ISSN 2070-0482, Mathematical Models and Computer Simulations, 2020, Vol. 12, No. 4, pp. 597–612. © Pleiades Publishing, Ltd., 2020.
Russian Text © The Author(s), 2019, published in Matematicheskoe Modelirovanie, 2019, Vol. 31, No. 6, pp. 55–81.

On Solving Second-Order Linear Elliptic Equations

A. V. Shilkov*

Keldysh Institute of Applied Mathematics, Russian Academy of Sciences, Moscow, 125047 Russia

**e-mail: ale-shilkov@yandex.ru*

Received November 19, 2018; revised November 19, 2018; accepted December 10, 2018

Abstract—A method is presented for solving interior boundary-value problems for second-order elliptic equations by transition to ray variables. The domain is divided into cells within which the coefficients and sources have the smoothness and continuity properties necessary for the existence of a regular classical solution in the cell. The finite discontinuities of the coefficients (if any) are located on the cell boundaries. The regular solution in the cell is sought in the form of a superposition of the contributions made by volume and boundary sources placed on the rays arriving at the given point from the cell boundaries. Next, a finite analytic scheme for the numerical solution of the boundary value problem in a domain with discontinuous coefficients and sources is constructed by matching the regular solutions emerging from cells at the cell boundaries. The scheme exhibits no hard dependence of the accuracy of approximation on the sizes and shape of the cells, which is inherent in finite-difference schemes.

Keywords: elliptic equations, boundary-value problem, method of ray variables, numerical methods, finite analytic scheme

DOI: 10.1134/S2070048220040171

1. INTRODUCTION

The paper describes a method for solving an elliptic equation of a fairly general form by reducing the boundary-value problem to a boundary-value problem for a system of kinetic equations. The kinetic equations possess characteristics. The solution to the system is sought by the transition to ray (characteristic) variables.

The introduction contains the formulation of boundary value problems for an elliptic equation in a region with discontinuous coefficients and a source. We partition the region into homogeneous cells, formulate the conditions for sewing classical solutions emerging from neighboring cells and the boundary conditions on the outer boundary of the domain.

In the first section, the correspondence is established between the elliptic equation and the system of kinetic equations describing the propagation of disturbances along the rays. The equations are formulated with respect to the distribution function of the disturbances. The zero moment of the distribution (the integral over the angular variables) is equal to the sought function satisfying the elliptic equation and the boundary conditions.

The kinetic equations have characteristics (hereinafter referred to as rays), which intersect the region from the entry point to the exit point. On the ray, the equations take the form of ordinary differential equations of the second order. In the second section, we pass to ray variables, formulate a one-dimensional boundary-value problem on the ray (instead of the original multidimensional boundary-value problem) and stitching condition for sewing solutions at the intersection points of the cell boundaries. Formulas are presented for solving typical problems on a chord: a ray segment between the cell boundaries. It is shown that imposing the Dirichlet or Neumann conditions at the ends of the chord allows satisfying the stitching condition.

In the third section, a finite analytic scheme is constructed for numerical solution of a one-dimensional boundary-value problem on the ray, and, therefore, for finding a solution to a multidimensional boundary-value problem in the domain. If we solve the set of problems on all the rays crossing the domain, we will be able to find the distribution of perturbations in the domain. Numerical integration of the distribution over angular variables provides a solution to the original boundary-value problem for the elliptic equation.

Analytic schemes that use analytical formulas of the differential equation solutions inside cells and match these solutions on the cell boundaries have been addressed in relatively few publications (see [1–15] and references therein). The schemes developed in the first publications [2, 3] were called exact schemes. The term finite analytic scheme appeared later [8].

Basic Equations

Suppose that a process is described by the following system of linear partial differential equations in a bounded region V of the three-dimensional Euclidean space:

$$\begin{cases} \dfrac{\partial W^i(\mathbf{r})}{\partial r^i} + \varkappa_0(\mathbf{r})U(\mathbf{r}) = Q(\mathbf{r}), \\ \dfrac{\partial [D^{ij}(\mathbf{r})U(\mathbf{r})]}{\partial r^j} + \varkappa_1(\mathbf{r})W^i(\mathbf{r}) = 0, \quad 1 \le i \le 3. \end{cases}$$
$$D^{ij} = D^{ji}, \quad b_D < D^{ij}\Omega^i\Omega^j < c_D, \quad |\mathbf{\Omega}| = 1, \tag{1}$$
$$|Q| < c_Q, \quad |\varkappa_0| < c_0, \quad b_1 < \varkappa_1 < c_1, \quad b_{(\cdot)}, c_{(\cdot)} > 0.$$

Here, $\mathbf{r} = (r^1, r^2, r^3)$ is the coordinate vector, $U(\mathbf{r})$ stands for the sought scalar function (hereinafter, referred to as density), $W^i(\mathbf{r})$ designates the sought vector function (flux), $Q(\mathbf{r})$ is the specified function (distributed source), $D^{ij}(\mathbf{r})$ denotes a symmetric positive definite tensor, and $\mathbf{\Omega}$ is the unit vector. In (1) and subsequent formulas, repeating coordinate indices $i, j, k, \ldots$ imply summation. Coefficients of equations $\varkappa_0(\mathbf{r})$ and $\varkappa_1(\mathbf{r})$ the source and tensor are bounded by positive constants $b_{(\cdot)}$ and $c_{(\cdot)} > 0$. Coefficient $\varkappa_0$ can have any sign, while coefficient $\varkappa_1$ can only assume positive values. The coefficients, the source, and the tensor can have finite discontinuities on 2-dimensional surfaces. The domain may be other than singly connected and can have, for instance, a toroidal shape. We restrict ourselves to considering a space of dimension $N = 3$. Generalization to spaces of greater dimension can be performed without difficulty.

The need to solve Eqs. (1) arises in the mathematical modeling of many problems in physics and technology. Examples are heat transfer in technical installations, neutron transport in nuclear reactors, radiative transfer in gases and plasma, simulation of chemical reactions, the spread of reagents and contaminants in natural media, mechanical vibrations of structures, and the propagation of electromagnetic waves.

If we express the flux $W^i(\mathbf{r})$ from the second equation of system (1) and substitute it into the first equation, then in the areas of smoothness/continuity of the functions of the system, it is reduced to a second-order equation, i.e.,

$$-\frac{\partial}{\partial r^i}\left[\frac{1}{\varkappa_1}\frac{\partial [D^{ij}U]}{\partial r^j}\right] + \varkappa_0 U = Q. \tag{2}$$

Since the tensor $D^{ij} = D^{ji}$ is positive definite, the operator of Eq. (2) is reduced locally (at a point) to an elliptic type operator. Indeed, the tensor can be reduced to a diagonal form with positive numbers on the diagonal by turning the local coordinate system. The further extension along the coordinate axes will change Eq. (2) (locally) to the stationary Schrödinger equation; in the case of constant coefficients, to the Helmholtz equation; and in the case $|\varkappa_0| \ll \varkappa_1$, to the Poisson equation.

Partition of the Region into Cells

We restrict ourselves to considering problems in which the region V admits splitting into disjoint subregions V_α, $\alpha = 1, \ldots, A$ (referred to hereinafter as cells) that tightly cover the region. The surfaces of the discontinuities of the coefficients and the source (if any), as well as the changes in sign of coefficient $\varkappa_0$, are located along the boundaries of the cells. The cells can be assigned to one of the following three types:

cells A with a positive coefficient $\varkappa_0$, in which $\varkappa_0/\varkappa_1 > c$, where c is a small positive number $0 < c \ll 1$;

degenerate cells B, in which $|\varkappa_0|/\varkappa_1 \le c$ and the solution of Eqs. (1) is close to the solution of Eq. (3); i.e., the Poisson equation at $\varkappa_0 = 0$;

cells C with a negative coefficient $\varkappa_0$ in which $\varkappa_0/\varkappa_1 < -c$.

For the cells, the conditions required for the existence of a classical solutions in them are satisfied. In particular, within a cell, the coefficients and the source are continuous functions and the tensor $D^{ij}(\mathbf{r})$ is a smooth function. The boundary of the cell Γ_α consists of smooth surfaces in which the external normal vector $\mathbf{n}_\alpha$ changes continuously and the cell is visible at the solid angle 2π equal to half the full angle 4π. At the intersection points of the surfaces, the cell is visible at an angle that is greater than the number b_ω and less than $4\pi - b_\omega$.

Without loss of generality, we further assume that the tensor $D^{ij}(\mathbf{r})$ is a normalized tensor with the unit trace

$$\mathrm{Sp}D^{ij} = \delta^{ij}D^{ij}(\mathbf{r}) = 1, \quad \mathbf{r} \in V. \tag{3}$$

If this is not so and the trace of the tensor is equal to the function $d(\mathbf{r})$, then the substitution $D^{ij}/d \to D^{ij}$, $Ud \to U$, and $\varkappa_0/d \to \varkappa_0$ yields the required normalization.

Stitching Conditions

On smooth surfaces of the internal boundaries of the region (the boundaries between the cells), we define the natural stitching conditions for the solutions emerging from adjacent cells V_α and V_β:

$$\begin{gathered} n_\alpha^i W_\alpha^i + n_\beta^i W_\beta^i = n_\alpha^i[W_\alpha^i - W_\beta^i] = 0, \quad \mathbf{r} \in \Gamma_\alpha, \Gamma_\beta, \\ D_\alpha^{ij} n_\alpha^i n_\alpha^j U_\alpha = D_\beta^{ij} n_\beta^i n_\beta^j U_\beta, \quad \mathbf{n}_\alpha = -\mathbf{n}_\beta. \end{gathered} \tag{4}$$

The first equation expresses the equality of the flux components normal to the boundary, and the second one expresses the equality of the normal components to the tensor function $D^{ij}U$. (In many applications, $D^{ij}U$ means the pressure tensor.) In (4), it is taken into account that the vectors of the external normal of the cells are opposite in direction. The natural stitching conditions are consistent with the equations of system (1).

Conditions on the External Boundary

On smooth surfaces of the outer boundary of the region V, the following conditions are imposed on the solutions of system (1):

$$\frac{1+\chi}{1-\chi} G_{ij} n^j \left[W^i - W_{\mathrm{ent}}^i\right] = U - U_{\mathrm{ent}}, \quad \mathbf{r}_\Gamma \in \Gamma, \tag{5}$$

$$G_{ij} = G_{ji}, \quad 0 < b_G < G_{ij}\Omega^i\Omega^j < c_G, \quad |\mathbf{\Omega}| = 1, \quad -1 < \chi \le 1. \tag{6}$$

Here, $U_{\mathrm{ent}}(\mathbf{r}_\Gamma)$, $W_{\mathrm{ent}}^i(\mathbf{r}_\Gamma)$, $G_{ij}(\mathbf{r}_\Gamma)$, and $\chi(\mathbf{r}_\Gamma)$ are the specified bounded piecewise continuous functions and $n^i(\mathbf{r}_\Gamma)$ is the vector of the external normal. The cell index α is omitted U_{ent} and W_{ent}^i have the meaning of boundary sources, G_{ij} denote the positively definite tensor, and χ means the coefficient of the boundary reflection. At $\chi = 0$, there is no reflection. If $\chi(\mathbf{r}_\Gamma) \to -1$ on a part of the boundary, then conditions (5) change over to the Dirichlet boundary conditions. If $G_{ij}(\mathbf{r}_\Gamma) = k(\mathbf{r}_\Gamma)D_{ij}(\mathbf{r}_\Gamma)$ on a part of the boundary, then conditions (5) correspond to the boundary conditions of the generalized Neumann problem [16, p. 436; 17, p. 158; 18, p. 9]. Here, $k(\mathbf{r}_\Gamma)$ is the piecewise continuous function, D_{ij} is the cotensor (inverse tensor) of the tensor D^{ij}: $D_{jk}D^{ki} = D^{ki}D_{jk} = \delta^{ij}$. The case $\chi \to 1$ corresponds to the conditions of the classical Neumann problem. The case $G_{ij} \ne kD_{ij}$ and $\chi \ne -1$ is called skew-derivative conditions. The general case is called mixed boundary conditions. Further, the term *Dirichlet boundary conditions* denote the situation when the Dirichlet conditions are set on a section of the boundary. It should be distinguished from the term *Dirichlet boundary-value problem* when this type of condition is posed on the entire boundary.

Existence and Uniqueness of the Solution

The existence of a classical solution of a boundary-value problem in one cell V_α and generalized solutions in the full domain V with discontinuous coefficients are discussed in [16–22].

Let the general mixed boundary conditions (5) be imposed on the boundary of some cell. We represent the solution of the boundary-value problem (1) and (5) in a cell as the sum $U = U_0 + U_\lambda$, $W^i = W_0^i + W_\lambda^i$. The functions U_0 and W_0^i satisfy the inhomogeneous problem (1) and (5) with the source $Q' = Q + [\lambda^2 - 1]\varkappa_0 U_\lambda$, and the functions U_λ and W_λ^i satisfy the homogeneous boundary-value problem

$$\begin{cases} \dfrac{\partial W_\lambda^i}{\partial r^i} + \lambda^2 \varkappa_0 U_\lambda = 0, \\ \dfrac{\partial [D^{ij} U_\lambda]}{\partial r^j} + \varkappa_1 W_\lambda^i = 0, \end{cases} \quad \frac{1+\chi}{1-\chi} G_{ij} n^j W_\lambda^i \Big|_{\Gamma_\alpha} = U_\lambda \Big|_{\Gamma_\alpha}, \tag{7}$$

where $\lambda > 0$ is an eigenvalue. The solutions in a cell of type A, B, or C are subject to the following alternatives [16, p. 443; 17, p. 157; 18, p. 9].

(1) In cells A and B, the homogeneous problem (7) has no solutions except for the trivial solution $U_\lambda = W_\lambda^i = 0$. The inhomogeneous problem (1) and (5) has a unique solution. The classical Neumann problem in the degenerate cell B is an exception. The problem is solvable if the total power of the sources Q is equal to the total flux across the boundary. In this case, the density U is determined with an accuracy up to a constant.

(2) The cell C, $\varkappa_0/\varkappa_1 < -c$, can have a number of eigenvalues $\lambda_2, \ldots > 0$ and nontrivial solutions to the homogeneous problem (7) corresponding to them and that are interpreted as natural oscillations. The set of solutions to a homogeneous problem consists of linear combinations of natural oscillations. Let some combination of natural oscillations be excited in the cell. Then the inhomogeneous boundary-value problem (1) and (5) has a unique solution if the source Q is orthogonal to any oscillation of the given combination. If the source orthogonality condition is not satisfied, then the inhomogeneous problem has only the trivial solution.

Some analytical solutions of an inhomogeneous boundary-value problem for the cell of a simple shape and particular values of the tensor $D^{ij} = D\delta^{ij}$, source, coefficients, and boundary conditions were obtained in [17, p. 125; 23, p. 140; 24, p. 896] using the method of Green functions or the method of separation of variables. The coefficients and sources of the equations are assumed to be constant, the cell has the shape of a parallelepiped or a flat, cylindrical, or spherical layer.

The aim of this paper is to find the solution to the inhomogeneous boundary-value problem (1)–(5) in the region V with cells of type A, B, and C on the assumption that the solution exists and it is unique.

2. DISTRIBUTION OF DISTURBANCIES

We introduce the even and odd $\psi^\pm(\mathbf{\Omega}, \mathbf{r}) = \pm\psi^\pm(-\mathbf{\Omega}, \mathbf{r})$ distributions of disturbances. The distributions are functions of coordinates and the unit vector $\mathbf{\Omega} = (\Omega^1, \Omega^2, \Omega^3) = (\sqrt{1-\mu^2}\cos\alpha, \sqrt{1-\mu^2}\sin\alpha, \mu)$ ($d\mathbf{\Omega} = d\alpha d\mu$) determining the direction of disturbance propagation; α $(0 \le \alpha < 2\pi)$ is an azimuthal angle measured from the axis r_1 in the plane $r_1 \times r_2$; and μ $(-1 \le \mu \le 1)$ stands for the cosine of the angle between the vector $\mathbf{\Omega}$ and the axis r_3. The distributions are defined in the domain $\mathbf{r} \times \mathbf{\Omega}$, $\mathbf{r} \in V$, $\mathbf{\Omega} \in 4\pi$, of the five-dimensional Euclidean space.

We normalize the scalar and vector angular moments of the distributions for the desired density and the desired flux, respectively:

$$U(\mathbf{r}) = \int_{4\pi} \psi^+(\mathbf{\Omega}, \mathbf{r}) d\mathbf{\Omega}, \quad W^i(\mathbf{r}) = \int_{4\pi} \Omega^i \frac{\psi^-(\mathbf{\Omega}, \mathbf{r})}{h} d\mathbf{\Omega},$$
$$h(\mathbf{\Omega}, \mathbf{r}) = \sqrt{g_{ij}(\mathbf{r})\Omega^i\Omega^j}, \quad g_{ij}(\mathbf{r}) = g_{ji}(\mathbf{r}), \quad g_{ij}\Omega^i\Omega^j \ge 3b/2 > 0, \tag{8}$$

where $h(\mathbf{\Omega},\mathbf{r}) > 0$ is a positive even $h(\mathbf{\Omega}) = h(-\mathbf{\Omega})$ function calculated through some symmetric metric tensor $g_{ij}(\mathbf{r})$. In addition to the lower moments (8), we also need the moments of the second—$\Psi_2^{ij}(\mathbf{r})$ and third $-\ \Psi_3^{ijk}(\mathbf{r})$ order distribution

$$\Psi_2^{ij}(\mathbf{r}) = \int\limits_{4\pi} P_2^{ij}(\mathbf{\Omega})\psi^+(\mathbf{\Omega},\mathbf{r})d\mathbf{\Omega},\quad \Psi_2^{ij}\delta^{ij} = 0,$$
$$\Psi_3^{ijk}(\mathbf{r}) = \int\limits_{4\pi} P_3^{ijk}(\mathbf{\Omega})\frac{\psi^-(\mathbf{\Omega},\mathbf{r})}{h}d\mathbf{\Omega},\quad \Psi_3^{ijk}\delta^{ij} = \Psi_3^{ijk}\delta^{ik} = \Psi_3^{ijk}\delta^{jk} = 0, \tag{9}$$
$$P_2^{ij}(\mathbf{\Omega}) = \frac{3\Omega^i\Omega^j - \delta^{ij}}{2},\quad P_3^{ijk}(\mathbf{\Omega}) = \frac{5\Omega^i\Omega^j\Omega^k - \Omega^i\delta^{jk} - \Omega^j\delta^{ik} - \Omega^k\delta^{ij}}{2}.$$

Here, $P_n^{ij\ldots}(\mathbf{\Omega})$ are the spherical Legendre tensors ($P_0 = 1,\ P_1^i = \Omega^i$). The tensor of the second order moment Ψ_2^{ij} is expressed in terms of the zero-order moment U and the symmetric normalized tensor C^{ij}, which is a linear fractional of the even distribution of disturbances

$$\Psi_2^{ij}(\mathbf{r}) = \frac{3C^{ij}(\mathbf{r}) - \delta^{ij}}{2}U(\mathbf{r}),\quad C^{ij} = C^{ji},\quad C^{ij}\delta^{ij} = 1,$$
$$C^{ij}(\mathbf{r}) = C^{ij}(\psi^+) = \int\limits_{4\pi}\Omega^i\Omega^j\psi^+(\mathbf{\Omega},\mathbf{r})d\mathbf{\Omega}\Big/\int\limits_{4\pi}\psi^+(\mathbf{\Omega},\mathbf{r})d\mathbf{\Omega}. \tag{10}$$

Kinetic Problem

Let the distributions satisfy the system of even-odd kinetic equations be consistent with Eqs. (1)

$$\begin{cases}\dfrac{\Omega^i}{h}\dfrac{\partial\psi^-(\mathbf{\Omega},\mathbf{r})}{\partial r^i} + \varkappa_0(\mathbf{r})\psi^+(\mathbf{\Omega},\mathbf{r}) = q_0(\mathbf{r}) \equiv \dfrac{Q(\mathbf{r})}{4\pi} + \dfrac{K_0(\mathbf{r})}{4\pi},\\[2ex] \dfrac{\Omega^j}{h}\dfrac{\partial\psi^+(\mathbf{\Omega},\mathbf{r})}{\partial r^j} + \varkappa_1(\mathbf{r})\psi^-(\mathbf{\Omega},\mathbf{r}) = q_1(\mathbf{\Omega},\mathbf{r}) \equiv \dfrac{3}{4\pi}\dfrac{\Omega^i K_1^i(\mathbf{r})}{h}.\end{cases} \tag{11}$$

Here, $K_0(\mathbf{r})$ and $K_1^i(\mathbf{r})$ are scalar and vector sources supplementing the main source $Q(\mathbf{r})$, and $h(\mathbf{\Omega},\mathbf{r})$ is the function introduced in (8). Equations (11) describe the propagation of disturbances along the rays. Disturbances are generated by the main and complementary sources.

The solution of system (11) is assumed to satisfy the following stitching conditions on the internal boundaries between the adjacent cells V_α and V_β:

$$\lim_{l\to 0}\left[\psi^\pm(\mathbf{\Omega},\mathbf{r}_\alpha + \mathbf{\Omega}l) - \psi^\pm(\mathbf{\Omega},\mathbf{r}_\alpha - \mathbf{\Omega}l)\right] = 0,\quad \mathbf{\Omega}\mathbf{n}_\alpha \neq 0, \tag{12}$$

where l is the distance measured along the ray $\mathbf{r}_\alpha + \mathbf{\Omega}l$ crossing a smooth surface of a common boundary at the point $\mathbf{r}_\alpha \in \Gamma_\alpha,\Gamma_\beta$ in direction $\mathbf{\Omega}$ that is different from the tangential direction. The point $\mathbf{r}_\alpha - \mathbf{\Omega}l$ belongs to the cell V_α; and the point $\mathbf{r}_\alpha + \mathbf{\Omega}l$, to the cell V_β.

At the external boundaries of the domain, we impose the following conditions on the solutions of the system:

$$\frac{1+\chi}{1-\chi}G_{ij}\Omega^i n^j\frac{\psi^-(\mathbf{\Omega},\mathbf{r}_\Gamma) - \psi^-_{\text{ent}}(\mathbf{\Omega},\mathbf{r}_\Gamma)}{h(\mathbf{\Omega},\mathbf{r}_\Gamma)} = \psi^+(\mathbf{\Omega},\mathbf{r}_\Gamma) - \psi^+_{\text{ent}}(\mathbf{\Omega},\mathbf{r}_\Gamma),$$
$$\psi^+_{\text{ent}}(\mathbf{\Omega},\mathbf{r}_\Gamma) = \frac{U_{\text{ent}}(\mathbf{r}_\Gamma)}{4\pi},\quad \frac{\psi^-_{\text{ent}}(\mathbf{\Omega},\mathbf{r}_\Gamma)}{h(\mathbf{\Omega},\mathbf{r}_\Gamma)} = \frac{3}{4\pi}\Omega^i W^i_{\text{ent}}(\mathbf{r}_\Gamma). \tag{13}$$

The integration of the stitching conditions (12) over angular variables with weights $\mathbf{\Omega}\mathbf{n}_\alpha/h$ and $[\mathbf{\Omega}\mathbf{n}_\alpha]^2$ gives the stitching conditions for the solutions of the initial problem (4), and integration of the boundary conditions (13) with the unit weight gives the boundary conditions (5).

Let us select the metric tensor $g_{ij}(\mathbf{r})$ and complementary sources $K_0(\mathbf{r})$ and $K_1^i(\mathbf{r})$ so that the result of integrating Eqs. (11) over angular variables exactly reproduces Eqs. (1).

It is reasonable to set the **metric tensor** in the form

$$g_{ij}(\mathbf{r},a) = g_{ij}(\psi^+) = \frac{5-3a}{2} D_{ik}(\mathbf{r}) C^{jk}(\psi^+) - \frac{1-a}{2} C^{pq} D_{pq} \delta_{ij},$$
$$g = \mathrm{Sp} g_{ij} = \delta^{ij} g_{ij} = C^{pq} D_{pq}, \quad D_{kj} D^{ik} = D^{ik} D_{kj} = \delta^{ij}, \tag{14}$$

where $D_{ij}(\mathbf{r})$ is a symmetric positive cotensor (inverse tensor) of the tensor $D^{ij}(\mathbf{r})$ (1), $C^{ij}(\psi^+)$ is a symmetric normalized tensor (10), a is the correction parameter being selected, and g is the trace of the metric tensor. The metric tensor (14), just as the tensors D^{ij} and C^{ij} (see (3), (10)), is a normalized tensor. Just as the tensor C^{ij}, it is a fractional linear functional of an even distribution ψ^+. If $C^{ij} = D^{ij}$, then $g_{ij}(\psi^+)$ turns into the unit metric tensor of the Euclidean space δ_{ij} ($\mathrm{Sp}\delta_{ij} = N = 3$).

The correction parameter $a(\mathbf{r})$ is introduced in (14) to ensure that the inequality $g_{ij}(\mathbf{r},a)\Omega^i\Omega^j \geq 3b/2 > 0$ is satisfied under strong deviations of the tensor $C^{ij}(\psi^+)$ from the tensor $D^{ij}(\mathbf{r})$; where $b \approx 0.1$ is a given small number. This parameter equals zero, $a = 0$, if the inequality $\min g_{ij}(\mathbf{r},0)\Omega^i\Omega^j \geq 3b/2$, $|\mathbf{\Omega}| = 1$ holds. Otherwise (large deviations C^{ij} from D^{ij}), the parameter is found from the equation $\min g_{ij}(\mathbf{r},a)\Omega^i\Omega^j = 3b/2$. By solving the equation, we find

$$a(\mathbf{r}) = \begin{bmatrix} 0, & [g+3b]/5 \leq CD \leq g/3 \\ [g-5CD+3b]/[g-3CD], & 0 < CD < [g+3b]/5 \end{bmatrix},$$
$$CD(\mathbf{r}) = \min_{\mathbf{\Omega}} C^{ik} D_{kj} \Omega^i \Omega^j, \quad b \approx 0.1, \quad g(\mathbf{r}) = C^{pq} D_{pq}, \tag{15}$$

where $CD(\mathbf{r}) > 0$ is the minimum eigenvalue of the tensor $C^{ik} D_{kj}$.

Example. Let $D^{ij} = \delta^{ij}/3$. Then

$$g_{ij}(\mathbf{r}) = \frac{3[5-3a]}{2} C^{ij}(\mathbf{r}) - \frac{3[1-a]}{2} \delta_{ij}, \quad g(\mathbf{r}) = g_{ij}\delta_{ij} = 3,$$
$$h^2(\mathbf{\Omega},\mathbf{r}) = g_{ij}(\mathbf{r})\Omega^i\Omega^j = \frac{3[5-3a]}{2} C^{ij}\Omega^i\Omega^j - \frac{3[1-a]}{2}.$$

The adjusting factor assumes the values. The correction parameter is

$$a(\mathbf{r}) = \begin{bmatrix} 0, & [1+b]/5 \leq C \leq 1/3 \\ [1-5C+b]/[1-3C], & 0 < C < [1+b]/5 \end{bmatrix}, \quad \begin{matrix} b \approx 0.1, \\ C(\mathbf{r}) = \min\limits_{\mathbf{\Omega}} C^{ij}\Omega^i\Omega^j. \end{matrix}$$

$C(\mathbf{r}) > 0$ is the minimum eigenvalue of the tensor $C^{ij}(\psi^+)$.

The correspondence to the first equation of system (1): by integrating the even kinetic equation (11) over the angular variables

$$\int\limits_{4\pi} \left[\Omega^i \frac{\partial}{\partial r^i} \frac{\psi^-}{h} + \frac{\psi^-\Omega^i}{h^2} \frac{\partial h}{\partial r^i} + \varkappa_0 \psi^+ \right] d\mathbf{\Omega} = [Q + K_0] \frac{1}{4\pi} \int\limits_{4\pi} d\mathbf{\Omega},$$

we obtain Eq. (1) if we define the complementary source as

$$K_0(\mathbf{r}) = \int\limits_{4\pi} \frac{\Omega^i\psi^-}{h} \frac{\partial \ln h}{\partial r^i} d\mathbf{\Omega} = \frac{W^i}{2} \frac{\partial \ln g}{\partial r^i} + \int\limits_{4\pi} \frac{\Omega^i\psi^-}{h} \frac{\partial \ln \sqrt{3 g_{ij}\Omega^i\Omega^j/g}}{\partial r^i} d\mathbf{\Omega}.$$

To represent the source in a form convenient for calculations, we expand the logarithmic function in a series in spherical tensors. Due to the evenness of the function in $\mathbf{\Omega}$ only even harmonics will be included in the expansion, i.e.,

$$\ln\sqrt{3g_{ij}(\mathbf{r})\Omega^i\Omega^j/g(\mathbf{r})} \simeq \sum_{n=0}^{\infty}\frac{4n+1}{4\pi}\underbrace{\Omega^i\Omega^j\ldots\Omega^q}_{2n} H_{2n}^{ij\ldots q}(\mathbf{r}),$$

$$H_{2n}^{ij\ldots q}(\mathbf{r}) = \int_{4\pi} P_{2n}^{ij\ldots q}(\mathbf{\Omega})\ln\sqrt{3g_{ij}(\mathbf{r})\Omega^i\Omega^j/g(\mathbf{r})}d\mathbf{\Omega},$$

$$H_0 = \int_{4\pi}\ln\sqrt{3g_{ij}\Omega^i\Omega^j/g}d\mathbf{\Omega},\quad H_2^{ij} = \int_{4\pi}\frac{3\Omega^i\Omega^j - \delta^{ij}}{2}\ln\sqrt{3g_{ij}\Omega^i\Omega^j/g}d\mathbf{\Omega},\ldots,$$

Here, $H_{2n}^{ij\ldots q}(\mathbf{r})$ are the even moments of the function, and $P_{2n}^{ij\ldots q}(\mathbf{\Omega})$ are the spherical Legendre tensors (9). By substituting the expansion in the integral we obtain

$$K_0 = \frac{W^i}{2}\frac{\partial \ln g}{\partial r^i} + W^j\frac{\partial}{\partial r^i}\left[\frac{H_0\delta^{ij}}{4\pi} + \frac{2H_2^{ij}}{4\pi}\right] + \Psi_3^{jkl}\frac{\partial}{\partial r^i}\left[\frac{2H_2^{kl}\delta^{ij}}{4\pi} + \frac{8}{5}\frac{H_4^{ijkl}}{4\pi}\right] + \ldots. \tag{16}$$

The source $K_0(\psi^{\pm})$ is expressed in terms of the moments W^i and $\Psi_{2n+1}^{ij\ldots}$ of the odd distribution ψ^- and derivatives of even moments $H_{2n}^{ij\ldots q}(\mathbf{r})$.

Series (16) rapidly converges. We restrict ourselves to solving problems in which for calculating the complementary source $K_0(\psi^{\pm})$ it suffices to take into account expansion terms containing moments W^i, Ψ_3^{ijk} and H_0, H_2^{ij}.

Correspondence to the Second Equation of (1): before integration, we multiply the second (odd) equation (11) by the factor $h\Omega^i$:

$$\int_{4\pi}\left[\Omega^i\Omega^j\frac{\partial\psi^+}{\partial r^j} + \varkappa_0 g_{jk}\Omega^i\Omega^j\Omega^k\frac{\psi^-}{h}\right]d\mathbf{\Omega} = K_1^j\frac{3}{4\pi}\int_{4\pi}\Omega^i\Omega^j d\mathbf{\Omega}.$$

By using (8) and (9) the equation is converted to the form

$$K_1^i(\psi^{\pm}) = \frac{\partial[C^{il}U]}{\partial r^l} + \varkappa_1\left[\frac{2g_{ij} + g\delta_{ij}}{5}W^j + \frac{2}{5}g_{jk}\Psi_3^{ijk}\right].$$

Taking into account the equalities

$$\frac{\partial[C^{il}U]}{\partial r^l} = \frac{\partial[C^{ik}D_{jk}D^{jl}U]}{\partial r^l} = C^{ik}D_{jk}\frac{\partial[D^{jl}U]}{\partial r^l} + D^{jl}U\frac{\partial[C^{ik}D_{jk}]}{\partial r^l},$$

$$2g_{ij} + g\delta_{ij} = 5C^{ik}D_{jk} - a[3C^{ik}D_{jk} - g\delta_{ij}],$$

and the second equation in (1), we obtain the following expression for the complementary source

$$K_1^i(\psi^{\pm}) = D^{jl}\frac{\partial[C^{ik}D_{kl}]}{\partial r^j}U + \frac{2\varkappa_1}{5}\left[g_{jk}\Psi_3^{ijk} - \frac{a}{2}[3C^{ik}D_{jk} - g\delta_{ij}]W^j\right]. \tag{17}$$

Discussion

Equations (1)–(5) follow from Eqs. (11)–(17). Therefore, the solution of the boundary-value problem (1)–(5) under the conditions of its existence and uniqueness can be obtained from the solution of the kinetic problem.

The kinetic equations (11) are integrodifferential equations with weak nonlinearity. This nonlinearity is due to the weak fractional linear dependence of the tensors $g_{ij}(\psi^+)$ and $C^{ij}(\psi^+)$ on the even distribution. The complementary sources $K_0(\psi^{\pm})$, $K_1^i(\psi^{\pm})$ on the right-hand side of Eqs. (16) and (17) are linear combinations of moments U, W^i, Ψ_2^{ijk}, and Ψ_3^{ijk}, i.e., of the angular integrals (8) and (9) of the sought distributions.

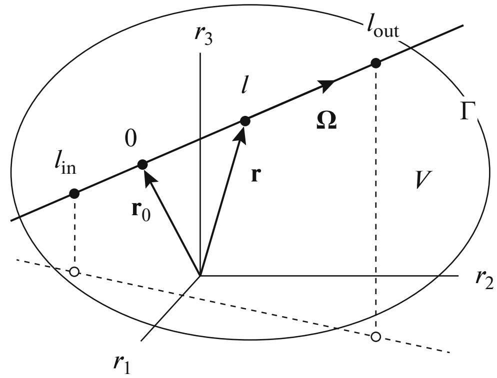

Fig. 1. Ray $\mathbf{r} = \mathbf{r}_0 + \mathbf{\Omega} l$ enters the region at point $\mathbf{r}_{in}$ with the ray coordinate l_{in}, passes in the direction $\mathbf{\Omega}$ through the observation point $\mathbf{r}$ with the ray coordinate l and leaves the region at point $\mathbf{r}_{out}$ with coordinate l_{out}. $\mathbf{r}_0$ is the base point.

Similar linear integrodifferential equations containing moments U, W^i, and $\Psi_n^{ij...}$ but not containing the metric tensor $g^{ij}(\psi^+)$ arise in the transport of particles—neutrons and photons (see, for example, [1, 12, 13]). The solution of the equations can be found numerically by iteration over the values of the moments and tensors. From the 75-year practice of solving transport equations (it started with the Atomic Project), it is known that iterations converge quickly if the complementary sources are relatively small compared to the main source $Q(\mathbf{r})$:

$$\left|K_0(\psi^{\pm})\right|, \quad \frac{1}{\varkappa_1}\left|\frac{\partial K_1^i(\psi^{\pm})}{\partial r^i}\right| < A\left|Q(\mathbf{r})\right|, \quad A \le 1-10, \quad \mathbf{r} \in V. \tag{18}$$

In this case, to solve the integrodifferential equations, it is sufficient to apply the so-called simple iterations, in which the tensors and moments are calculated from the values of the distributions at the previous iteration. The convergence rate of simple iterations is higher the smaller constant A. (If condition (18) is not fulfilled, more complicated iterative methods are used in particle transport problems.)

Analysis of the complementary sources (16) and (17) of our kinetic problem (11)–(13) shows that they are relatively small, and for many applied problems condition (18) is fully satisfied. In fact, the main moment of the zeroth order U enters with the small coefficient only in the source K_1^i (17). The first-order moment W^i is included in both complementary sources, albeit, with small coefficients.

3. TRANSITION TO RAY VARIABLES

A distinctive property of Eqs. (11) is the presence of characteristics. This will allow us to progress in integrating these equations, i.e., in constructing analytical formulas for the implicit representation of the boundary value problem solution and in designing finite analytic discrete schemes and algorithms for solving the problem.

Ray Variables

Let us consider the point of the domain $\mathbf{r}_0 \in V$ (Fig. 1). Only one ray $\mathbf{r} = \mathbf{r}_0 + \mathbf{\Omega} l$ passes through the point in direction $\mathbf{\Omega}$, $|\mathbf{\Omega}| = 1$, where l is the distance from point $\mathbf{r}_0$ to the point on ray $\mathbf{r}$ (hereinafter referred to as the dimensional ray variable or alternatively called the characteristic variable).

Also we introduce the *dimensionless ray variable* ξ on the ray:

$$\xi(l) = \xi(\mathbf{r}_0, \mathbf{r}) = \int_0^l \varkappa(\mathbf{r}') h(\mathbf{\Omega}, \mathbf{r}') dl', \quad \mathbf{r}' = \mathbf{r}_0 + \mathbf{\Omega} l',$$
$$\varkappa(\mathbf{r}) = \begin{bmatrix} \sqrt{|\varkappa_0 \varkappa_1|}, & \text{cells A and B}, \\ \varkappa_1, & \text{cell B } (\varkappa_0 \approx 0), \end{bmatrix}, \quad \varkappa(\mathbf{r}) > 0. \tag{19}$$

Here, $\varkappa(\mathbf{r})$ is the positive coefficient equal to geometric mean $|\varkappa_0|$ and $\varkappa_1$. In the degenerate cell B, we set $\varkappa = \varkappa_1$. Differentials of ray variables and Cartesian coordinates are connected by the relations

$$d\xi = \varkappa h dl = \varkappa h \frac{dr^1}{\Omega^1} = \varkappa h \frac{dr^2}{\Omega^2} = \varkappa h \frac{dr^3}{\Omega^3}, \quad \frac{\Omega^i}{h}\frac{\partial}{\partial r^i} = \frac{1}{h}\frac{\partial}{\partial l} = \varkappa \frac{\partial}{\partial \xi}. \tag{20}$$

Boundary-Value Problem on a Ray

It follows from (20) that the rays are characteristics of Eqs. (11). On the ray $\mathbf{r} = \mathbf{r}_0 + \mathbf{\Omega} l$, the system takes the form of a system of ordinary differential equations of the second order:

$$\begin{cases} \dfrac{d\psi^-}{d\xi} + \dfrac{\varkappa_0}{\varkappa}\psi^+ = \dfrac{q_0}{\varkappa}, \quad q_0(\xi) = q_0(\mathbf{r}(\xi)) = \dfrac{Q}{4\pi} + \dfrac{K_0}{4\pi}, \\ \dfrac{d\psi^+}{d\xi} + \dfrac{\varkappa_1}{\varkappa}\psi^- = \dfrac{q_1}{\varkappa}, \quad q_1(\xi) = q_1(\mathbf{\Omega}, \mathbf{r}(\xi)) = \dfrac{3}{4\pi}\dfrac{\Omega^i K_1^i}{h}. \end{cases} \tag{21}$$

The ray coordinates l_α and ξ_α $(\text{in} \le \alpha \le \text{out})$ of points $\mathbf{r}_\alpha$, at which the ray crosses the boundaries of the cells, will be indicated by index α; l_{in}, ξ_{in} and l_{out}, ξ_{out} are the coordinates of the entry point and the exit point of the ray from the region. The numbering order of the points is consistent with the growth direction of the ray variable. On the ray of the opposite direction $-\mathbf{\Omega}$, the same points are numbered in reverse order. In these notations, the stitching conditions for the solutions on the internal boundaries (12) have the form

$$\psi^{\pm}(\xi_\alpha + 0) = \psi^{\pm}(\xi_\alpha - 0), \quad \mathbf{\Omega n}_\alpha \ne 0 \quad (\text{in} < \alpha < \text{out}). \tag{22}$$

The mixed conditions on the external boundaries of the region (13) are written in the form

$$\begin{aligned} -\theta_{\text{in}}\left[\psi^-(\xi_{\text{in}}) - \psi^-_{\text{ent}}(\xi_{\text{in}})\right] &= \psi^+(\xi_{\text{in}}) - \psi^+_{\text{ent}}(\xi_{\text{in}}), \\ \theta_{\text{out}}\left[\psi^-(\xi_{\text{out}}) - \psi^-_{\text{ent}}(\xi_{\text{out}})\right] &= \psi^+(\xi_{\text{out}}) - \psi^+_{\text{ent}}(\xi_{\text{out}}), \end{aligned} \tag{23}$$

where parameters θ and $\psi^{\pm}_{\text{ent}}$ take the values

$$\begin{gathered} \theta_{\text{in}} = -\frac{1+\chi_{\text{in}}}{1-\chi_{\text{in}}}\frac{G_{ij}(\xi_{\text{in}})\Omega^i n^j_{\text{in}}}{h(\xi_{\text{in}})}, \quad \theta_{\text{out}} = \frac{1+\chi_{\text{out}}}{1-\chi_{\text{out}}}\frac{G_{ij}(\xi_{\text{out}})\Omega^i n^j_{\text{out}}}{h(\xi_{\text{out}})}, \\ \psi^+_{\text{ent}} = \frac{U_{\text{ent}}}{4\pi}, \quad \frac{\psi^-_{\text{ent}}}{h} = \frac{3}{4\pi}\Omega^i W^i_{\text{ent}}. \end{gathered} \tag{24}$$

When arranging the signs, we took into account the sign of the scalar product, $\mathbf{\Omega n}$: $\mathbf{\Omega n}_{\text{in}} < 0$ at the entry point of the ray into the region, and $\mathbf{\Omega n}_{\text{out}} > 0$ at the exit point.

The segment of the ray $\xi_{\alpha-1} \le \xi \le \xi_\alpha$ $(l_{\alpha-1} \le l \le l_\alpha)$ between the intersection points of the cell boundaries will be called a chord. At the cell boundaries, the coefficients and sources of the system of equations (21) may be subject to discontinuities and/or sign changes. Therefore, the classical solution of the system in the general case exists only on the chord. A solution composed from classical solutions defined on chords (if such solutions exist) and satisfying the stitching conditions (22) on the boundaries between cells (internal boundaries) and conditions (23), (24) on the external boundary of the region will be called the *solution of the boundary value poblem* (21)–(24) *on the ray* $\mathbf{r} = \mathbf{r}_0 + \mathbf{\Omega} l$, $l_{\text{in}} \le l \le l_{\text{out}}$.

Note. With regard to the formal integration of the equations, in the cells V_α of domain V, we can define smooth Riemannian manifolds with a metric tensor $g_{ij}(\mathbf{r})$ and connectivity [25, 26, p.359] and substitute in (11) and (21) the usual differentiation along directions for a covariant differentiation along the geodesic lines of the manifolds. Then direct rays will turn into segments of geodesic ones. The magnitude of the complementary sources $K_0(\mathbf{r})$ and $K_1^i(\mathbf{r})$ will decrease as they will lose some combinations of derivatives of the metric tensor $g_{ij}(\mathbf{r})$. In this paper, intending to develop simple numerical methods for finding a solution to a system, we will remain in Euclidean space because calculating geodesic lines in cells by itself takes up additional computer resources.

Below in this section, we will deal with the problem of finding a classical solution for the system of equations (21) on a chord. The problem of finding the solution on the ray is discussed in the next section.

The solution of Eqs. (21) on the chord can easily be written in the degenerate cell B. In cells A and C, the cases of integration of the system are more diverse and are determined by the dependences of the coefficients $\varkappa_0(\mathbf{r})$ and $\varkappa_1(\mathbf{r})$. We confine ourselves to the case often encountered in practice, when the coefficients change similarly or almost similarly to each other: $\varkappa_0/\varkappa_1 \approx \text{const}$. Let the derivatives of the coefficients in the directions be bounded by a small constant c_R:

$$\left|\Omega^i \frac{\partial}{\partial r^i}\left[\frac{\varkappa_0(\mathbf{r})}{\varkappa_1(\mathbf{r})}\right]\right| \le c_R \varkappa_1(\mathbf{r}), \quad c_R < 0.3, \quad \mathbf{r} \in V_\alpha. \tag{25}$$

Note. In many practical problems, the condition (25) can be fulfilled by reducing the size of the cells. If this fails, then the ratio of the coefficients on the chord $\xi_{\alpha-1} \le \xi \le \xi_\alpha$ can be approximated by the exponential function $\varkappa_0/\varkappa_1 \approx Be^{k\xi}$. This case is reduced to the case $\varkappa_0/\varkappa_1 \approx \text{const}$ by a simple substitution of the sought functions.

Basic Solution on the Chord

It will be assumed that at the ends of the chord $\xi_{\alpha-1} \le \xi \le \xi_\alpha$, mixed boundary conditions are imposed, similar to the conditions on the outer boundary (23) ($\text{in} = \alpha - 1$, $\text{out} = \alpha$) but without specifying parameters $\theta_{\alpha-1}$, θ_α, and $\psi^\pm_{\text{ent}}$ by values (24):

$$\begin{gathered} -\theta_{\alpha-1}\left[\psi^-(\xi_{\alpha-1}) - \psi^-_{\text{ent}}(\xi_{\alpha-1})\right] = \psi^+(\xi_{\alpha-1}) - \psi^+_{\text{ent}}(\xi_{\alpha-1}), \\ \theta_\alpha\left[\psi^-(\xi_\alpha) - \psi^-_{\text{ent}}(\xi_\alpha)\right] = \psi^+(\xi_\alpha) - \psi^+_{\text{ent}}(\xi_\alpha). \end{gathered} \tag{26}$$

The parameters take values (24) if the endpoits of the chord lie on the outer boundary of the domain. If one of the points or both points lie on the inner boundary between the cells, then at these points we will pass onto a special case of conditions (26): the Dirichlet conditions ($\theta_\alpha \to 0$). This will satisfy the stitching conditions of solutions (22) at the internal boundaries.

We introduce the elementary functions of the ray variable on the chord:

$$\begin{gathered} s^{\rightarrow}_{\alpha-1}(\xi) = \frac{\varkappa_1(\xi)}{\varkappa(\xi)}\begin{bmatrix}\sinh(\xi-\xi_{\alpha-1})\\ \sin(\xi-\xi_{\alpha-1})\end{bmatrix} + \theta_{\alpha-1}\begin{bmatrix}\cosh(\xi-\xi_{\alpha-1})\\ \cos(\xi-\xi_{\alpha-1})\end{bmatrix}, \\ s^{\leftarrow}_{\alpha}(\xi) = \frac{\varkappa_1(\xi)}{\varkappa(\xi)}\begin{bmatrix}\sinh(\xi_\alpha-\xi)\\ \sin(\xi_\alpha-\xi)\end{bmatrix} + \theta_{\alpha}\begin{bmatrix}\cosh(\xi_\alpha-\xi)\\ \cos(\xi_\alpha-\xi)\end{bmatrix}, \\ c^{\rightarrow}_{\alpha-1}(\xi) = \begin{bmatrix}\cosh(\xi-\xi_{\alpha-1})\\ \cos(\xi-\xi_{\alpha-1})\end{bmatrix} + \theta_{\alpha-1}\frac{\varkappa(\xi)}{\varkappa_1(\xi)}\begin{bmatrix}\sinh(\xi-\xi_{\alpha-1})\\ -\sin(\xi-\xi_{\alpha-1})\end{bmatrix}, \\ c^{\leftarrow}_{\alpha}(\xi) = \begin{bmatrix}\cosh(\xi_\alpha-\xi)\\ \cos(\xi_\alpha-\xi)\end{bmatrix} + \theta_{\alpha}\frac{\varkappa(\xi)}{\varkappa_1(\xi)}\begin{bmatrix}\sinh(\xi_\alpha-\xi)\\ -\sin(\xi_\alpha-\xi)\end{bmatrix}, \\ x_\alpha(\xi) = s^{\rightarrow}_{\alpha-1}(\xi)c^{\leftarrow}_{\alpha}(\xi) + c^{\rightarrow}_{\alpha-1}(\xi)s^{\leftarrow}_{\alpha}(\xi) = \frac{\varkappa_1(\xi)}{\varkappa(\xi)}\begin{bmatrix}\sinh\Delta\xi_\alpha\\ \sin\Delta\xi_\alpha\end{bmatrix} \\ + [\theta_{\alpha-1}+\theta_\alpha]\begin{bmatrix}\cosh\Delta\xi_\alpha\\ \cos\Delta\xi_\alpha\end{bmatrix} + \theta_{\alpha-1}\theta_\alpha\frac{\varkappa(\xi)}{\varkappa_1(\xi)}\begin{bmatrix}\sinh\Delta\xi_\alpha\\ -\sin\Delta\xi_\alpha\end{bmatrix} \approx \text{const}. \end{gathered} \tag{27}$$

Here, $\Delta\xi_\alpha = \xi_\alpha - \xi_{\alpha-1}$ is the chord length. The upper line refers to cell A ($\varkappa_0/\varkappa_1 > c$); and the lower line, to the cell C ($\varkappa_0/\varkappa_1 < -c$). In the degenerate cell B ($|\varkappa_0|/\varkappa_1 \le c$), functions (27) continuously transform into functions

$$\begin{gathered} s^{\rightarrow}_{\alpha-1}(\xi) = \xi - \xi_{\alpha-1} + \theta_{\alpha-1}, \quad s^{\leftarrow}_{\alpha}(\xi) = \xi_\alpha - \xi + \theta_\alpha, \\ c^{\rightarrow}_{\alpha-1}(\xi) = c^{\leftarrow}_{\alpha}(\xi) = 1, \quad x_\alpha = \xi_\alpha - \xi_{\alpha-1} + \theta_{\alpha-1} + \theta_\alpha = \text{const}, \end{gathered} \tag{28}$$

where the ray coordinate ξ is calculated given the coefficient $\varkappa = \varkappa_1$ (see (19)).

The general solution of system (21) and (26) on the chord $\xi_{\alpha-1} \le \xi \le \xi_\alpha$ is

$$\psi^+(\xi) \approx \frac{s_\alpha^{\leftarrow}(\xi)}{x_\alpha(\xi)}\left[I_{\alpha-1} + J_{\alpha-1}^{\rightarrow}(\xi)\right] + \frac{s_{\alpha-1}^{\rightarrow}(\xi)}{x_\alpha(\xi)}\left[I_\alpha + J_\alpha^{\leftarrow}(\xi)\right], \tag{29}$$

$$\begin{gathered}\psi^-(\xi) \approx \frac{c_\alpha^{\leftarrow}(\xi)}{x_\alpha(\xi)}\left[I_{\alpha-1} + J_{\alpha-1}^{\rightarrow}(\xi)\right] - \frac{c_{\alpha-1}^{\rightarrow}(\xi)}{x_\alpha(\xi)}\left[I_\alpha + J_\alpha^{\leftarrow}(\xi)\right],\\ I_{\alpha-1} = \psi_{\text{ent}}^+(\xi_{\alpha-1}) + \theta_{\alpha-1}\psi_{\text{ent}}^-(\xi_{\alpha-1}),\quad I_\alpha = \psi_{\text{ent}}^+(\xi_\alpha) - \theta_\alpha\psi_{\text{ent}}^-(\xi_\alpha),\end{gathered} \tag{30}$$

$$\begin{gathered}J_{\alpha-1}^{\rightarrow}(\xi) = \int\limits_{\xi_{\alpha-1}}^{\xi}\left[s_{\alpha-1}^{\rightarrow}(\xi')q_0(\xi') + c_{\alpha-1}^{\rightarrow}(\xi')q_1(\xi')\right]\frac{d\xi'}{\varkappa},\\ J_\alpha^{\leftarrow}(\xi) = \int\limits_{\xi}^{\xi_\alpha}\left[s_\alpha^{\leftarrow}(\xi')q_0(\xi') - c_\alpha^{\leftarrow}(\xi')q_1(\xi')\right]\frac{d\xi'}{\varkappa},\quad \frac{d\xi'}{\varkappa} = h(l')dl'.\end{gathered} \tag{31}$$

where $I_{\alpha-1}$ and $J_{\alpha-1}^{\rightarrow}(\xi)$ are the contributions of the boundary and distributed sources from the ray segment $\xi_{\alpha-1} \le \xi' < \xi$, and I_α, $J_\alpha^{\leftarrow}(\xi)$ stand for the contributions from the ray segments $\xi < \xi' \le \xi_\alpha$. The validity of (29) and (30) can be verified by substituting the formulas into Eqs. (21) and (26). The reader is free to verify this. When differentiating the functions of the ray variable (27), one should use conditions (25) and neglect the derivatives of the ratio of the coefficients $\varkappa/\varkappa_1$ in comparison with other functions. Formulas (29) and (30) represent the approximate solution that converges to the exact solution when constant c_R in (25) decreases. It can be shown that the error modulus of the formulas uniformly tends to zero on any cell chord in the limit $c_R \to 0$. At $c_R = 0$, approximate equalities turn into exact equalities. In the degenerate cell B, the solution is exact if the parameters $\theta_{\alpha-1}$ and θ_α do not tend to infinity simultaneously.

Solution of the Dirichlet Problem (Solution in an Inner Cell)

The boundary conditions of the Dirichlet problem are derived from the mixed boundary conditions (26) by passing in (26) to the limit $\theta_{\alpha-1}, \theta_\alpha \to 0$: $\psi^+(\xi_{\alpha-1}) = \psi_{\text{ent}}^+(\xi_{\alpha-1})$, $\psi^+(\xi_\alpha) = \psi_{\text{ent}}^+(\xi_\alpha)$. The solution of the problem on the chord $\xi_{\alpha-1} \le \xi \le \xi_\alpha$ is given by formulas (29) and (30) with parameters $\theta_{\alpha-1} = \theta_\alpha = 0$, $I_{\alpha-1} = \psi_{\text{ent}}^+(\xi_{\alpha-1})$, $I_\alpha = \psi_{\text{ent}}^+(\xi_\alpha)$. Let both ends of the chord lie on the inner boundaries between adjacent cells. If the constants $\psi_{\text{ent}}^+(\xi_{\alpha-1})$, $\psi_{\text{ent}}^+(\xi_\alpha)$ are selected equal to the values of the odd distribution emanating from the adjacent cells $\xi_{\alpha-2} \le \xi \le \xi_{\alpha-1}$ and $\xi_\alpha \le \xi \le \xi_{\alpha+1}$, we satisfy the stitching conditions (22) for even distributions. By sewing the odd distributions, we completely satisfy (22).

Solution in a Cell Adjacent to the Outer Boundary of the Domain

Consider the chord $\xi_{\alpha-1} \le \xi \le \xi_\alpha$ one of the endpoints of which belongs to the outer boundary of the domain, and the other to the inner border between cells. (The case when both ends of the chord lie on the outer boundary has been considered earlier.) Let the interior point be point ξ_α. We impose the Dirichlet conditions in it, assuming in (26) that $\theta_\alpha \to 0$, which yields $\psi^+(\xi_\alpha) = \psi_{\text{ent}}^+(\xi_\alpha)$. The solution to the problem is given by formulas (29) and (30) with parameters $\theta_\alpha = 0$ and $I_\alpha = \psi_{\text{ent}}^+(\xi_\alpha)$. If the constant $\psi_{\text{ent}}^+(\xi_\alpha)$ is specified as equal to the value of the even distribution emerging from the neighboring cell $\xi_\alpha \le \xi \le \xi_{\alpha+1}$, we satisfy the stitching condition (22) for even distributions $\psi^+(\xi_\alpha)$. By sewing further odd distributions $\psi^-(\xi_\alpha)$ we will completely satisfy (22).

Suppose that the classical Neumann conditions are set at the point ξ_α: $\theta_\alpha \to \infty$, $\psi^-(\xi_\alpha) = \psi_{\text{ent}}^-(\xi_\alpha)$. The solution is given by formulas (29) and (30) with the parameters $\theta_\alpha \to \infty$ and $I_\alpha = -\theta_\alpha\psi_{\text{ent}}^-(\xi_\alpha)$. If we set the constant $\psi^-_{ent}(\xi_\alpha)$ to be equal to the value of the odd distribution emerging from the neighboring cell, we satisfy the stitching condition (22) for the odd distributions.

Solution of the Neumann-Dirichlet Problem

We render the case considered above more specific by imposing the Neumann conditions at the endpoint $\xi_{\alpha-1}$ of the chord and the Dirichlet conditions at the endpoint ξ_α. For this purpose, we let $\theta_{\alpha-1} \to \infty$ and $\theta_\alpha \to 0$ in (26), which yields $\psi^-(\xi_{\alpha-1}) = \psi^-_{ent}(\xi_{\alpha-1})$, $\psi^+(\xi_\alpha) = \psi^+_{ent}(\xi_\alpha)$. The solution can easily be obtained from the formulas of the general solution (29) and (30) using the corresponding limit transition. Now we present it to the reader.

If point ξ_α lies on the inner boundary, then the boundary source $\psi^+_{ent}(\xi_\alpha)$ is sewn with an even distribution emerging from the neighboring cell. If the point ξ_α lies on the outer boundary, then $\psi^+_{ent}(\xi_\alpha)$ is given by (24). Similarly, we treat the boundary source $\psi^-_{ent}(\xi_{\alpha-1})$. It should be noted that to set the Neumann conditions on the external boundary, we pass to the limit $\chi_{in} \to 1$ in (24).

Solution of the Classical Neumann Problem

(The solution of the *generalized* Neumann problem [17, p. 158; 18, p. 9] is given by formulas (29) and (30) where parameters $\theta_{\alpha-1}$ and θ_α (24) are calculated with the tensor $G_{ij} = kD_{ij}$ and D_{ij} is the cotensor of the tensor D^{ij} (see(5)). We set at the endpoints of the chord $\xi_{\alpha-1} \le \xi \le \xi_\alpha$ the classical Neumann conditions, passing in (26) to the limits $\theta_{\alpha-1}, \theta_\alpha \to \infty$: $\psi^-(\xi_{\alpha-1}) = \psi^-_{ent}(\xi_{\alpha-1})$ and $\psi^-(\xi_\alpha) = \psi^-_{ent}(\xi_\alpha)$. The corresponding limit transitions in (29) and (30) yield the sought solution

$$\begin{aligned}
\psi^+(\xi) &\approx \frac{\varkappa_1(\xi)}{\varkappa(\xi)}\begin{bmatrix}\cosh(\xi_\alpha-\xi)/\sinh\Delta\xi_\alpha\\ -\cos(\xi_\alpha-\xi)/\sin\Delta\xi_\alpha\end{bmatrix}\left[\psi^-_{ent}(\xi_{\alpha-1}) + H^{\rightarrow}_{\alpha-1}(\xi)\right]\\
&\quad - \frac{\varkappa_1(\xi)}{\varkappa(\xi)}\begin{bmatrix}\cosh(\xi-\xi_{\alpha-1})/\sinh\Delta\xi_\alpha\\ -\cos(\xi-\xi_{\alpha-1})/\sin\Delta\xi_\alpha\end{bmatrix}\left[\psi^-_{ent}(\xi_\alpha) - H^{\leftarrow}_{\alpha}(\xi)\right],\\
\psi^-(\xi) &\approx \begin{bmatrix}\sinh(\xi_\alpha-\xi)/\sinh\Delta\xi_\alpha\\ \sin(\xi_\alpha-\xi)/\sin\Delta\xi_\alpha\end{bmatrix}\left[\psi^-_{ent}(\xi_{\alpha-1}) + H^{\rightarrow}_{\alpha-1}(\xi)\right]\\
&\quad + \begin{bmatrix}\sinh(\xi-\xi_{\alpha-1})/\sinh\Delta\xi_\alpha\\ \sin(\xi-\xi_{\alpha-1})/\sin\Delta\xi_\alpha\end{bmatrix}\left[\psi^-_{ent}(\xi_\alpha) - H^{\leftarrow}_{\alpha}(\xi)\right],\\
H^{\rightarrow}_{\alpha-1}(\xi) &= \int_{\xi_{\alpha-1}}^{\xi}\left[\begin{bmatrix}\cosh(\xi'-\xi_{\alpha-1})\\ \cos(\xi'-\xi_{\alpha-1})\end{bmatrix} q_0(\xi') + \frac{\varkappa}{\varkappa_1}\begin{bmatrix}\sinh(\xi'-\xi_{\alpha-1})\\ -\sin(\xi'-\xi_{\alpha-1})\end{bmatrix} q_1(\xi')\right]\frac{d\xi'}{\varkappa},\\
H^{\leftarrow}_{\alpha}(\xi) &= \int_{\xi}^{\xi_\alpha}\left[\begin{bmatrix}\cosh(\xi_\alpha-\xi')\\ \cos(\xi_\alpha-\xi')\end{bmatrix} q_0(\xi') - \frac{\varkappa}{\varkappa_1}\begin{bmatrix}\sinh(\xi_\alpha-\xi')\\ -\sin(\xi_\alpha-\xi')\end{bmatrix} q_1(\xi')\right]\frac{d\xi'}{\varkappa}.
\end{aligned} \tag{32}$$

The solution is valid in cells A and C and does not apply to the degenerate cell B. A solution to the classical Neumann problem in cell B exists if the sources q_0 and q_1 satisfy the condition of the problem's solvability.

Implicit Representation of a Solution to the Boundary-value Problem in a Cell

In the simplest case when domain V consists of one cell, we can formulate a system of integral equations to find a solution to the original boundary-value problem (1)–(5) under constraints (25). The domain may be non-simply connected. Let us substitute the general solution (29) and (30) in (8)–(10), setting in $= \alpha - 1$, out $= \alpha$,

$$
\begin{aligned}
U(\mathbf{r}) &= \int_{4\pi} \psi^{+}(\mathbf{\Omega},\mathbf{r})d\mathbf{\Omega} \approx \int_{4\pi} \frac{s_{\text{out}}^{\leftarrow}(\mathbf{\Omega},\mathbf{r})}{x(\mathbf{\Omega},\mathbf{r})}\left[I_{\text{in}}(\mathbf{\Omega},\mathbf{r}) + J_{\text{in}}^{\rightarrow}(\mathbf{\Omega},\mathbf{r})\right]d\mathbf{\Omega} \\
&\quad + \int_{4\pi} \frac{s_{\text{in}}^{\rightarrow}(\mathbf{\Omega},\mathbf{r})}{x(\mathbf{\Omega},\mathbf{r})}\left[I_{\text{out}}(\mathbf{\Omega},\mathbf{r}) + J_{\text{out}}^{\leftarrow}(\mathbf{\Omega},\mathbf{r})\right]d\mathbf{\Omega}, \\
W^{i}(\mathbf{r}) &= \int_{4\pi} \Omega^{i}\frac{\psi^{-}(\mathbf{\Omega},\mathbf{r})}{h(\mathbf{\Omega},\mathbf{r})}d\mathbf{\Omega} \approx \int_{4\pi} \frac{\Omega^{i}c_{\text{out}}^{\leftarrow}(\mathbf{\Omega},\mathbf{r})}{h(\mathbf{\Omega},\mathbf{r})x(\mathbf{\Omega},\mathbf{r})}\left[I_{\text{in}}(\mathbf{\Omega},\mathbf{r}) + J_{\text{in}}^{\rightarrow}(\mathbf{\Omega},\mathbf{r})\right]d\mathbf{\Omega} \\
&\quad - \int_{4\pi} \frac{\Omega^{i}c_{\text{in}}^{\rightarrow}(\mathbf{\Omega},\mathbf{r})}{h(\mathbf{\Omega},\mathbf{r})x(\mathbf{\Omega},\mathbf{r})}\left[I_{\text{out}}(\mathbf{\Omega},\mathbf{r}) + J_{\text{out}}^{\leftarrow}(\mathbf{\Omega},\mathbf{r})\right]d\mathbf{\Omega}, \\
\Psi_{3}^{ijk}(\mathbf{r}) &= \int P_{3}^{ijk}(\mathbf{\Omega})\frac{\psi^{-}}{h}d\mathbf{\Omega} = \ldots, \quad C^{ij}(\mathbf{r}) = \int \Omega^{i}\Omega^{j}\psi^{+}d\mathbf{\Omega}\Big/\int \psi^{+}d\mathbf{\Omega} = \ldots, \\
I_{\text{in}} &= \psi_{\text{ent}}^{+}(\xi_{\text{in}}) + \theta_{\text{in}}\psi_{\text{ent}}^{-}(\xi_{\text{in}}), \quad I_{\text{out}} = \psi_{\text{ent}}^{+}(\xi_{\text{out}}) - \theta_{\text{out}}\psi_{\text{ent}}^{-}(\xi_{\text{out}}).
\end{aligned} \tag{33}
$$

When the total solid angle 4π is traversed, the contributions of each distributed and boundary source are counted twice: once on the ray $\mathbf{\Omega}$ and for the second time on the ray of the opposite direction $-\mathbf{\Omega}$. The parameters θ_{in}, θ_{out}, and $\psi_{\text{ent}}^{\pm}$ are given by formulas (24). The sources q_0 and q_1 (11), (16), (17) included in the ray integrals $J_{\text{in}}^{\rightarrow}$ and $J_{\text{out}}^{\leftarrow}$ (31) depend on the distribution moments. The coordinate ξ (19) depends on tensor C^{ij}. The solution of the integral equations can be found by simple iterations (see (18)).

Note. Some of the integrals in (33) have the form $\int\int \ldots dl'\, d\mathbf{\Omega}$. If we change from the spherical coordinates to the Cartesian coordinates $dl'\, d\mathbf{\Omega} \to d\mathbf{r}'\big/|\mathbf{r}-\mathbf{r}'|^2$, then these integrals are transformed into integrals over the cell volume. The change of variables should only be done if it simplifies the calculation. In the general case, it is advisable to calculate the integrals by applying the quadrature formulas on the unit sphere and quadrature formulas on the chord.

4. FINITE ANALYTIC SCHEME

A finite-analytic scheme is a discrete approximation of the solution of a problem constructed by stitching the exact (almost exact) solutions of the differential equations emerging from the cells. The analytic solutions are stitched on the boundaries of the cells into which the region is divided. The term arose as an analog of the term *finite difference scheme* [8]. It should be noted that the latter is a discrete approximation constructed by stitching finite differences of the solution in the cells.

We will construct a finite analytic scheme for finding a solution to the system of ordinary differential equations (21)–(24) on the set of rays that cross the region.

Elements of the Scheme

Like other discrete schemes (finite difference schemes, finite element schemes), the finite analytic scheme has a stencil. A stencil is the set of points located on the smooth surfaces of the cell boundary and inside the cell. The points inside the cell will be called *central* points, the points on the outer boundary of the region (if the cell is adjacent to the boundary) will be referred to as *external* points. The central and external points belong to the stencil of only one cell. In addition to these points, there are *internal* points in the stencil located on the smooth surfaces of the internal boundary separating two cells. Any internal point is simultaneously part of two stencils of adjacent cells.

The set of rays passes through the stencil points of all cells. Several rays pass through one point of the stencil. It is advisable to choose the direction and number of the rays so as to ensure the calculation of the moments U, W^{i}, Ψ_{2}^{ij}, and Ψ_{3}^{ijk} at this point with an accuracy that meets the requirements of the problem being solved. The moments (angular integrals) are calculated using the quadrature formula defined on the surface of the unit sphere. In order to calculate the moments at all points of all stencils, the set of typical quadrature formulas is constructed. Methods for constructing quadrature formulas on the unit sphere are developed in the theory of particle transport.

One ray crosses the borders of cells at a series of points. It is advisable to choose the shape of the cells and arrange the stencil points so as to minimize the number of "additional" points that do not belong to stencils in this series.

In the particle transport theory, long rays that cross the entire region are called long characteristics. The best spatial-angular grids with long characteristics in the sense of minimizing the number of additional points and minimizing the number of typical quadrature formulas are regular grids constructed on cells of a regular symmetrical shape. These are regular tetrahedra, prisms, which are based on regular hexagons, rectangles, regular triangles, and others. However, using regular grids, it is often impossible to convey the position of the discontinuities of coefficients and sources in the problem, as well as the position of the external region boundary. Therefore, along with regular grids, irregular grids are also often used.

The scheme's elements also include quadrature formulas for calculating integrals from sources along rays $J_\alpha^{\leftrightarrow}$ (31) between the points of the stencil.

Equations of the Scheme

Let the ray $\mathbf{r} = \mathbf{r}_0 + \mathbf{\Omega} l$ cross the internal boundaries between the cells at the points $\mathbf{r}_\alpha(l_\alpha)$, $\xi_\alpha = \xi_\alpha(l_\alpha)$. These are the internal stencil points and *additional points* (if any). We set the Dirichlet conditions at all these points and stitch the even distributions in accordance with stitching conditions (22):

$$\theta_\alpha = 0,\quad \psi^+_{\text{ent}}(\xi_\alpha) = \psi^+(\xi_\alpha) \quad (\text{in} + 1 \le \alpha \le \text{out} - 1). \tag{34}$$

Next, we write formula (30) for an odd distribution $\psi^-(\xi)$ at one end and the other end of each chord $\xi_{\alpha-1}$, ξ_α. Then the odd distributions emanating from neighboring cells are sewn. As a result, we obtain equations of a finite analytic discrete scheme on the ray:

$$\begin{cases} \psi^-(\xi_\alpha) \approx -[e_\alpha^{\rightarrow} + k_\alpha^{\rightarrow}]\psi^+(\xi_\alpha) + B_\alpha^{\rightarrow}, \\ \psi^-(\xi_\alpha) \approx [e_{\alpha+1}^{\leftarrow} + k_{\alpha+1}^{\leftarrow}]\psi^+(\xi_\alpha) - e_{\alpha+1}^{\leftarrow}\psi^+(\xi_{\alpha+1}) - B_{\alpha+1}^{\leftarrow}, \end{cases} \quad (\alpha = \text{in} + 1),$$
$$\begin{cases} \psi^-(\xi_\alpha) = e_\alpha^{\rightarrow}\psi^+(\xi_{\alpha-1}) - [e_\alpha^{\rightarrow} + k_\alpha^{\rightarrow}]\psi^+(\xi_\alpha) + B_\alpha^{\rightarrow}, \\ \psi^-(\xi_\alpha) = [e_{\alpha+1}^{\leftarrow} + k_{\alpha+1}^{\leftarrow}]\psi^+(\xi_\alpha) - e_{\alpha+1}^{\leftarrow}\psi^+(\xi_{\alpha+1}) - B_{\alpha+1}^{\leftarrow}, \end{cases} \quad (\text{in} + 2 \le \alpha \le \text{out} - 2), \tag{35}$$
$$\begin{cases} \psi^-(\xi_\alpha) = e_\alpha^{\rightarrow}\psi^+(\xi_{\alpha-1}) - [e_\alpha^{\rightarrow} + k_\alpha^{\rightarrow}]\psi^+(\xi_\alpha) + B_\alpha^{\rightarrow}, \\ \psi^-(\xi_\alpha) = [e_{\alpha+1}^{\leftarrow} + k_{\alpha+1}^{\leftarrow}]\psi^+(\xi_\alpha) - B_{\alpha+1}^{\leftarrow}, \end{cases} \quad (\alpha = \text{out} - 1),$$

where $e_\alpha^{\leftrightarrow}$, $k_\alpha^{\leftrightarrow}$, and $B_\alpha^{\leftrightarrow}$ are the coefficients of the scheme:

$$\begin{aligned} e_\alpha^{\rightarrow} &= \frac{1}{x_\alpha(\xi_\alpha)},\quad k_\alpha^{\rightarrow} = \frac{c_{\alpha-1}^{\rightarrow}(\xi_\alpha) - 1}{x_\alpha(\xi_\alpha)} \quad (\text{in} + 1 \le \alpha \le \text{out} - 1), \\ e_\alpha^{\leftarrow} &= \frac{1}{x_\alpha(\xi_{\alpha-1})},\quad k_\alpha^{\leftarrow} = \frac{c_\alpha^{\leftarrow}(\xi_{\alpha-1}) - 1}{x_\alpha(\xi_{\alpha-1})} \quad (\text{in} + 2 \le \alpha \le \text{out}), \end{aligned} \tag{36}$$

$$\begin{aligned} B_\alpha^{\rightarrow} &= \frac{J_{\alpha-1}^{\rightarrow}(\xi_\alpha)}{x_\alpha(\xi_\alpha)} + e_{\text{in}+1}^{\rightarrow}\begin{bmatrix} 0,\ \ \text{in} + 2 \le \alpha \le \text{out} - 1 \\ \psi^+_{\text{ent}}(\xi_{\text{in}}) + \theta_{\text{in}}\psi^-_{\text{ent}}(\xi_{\text{in}}),\ \ \alpha = \text{in} + 1 \end{bmatrix}, \\ B_\alpha^{\leftarrow} &= \frac{J_\alpha^{\leftarrow}(\xi_{\alpha-1})}{x_\alpha(\xi_{\alpha-1})} + e_{\text{out}}^{\leftarrow}\begin{bmatrix} 0,\ \ \text{in} + 2 \le \alpha \le \text{out} - 1 \\ \psi^+_{\text{ent}}(\xi_{\text{out}}) - \theta_{\text{out}}\psi^-_{\text{ent}}(\xi_{\text{out}}),\ \ \alpha = \text{out} \end{bmatrix}. \end{aligned} \tag{37}$$

If we eliminate the fluxes $\psi^-(\xi_\alpha)$, the equations of scheme (35) are transformed to a closed system of algebraic equations for $\psi^+(\xi_\alpha)$ with a square tridiagonal matrix $n \times n$, where $n = \text{out} - \text{in} - 1$ is the number of points (34) at which the analytical solutions are stitched. Since the boundary-value problem on the ray is one-dimensional, the dimensionality of the system is small.

Numerical Solution

Suppose that the system of algebraic equations of the finite analytic scheme is solvable. Due to the small dimensionality of the system, its solution can be found by direct (non-iterative) algorithms of computational linear algebra, such as the Gaussian elimination method with the choice of the maximum element and the tridiagonal matrix algorithm (economical version of the Gauss method for systems with a three-diagonal matrix). After finding the even and odd distributions $\psi^{\pm}(\xi_{\alpha})$ at the points at which the ray crosses the boundaries between the cells, the solution is found at the outer and central points, as well as in the continuum of the points of the ray (if required). The solution on the ray is recovered using formulas (29)–(32).

Note. The solvability of system (35) and (36) is closely related to the existence of solutions in the cells. The following statement holds true. If a ray intersects only cells of type A and B (in these cells there always exists a unique solution to the inhomogeneous boundary-value problem), then the system of equations is always solvable and the solution can be found by the tridiagonal matrix algorithm. In fact, in cells A and B, the inequalities $e_{\alpha}^{\leftrightarrow} > 0$, $k_{\alpha}^{\leftrightarrow} \geq 0$ are satisfied. At the same time, the same inequalities provide a condition for the stability of the tridiagonal matrix algorithm.

Similar systems (35) and (36) (boundary-value problems on the ray) are solved on all rays that make up the spatial-angular grid of the problem.

Next, at the points of the stencils of all cells, using the quadrature formulas, the moments of distributions U, W^i, Ψ_2^{ij}, Ψ_3^{ijk} and tensors C^{ij}, g_{ij} are calculated and the values of the sources q_0 and q_1 entering into the equations of the kinetic problem are specified. This completes the simple iteration step. Iterations to refine the moments and tensors are carried out until convergence.

DISCUSSION

The finite analytic scheme constructed above has some advantages and disadvantages in comparison with difference schemes and finite element schemes.

The accuracy of finite difference schemes and finite element schemes substantially depends on the size and shape of the cells. In a number of problems, the size cell has to be significantly reduced to achieve the required accuracy of the solution. In the finite analytic scheme, there is no hard dependence of the approximation accuracy on the cell size. The accuracy of the scheme is determined by condition (25), the accuracy of the quadrature formulas for calculating the integrals along rays (31) and (37), and the accuracy of the quadrature formulas on the unit sphere for calculating the angular integrals. The number of rays passing through the stencil point should not be too small (not less than ~10). Calculations can be performed on coarse (sparse) spatial grids.

The change to ray variables reduces the solution of the multidimensional boundary-value problem to the solution of a series of one-dimensional problems on the rays. The system of algebraic equations of scheme (35) has a small dimensionality. Therefore, its solution can be found by direct algorithms (the Gaussian elimination technique or the tridiagonal matrix algorithm). The dimensionality of the system of algebraic equations of finite difference and finite element schemes can be large (multidimensionality of the problem, small cell size). Systems of large dimensionality are to be solved by iterative algorithms of computational linear algebra [27]. If the region contains many degenerate cells of type B and/or close to them, then the iterations converge slowly.

The finite analytic scheme has additional elements that are not used in the finite-difference and finite-element schemes. These are rays that make up the spatial-angular grid, quadrature formulas on the unit sphere, the metric tensor, and iteration over the values of moments and tensors. Therefore, this scheme will show its maximum efficiency when solving applied problems in which these additional elements are minimal, for example, problems with regular spatial-angular grids built on the basis of cells of regular symmetric shape.

FUNDING

This study was supported by the Russian Science Foundation, project no. 14-11-00699.

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

REFERENCES

1. V. S. Vladimirov, "Numerical solution of the kinetic equation for a sphere," in *Computational Mathematics, Collection of Articles,* Ed. by A. A. Abramov (Akad. Nauk SSSR, Moscow, 1958), No. 3, pp. 3–33.
2. A. N. Tikhonov and A. A. Samarskii, "Homogeneous difference schemes of a high degree of accuracy on nonuniform nets," USSR Comput. Math. Math. Phys. **1**, 465–486 (1962).
3. A. A. Samarskii, *Introduction to the Theory of Differential Schemes* (Nauka, Moscow, 1971) [in Russian].
4. A. M. Il'in, "Differencing scheme for a differential equation with a small parameter affecting the highest derivative," Math. Notes Acad. Sci. USSR **6**, 596–602 (1969).
5. K. V. Emelyanov, "Difference scheme for a three-dimensional elliptic equation with a small parameter in the highest derivatives," Tr. Inst. Mat. Mekh. Ural. Nauch. Tsentra AN SSSR, No. **11**, 30–42 (1973) [in Russian].
6. Yu. N. Kulikov and B. N. Chetverushkin, "An implicit difference method for temperature determination in problems of radiation gas dynamics," USSR Comput. Math. Math. Phys. **13**, 174–186 (1974).
7. A. A. Charakhchyan, "Calculation of unsteady, spherically symmetric flows of radiating gray gas," in *The Dynamics of the Emitting Gas,* Ed. by V. V. Aleksandrov (Vychisl. Tsentr AN SSSR, Moscow, 1974), No. 1, pp. 54–74 [in Russian].
8. C. J. Chen, Naseri-H. Nashet, and K. S. Ho, "Finite analytic numerical solution of heat transfer in two dimensional cavity flow," J. Numer. Heat Transfer **4**, 179–197 (1981).
9. Y. Sun and J. Militzer, "The piecewise parabolic finite analytic method. Part I: Theory," Appl. Math. Modell. **16**, 576–588 (1992).
10. C. J. Chen, R. A. Bernatz, K. D. Carlson, W. Lin, and D. G. de Vahl, "Finite analytic method in flows and heat transfer," Appl. Mech. Rev. **55** (2), B34–B34 (2002).
11. J. P. Pontaza, H. C. Chen, and J. N. Reddy, "A local-analytic-based discretization procedure for the numerical solution of incompressible flows," Num. Methods Fluids **49**, 657–699 (2005).
12. A. V. Shilkov, "Even and odd parity kinetic equations of particle transport. 2: A finite analytic characteristic scheme for one dimensional problems," Math. Models Comput. Simul. **7**, 36–50 (2015).
13. A. V. Shilkov, "Even and odd parity kinetic equations of particle transport. 3: Finite analytic schemes on tetrahedral," Math. Models Comput. Simul. **7**, 409–429 (2015).
14. M. M. Basko and I. P. Tsygvintsev, "A hybrid model of laser energy deposition for multidimensional simulations of plasmas and metals," Comput. Phys. Commun. **214**, 59–70 (2017).
15. A. V. Shilkov, "On the solution of elliptic equations by the ray variable method," KIAM Preprint No. 119 (Keldysh Inst. Appl. Math., Moscow, 2017).
16. N. S. Koshlyakov, E. B. Gliner, and M. M. Smirnov, *Partial Differential Equations of Mathematical Physics* (Vyssh. Shkola, Moscow, 1970) [in Russian].
17. V. M. Babich, M. B. Kapilevich, and S. G. Mikhlin, *Linear Equations of Mathematical Physics. The Handbook* (Nauka, Moscow, 1964) [in Russian].
18. C. Miranda, *Partial Differential Equations of Elliptic Type* (Springer, Berlin, Heidelberg, 1970).
19. R. Courant, and D. Hilbert, *Methods of Mathematical Physics,* Vol. 2: *Partial Differential Equations* (Wiley Interscience, New York, 1962).
20. E. M. Landis, *Second Order Partial Differential Equations of Elliptic and Parabolic Type,* Vol. 171 of *AMS Transl. of Math. Monographs* (AMS, Philadelphia, 1998).
21. Ladyzhenskaya O.A. and Ural'tseva N.N., *Linear and Quasilinear Elliptic Equations* (Academic Press, New York, 1968).
22. S. G. Mikhlin, *Linear Equations of Mathematical Physics* (Holt, Rinehart and Winston, 1967; Vyssh. Shkola, Moscow, 1977).
23. B. M. Budak, A. A. Samarskii, and A. N. Tikhonov, *A Collection of Problems in Mathematical Physics* (Pergamon, Oxford, 1964; Fizmatlit, Moscow, 2004).
24. A. D. Polyanin and V. E. Nazaikinskii, *Linear Partial Differential Equations for Engineers and Scientists,* 2nd ed. (CRC, Boca Raton, 2016; Fizmatlit, Moscow, 2001).
25. Yu. D. Burago and V. A. Zalgaller, *Introduction to Riemannian Geometry* (Nauka, St. Petersburg, 1994) [in Russian].
26. P. K. Rashevsky, *Riemannian Geometry and Tensor Analysis* (Nauka, Moscow, 1967) [in Russian].
27. J. W. Demmel, *Applied Numerical Linear Algebra* (SIAM, Philadelphia, PA, USA, 1997).

Translated by I. Pertsovskaya