \frac{\Omega_n^{ijk\ldots}(n)}{\pi^2[1+\delta_{0n}]} \int_0^\infty \nu^s \Sigma^s(E',\mathbf{r},t)\omega_n(E' \to E,\mathbf{r},t)\tau_n^{ijk\ldots}(n,E',\mathbf{r},t)dE', \tag{34}$$
$$\omega_n(E' \to E,\mathbf{r},t) = \int_{-1}^{1} \frac{w(E' \to E,\eta,\mathbf{r},t)T_n(\eta)}{\sqrt{1-\eta^2}} d\eta.$$

If we consider a wide class of functions $w(E' \to E,\eta,\mathbf{r},t)$ varying in space, in time, or from problem-to-problem, then the rate of convergence of expansion (24) and expansions of source (25) can vary depending on the changes in the function and the choice of the system of polynomials. The system of Chebyshev polynomials differs in the fact that it has a stably high rate of uniform convergence as order N increases. If we expand a bounded function continuous on $[-1,1]$, the rate of convergence is slower by a factor of at most $\ln N$ than the maximum possible rate in the class of polynomials [3, pp. 95, 448], [4, p. 111 of the Russian translation].

Comparison of the rates of convergence with an example of the Henyey-Greenstein indicatrix. As an example, we consider the Henyey–Greenstein indicatrix, which is often used to model the predominant forward-backward scattering of photons:

Table 3. The values $K_{2n}(g)$ for the expansion in Legendre polynomials

g	n										
	0	2	4	6	8	10	12	14	16	18	20
0.85	1.00	3.61	4.70	4.90	4.63	4.13	3.56	2.98	2.45	1.98	1.59
0.80	1.00	3.20	3.69	3.41	2.85	2.25	1.72	1.28	0.93	0.67	0.47
0.70	1.00	2.45	2.16	1.53	0.98	0.59	0.35	0.20	0.11	0.06	0.03
0.60	1.00	1.80	1.17	0.61	0.29	0.13	0.05	0.02	0.01		
0.50	1.00	1.25	0.56	0.20	0.07	0.02	0.01				
0.40	1.00	0.80	0.23	0.05	0.01						
0.30	1.00	0.45	0.07	0.01							
0.20	1.00	0.20	0.01								
0.10	1.00	0.05									
0.00	1.00										
g	n										
	22	24	26	28	30	32	34	36	38	40	
0.85	1.26	0.99	0.77	0.60	0.47	0.36	0.27	0.21	0.16	0.12	
0.80	0.33	0.23	0.16	0.11	0.08	0.05	0.03	0.02	0.02	0.01	
0.70	0.02	0.01									

$$w(\eta) = \frac{[1-\gamma][1-g^2]}{2[1-2g\eta+g^2]^{3/2}} + \frac{\gamma[1-g^2]}{2[1+2g\eta+g^2]^{3/2}}, \quad 0 \le g < 1. \tag{35}$$

The closer the parameter g to unity the higher the anisotropy of the scattering. For $g = 0$, the scattering is isotropic. The parameter γ, $0 \le \gamma \le 1$, sets the weight of backward scattering. For $\gamma = 0$, the scattering occurs mainly forward from $\eta \approx 1$. The expansion of the indicatrix in Legendre polynomials is

$$w(\eta) \approx \sum_{n=0}^{N} \frac{2n+1}{2}\omega_n P_n(\eta), \quad \omega_n = \int_{-1}^{1} w(\mu)P_n(\mu)d\mu = \left[1-\gamma+[-1]^n\gamma\right]g^n. \tag{36}$$

The expansion of the indicatrix in Chebyshev polynomials is

$$w(\eta) \approx \sum_{n=0}^{N} \frac{2\varepsilon_n}{\pi[1+\delta_{0n}]} T_n(\eta), \quad \varepsilon_n = \sum_{k=0}^{\infty} D_{nk}\omega_{n+2k},$$
$$D_{nk} = \frac{[n+2k+1/2]\Gamma(k+1/2)\Gamma(n+k+1/2)}{\Gamma(k+1)\Gamma(n+k+1)}. \tag{37}$$

We estimate the rate of convergence of expansions (36) and (37) using the ratio of the maximum of the nth harmonic to the first harmonic of the corresponding series as follows:

$$K_{2n}(g) = [4n+1]\frac{\omega_{2n}}{\omega_0} = [4n+1]g^{2n}, \quad K_{2n}(g) = \frac{2}{1+\delta_{0n}}\frac{\varepsilon_{2n}}{\varepsilon_0}.$$

The ratio values are shown in Tables 3 and 4.

As can be seen, with high scattering anisotropy, the expansion in Chebyshev polynomials converges one-and-a-half to two times faster than the expansion in Legendre polynomials. Therefore, when we solve problems of radiation transfer in a substance with predominant forward or backward scattering, it is advisable to use the expansion of the indicatrix in Chebyshev polynomials.

Formulas for the transformation of polynomials and tensors into a finite sum of polynomials and tensors of another system. We conclude the section with a number of useful auxiliary formulas, which we will use in the next section to transform the expansions from one system of classical polynomials to another.

Table 4. The values $K_{2n}(g)$ for the expansion in Chebyshev polynomials

g	n										
	0	2	4	6	8	10	12	14	16	18	20
0.85	1.00	1.80	1.50	1.21	0.96	0.75	3.56	0.44	2.45	0.25	0.19
0.80	1.00	1.68	1.28	0.92	0.65	0.45	1.72	0.21	0.34	0.10	0.06
0.70	1.00	1.41	0.85	0.48	0.26	0.14	0.35	0.04	0.14	0.01	0.01
0.60	1.00	1.12	0.50	0.21	0.09	0.03	0.05	0.01	0.02		
0.50	1.00	0.82	0.26	0.08	0.02	0.01	0.01				
0.40	1.00	0.55	0.11	0.02							
0.30	1.00	0.32	0.04								
0.20	1.00	0.15	0.01								
0.10	1.00	0.04									
0.00	1.00										

(1) The expansion of the Gegenbauer polynomial (tensor) of the system λ into a finite sum of polynomials (tensors) of the system ξ is

$$\begin{Bmatrix} C_n(\lambda;\eta), C_n^{ijk\ldots}(\lambda;\nu,\mathbf{\Omega}) \\ \Lambda_n^{ijk\ldots}(\lambda;\nu,\mathbf{r},t) \end{Bmatrix} = \sum_{s=0}^{\backslash n/2 \backslash} d_{n-2s}^{(n)}(\lambda \to \xi) \begin{Bmatrix} C_{n-2s}(\xi;\eta), C_{n-2s}^{ijk\ldots}(\xi;\nu,\mathbf{\Omega}) \\ \Lambda_{n-2s}^{ijk\ldots}(\xi;\nu,\mathbf{r},t) \end{Bmatrix},$$
$$d_{n-2s}^{(n)}(\lambda \to \xi) = \frac{\Gamma(\xi)[n-2s+\xi]\Gamma(s+\lambda-\xi)\Gamma(n-s+\lambda)}{\Gamma(\lambda)\Gamma(\lambda-\xi)\Gamma(s+1)\Gamma(n-s+\xi+1)}; \tag{38}$$

see [8, pp. 566, 570 of the Russian translation] (after transformation). Expansions of the Legendre polynomial and tensor follow from (38), if we set $\lambda, \xi = 1/2$.

(2) The expansion of the Gegenbauer polynomial (tensor) of the system λ into a finite sum of Chebyshev polynomials (tensors) is

$$\begin{Bmatrix} C_n(\lambda;\eta) \\ C_n^{ijk\ldots}(\lambda;\nu,\mathbf{\Omega}) \\ \Lambda_n^{ijk\ldots}(\lambda;\nu,\mathbf{r}) \end{Bmatrix} = \sum_{s=0}^{\backslash n/2 \backslash} d_{n-2s}^{(n)} \begin{Bmatrix} T_{n-2s}(\eta) \\ T_{n-2s}^{ijk\ldots}(\nu,\mathbf{\Omega}) \\ \tau_{n-2s}^{ijk\ldots}(\nu,\mathbf{r}) \end{Bmatrix} = \frac{1}{2}\sum_{s=0}^{n} e_{|n-2s|}^{(n)} \begin{Bmatrix} T_{|n-2s|}(\eta) \\ T_{|n-2s|}^{ijk\ldots}(\nu,\mathbf{\Omega}) \\ \tau_{|n-2s|}^{ijk\ldots}(\nu,\mathbf{r}) \end{Bmatrix},$$
$$d_{n-2s}^{(n)}(\lambda \to 0) = \frac{e_{n-2s}^{(n)}}{1+\delta_{2s,n}}, \quad e_{n-2s}^{(n)} = \frac{2\Gamma(s+\lambda)\Gamma(n-s+\lambda)}{\Gamma^2(\lambda)\Gamma(s+1)\Gamma(n-s+1)}. \tag{39}$$

see [2, pp. 177, 181 of the Russian translation], [3, pp. 121, 263], [5, pp. 103, 105 of the Russian translation], and [7, p. 517 of the Russian translation]. Expansions of the Legendre polynomial and tensor follow from (39), if we set $\lambda = 1/2$.

(3) Substituting (5) into (38) yields the expansion in spherical functions:

$$C_n(\lambda;\mathbf{\Omega}\mathbf{\Omega}') = \sum_{s=0}^{\backslash n/2 \backslash} d_{n-2s}^{(n)}(\lambda \to 1/2) \sum_{|l| \le n-2s} \frac{2Y_{n-2s}^{l}(\mathbf{\Omega})Y_{n-2s}^{l}(\mathbf{\Omega}')}{1+\delta_{0l}},$$
$$d_{n-2s}^{(n)}(\lambda \to 1/2) = \frac{\Gamma(1/2)[n-2s+1/2]\Gamma(s+\lambda-1/2)\Gamma(n-s+\lambda)}{\Gamma(\lambda)\Gamma(\lambda-1/2)\Gamma(s+1)\Gamma(n-s+3/2)}. \tag{40}$$

(4) The expansion of the Chebyshev polynomial and tensor ($\lambda = 0$) into a finite sum of Gegenbauer polynomials and tensors of the system ξ follows from (32) and (38):

$$\begin{Bmatrix} T_n(\eta) \\ T_n^{ijk\ldots}(\nu,\mathbf{\Omega}) \\ \tau_{ijk\ldots}^{(n)}(\nu,\mathbf{r},t) \end{Bmatrix} = [1+\delta_{0n}] \sum_{s=0}^{\backslash n/2 \backslash} d_{n-2s}^{(n)}(0 \to \xi) \begin{Bmatrix} C_{n-2s}(\xi;\eta) \\ C_{n-2s}^{ijk\ldots}(\xi;\eta) \\ \Lambda_{ijk\ldots}^{(n-2s)}(\xi;\eta,\mathbf{r},t) \end{Bmatrix},$$
$$d_{n-2s}^{(n)}(0 \to \xi) = \frac{n[n-2s+\xi]\Gamma(\xi)\Gamma(s-\xi)\Gamma(n-s)}{2\Gamma(-\xi)\Gamma(s+1)\Gamma(n-s+\xi+1)}. \tag{41}$$

(5) Substituting (5) into (41) gives the expansion in spherical functions:

$$T_n(\mathbf{\Omega\Omega'}) = [1+\delta_{0n}]\sum_{s=0}^{\backslash n/2\backslash} d_{n-2s}^{(n)}(0\to 1/2)\sum_{|l|\le n-2s}\frac{2Y_{n-2s}^l(\mathbf{\Omega})Y_{n-2s}^l(\mathbf{\Omega'})}{1+\delta_{0l}}. \quad (42)$$

A more complete list of transformation formulas is presented in [9].

4. TRANSFORMATION AND ECONOMIZATION OF EXPANSIONS

This section presents the economization (order reduction) method for the expansion of the particles' source by using the expansion transformations.

Suppose that the coefficients $\omega_n(\lambda)$, $0\le n\le N$, of the expansion of the indicatrix in series (24) in terms of the Gegenbauer polynomials $C_n(\lambda;\eta)$ of the system λ are known. Using the finite sums (38)-(41), we can calculate the coefficients $\varepsilon_n(\xi)$ of the expansion of the indicatrix in the polynomials $C_n(\xi;\eta)$ of the system $\xi\ne\lambda$. Substituting (38) and (39) into (24) and interchanging the order of summation, we obtain

$$w(\eta)\approx\sum_{n=0}^{N}\frac{\omega_n(\lambda)}{h_n(\lambda)}C_n(\lambda;\eta)\underset{\lambda\to\xi}{\approx}\sum_{n=0}^{N}\frac{\varepsilon_n(\xi)}{h_n(\xi)}\begin{Bmatrix}C_n(\xi;\eta)\\ T_n(\eta),\ \xi=0\end{Bmatrix}. \quad (43)$$

In a similar way, the expansion in Chebyshev polynomials can be transformed:

$$w(\eta)\approx\sum_{n=0}^{N}\frac{2\omega_n}{\pi[1+\delta_{0n}]}T_n(\eta)\underset{0\to\xi}{\approx}\sum_{n=0}^{N}\frac{\varepsilon_n(\xi)}{h_n(\xi)}C_n(\xi;\eta). \quad (44)$$

The coefficients of the transformed expansion are calculated by the formulas:

$$\varepsilon_n(\xi)=\sum_{k=0}^{\backslash[N-n]/2\backslash}D_{nk}(\lambda\to\xi)\,\omega_{n+2k}(\lambda), \quad (45)$$

$$D_{nk}(\lambda\to\xi)=\frac{\Gamma(\lambda)\Gamma(n+2\xi)[n+2k+\lambda]\Gamma(k+\lambda-\xi)\Gamma(n+k+\lambda)\Gamma(n+2k+1)}{2^{2\xi-2\lambda}\Gamma(\xi)\Gamma(\lambda-\xi)\Gamma(n+1)\Gamma(k+1)\Gamma(n+k+\xi+1)\Gamma(n+2k+2\lambda)},$$

$$D_{nk}(\lambda\to 0)=\frac{\pi e_n^{(n+2k)}}{2h_{n+2k}(\lambda)}=\frac{[n+2k+\lambda]\Gamma(k+\lambda)\Gamma(n+k+\lambda)\Gamma(n+2k+1)}{2^{1-2\lambda}\Gamma(k+1)\Gamma(n+k+1)\Gamma(n+2k+2\lambda)},$$

$$D_{nk}(0\to\xi)=-\frac{\xi\sin(\pi\xi)\Gamma(n+2\xi)}{2^{2\xi-1}\pi\Gamma(n+1)}\frac{[n+2k]\Gamma(k-\xi)\Gamma(n+k)}{[1+\delta_{0,n+k}]\Gamma(k+1)\Gamma(n+k+\xi+1)},$$

$$D_{nk}(0\to 1/2)=-\frac{1}{2\pi[1+\delta_{0,n+k}]}\frac{[n+2k]\Gamma(k-1/2)\Gamma(n+k)}{\Gamma(k+1)\Gamma(n+k+3/2)}.$$

The transition matrix elements D_{nk} can be easily calculated in Excel, MathLab, and Mathematica programs. After the transformation, it is necessary to correct the first coefficient $\varepsilon_0(\xi)$ of the expansion so that normalization (3) is performed, which ensures the implementation of the particles' conservation law.

The substitution of (43) and (44) into the particle source (2) and the application of the conformity between polynomials and symmetric spherical tensors (16)–(18) give the expansions

$$\begin{aligned}q^s(\varphi)&\approx\sum_{n=0}^{N}\frac{C_n^{ijk\ldots}(\xi;n,\mathbf{\Omega})}{2\pi h_n(\xi)}\int_0^\infty\nu^s\Sigma^s(E',\mathbf{r})\,\varepsilon_n(E'\to E,\mathbf{r})\Phi_n^{ijk\ldots}(E',\mathbf{r})dE'\\&\approx\sum_{n=0}^{N}\frac{\Omega_n^{ijk\ldots}(n)}{2\pi h_n(\xi)}\int_0^\infty\nu^s\Sigma^s(E',\mathbf{r})\,\varepsilon_n(E'\to E,\mathbf{r})\Lambda_n^{ijk\ldots}(\xi;n,E',\mathbf{r})dE'.\end{aligned} \quad (46)$$

In the case of the transformation $\lambda\to 1/2$, we find

$$q^s(\varphi) \approx \sum_{n=0}^{N} \frac{2n+1}{4\pi} P_n^{ijk\ldots}(n,\mathbf{\Omega}) \int_0^\infty \nu^s \Sigma^s(E',\mathbf{r}) \varepsilon_n(E' \to E,\mathbf{r}) \Phi_n^{ijk\ldots}(E',\mathbf{r}) dE'$$
$$\approx \sum_{n=0}^{N} \frac{2n+1}{4\pi} \Omega_n^{ijk\ldots}(n) \int_0^\infty \nu^s \Sigma^s(E',\mathbf{r}) \varepsilon_n(E' \to E,\mathbf{r}) \Psi_n^{ijk\ldots}(n,E',\mathbf{r}) dE' \quad (47)$$
$$\approx \sum_{n=0}^{N} \frac{2n+1}{4\pi} \frac{P_n^{ijk\ldots}(\mathbf{\Omega})}{a_n^{(n)}} \int_0^\infty \nu^s \Sigma^s(E',\mathbf{r}) \varepsilon_n(E' \to E,\mathbf{r}) \Psi_n^{ijk\ldots}(n,E',\mathbf{r}) dE'.$$

Using the addition theorem for spherical functions (5) and (6) gives the expansion in spherical functions

$$q^s(\varphi) \approx \sum_{n=0}^{N} \frac{2n+1}{4\pi} \sum_{l=-n}^{n} \frac{2Y_n^l(\mathbf{\Omega})}{1+\delta_{0l}} \int_0^\infty \nu^s \Sigma^s(E',\mathbf{r}) \varepsilon_n(E' \to E,\mathbf{r}) Z_n^l(E',\mathbf{r})\, dE'. \quad (48)$$

At first glance, (46)–(48) do not differ from expansions (25), (30), and (7). However, this is not the case. Expansions (46)–(48) are of order N, which is equal to the order of the original expansion (43) of the indicatrix in the system $C_n(\eta,\lambda)$ or that of expansion (44) in the system $T_n(\eta)$. The coefficients ε_n of the transformed expansion are combinations (45) of the original coefficients ω_n. Therefore, in the case of the transformation $\lambda = 0 \to \xi = 1/2$, expansion (47) is the product of the expansion of the indicatrix in Chebyshev polynomials by expansion (31) of the particle distribution in terms of Legendre tensors.

The choice of a good expansion of the indicatrix (for example, according to the Chebyshev system) allows us to reduce the order of the final expansions (46)–(48). This method is similar to the known economization method for power series by transforming them into an expansion in the Chebyshev system.

5. THE SYSTEM OF EQUATIONS FOR POWER MOMENTS

In this section, a tensor formulation of the system of equations for the power moments of the particle distribution function is given. Solving the system of equations provides an alternative way to find the particle distribution. The particle distribution function is restored by formula (31).

If we multiply the particle transport equation (1) by the components of the spherical Legendre tensors $P_n^{ijk\ldots}(n,\mathbf{\Omega})$ and integrate the result over the angular variables, we obtain an infinite system of equations for the power deviator moments $\Psi_n^{ijk\ldots}(n)$ (29):

$$\frac{1}{v}\frac{\partial \Psi_0}{\partial t} + \frac{\partial \Psi_1^i}{\partial r^i} + \Sigma\Psi_0 = F_0 + S_0,$$
$$\frac{1}{v}\frac{\partial \Psi_1^i}{\partial t} + \frac{\partial}{\partial r^j}\left[\frac{2}{3}\Psi_2^{ij} + \Psi_0\frac{\delta^{ij}}{3}\right] + \Sigma\Psi_1^i = S_1^i,$$
$$\frac{1}{v}\frac{\partial \Psi_2^{ij}}{\partial t} + \frac{3}{5}\frac{\partial \Psi_3^{ijk}}{\partial r^k} + \frac{\partial}{\partial r^k}\left[\frac{3}{5}\frac{\Psi_1^i\delta^{jk} + \Psi_1^j\delta^{ik}}{2} - \frac{2}{5}\frac{\Psi_1^k\delta^{ij}}{2}\right] + \Sigma\Psi_2^{ij} = S_2^{ij}, \quad (49)$$
$$\frac{1}{v}\frac{\partial \Psi_3^{ijk}}{\partial t} + \frac{4}{7}\frac{\partial \Psi_4^{ijkl}}{\partial r^l} + \frac{5}{7}\frac{\partial}{\partial r^l}\frac{\Psi_2^{ij}\delta^{kl} + \Psi_2^{ik}\delta^{jl} + \Psi_2^{jk}\delta^{il}}{3}$$
$$- \frac{2}{7}\frac{\partial}{\partial r^l}\frac{\Psi_2^{il}\delta^{jk} + \Psi_2^{jl}\delta^{ik} + \Psi_2^{kl}\delta^{ij}}{3} + \Sigma\Psi_3^{ijk} = S_3^{ijk}, \ldots .$$

The right-hand sides of the equations are equal to

$$S_n^{ijk\ldots} = \int_0^\infty \nu^s \Sigma^s(E',\mathbf{r},t) \omega_n(E' \to E,\mathbf{r},t) \Psi_n^{ijkl}(n,E',\mathbf{r},t)\, dE'$$
$$+ \int_{4\pi} P_n^{ijk\ldots}(n,\mathbf{\Omega})\, q^{\text{ext}}(E,\mathbf{\Omega},\mathbf{r},t) d\mathbf{\Omega}. \quad (50)$$

When we apply the method of economizing the expansions, the coefficients ω_n are replaced by the coefficients ε_n (45).

The closure of the system of Eqs. (49) is performed with neglecting the moments $\Psi_n^{ijk...}$ with numbers $n \geq 2k$, $k = 2, 3, \ldots$. Setting the boundary conditions for a system of equations is discussed, for example, in [10, 11].

CONCLUSIONS

The relation between the classes of even/odd polynomials and symmetric spherical tensors is established. Tensor expansions of the particle sources are obtained. The technique of economization (order reduction) of expansions is described. It is shown that, in problems of neutron and photon transport in a substance with predominant forward or backward scattering, it is expedient to use expansions in the Chebyshev system, which is characterized by a high rate of uniform convergence. The tensor formulation of the method of power moments for solving the radiation transfer equation is given.

FUNDING

This work was supported by the Russian Science Foundation, project no. 14-11-00699.

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Translated by I. Tselishcheva