The Effect of Electron-Ion Collisions on Breaking Cylindrical Plasma Oscillations

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Abstract—The influence of electron-ion collisions on breaking cylindrical nonlinear plasma oscillations is studied. Numerical calculations by the particle method and an analytic analysis by the perturbation method in the weak nonlinearity regime show that, with an increasing collision frequency, the time needed to break plasma oscillations increases. The threshold value of the collision frequency is found exceeding which the density singularity does not arise. In this case, the maximum of the electron density formed outside the axis of the oscillations, the growth of which in the regime of rare collisions leads to the breaking effect, after some growth begins to decrease due to the damping of the oscillations.

Keywords: plasma oscillations, breaking effect, electron-ion collisions, perturbation method, numerical simulation, particle method

1. INTRODUCTION

When studying plasma oscillations and waves, it should be borne in mind that in the absence of dissipation even a relatively small initial collective displacement of particles can lead to the appearance of a singularity of the electron density [1]. This effect is called breaking the oscillations. It was shown in the monograph [2] that the singularity, i.e., going to infinity, of the the electron density in the Euler description of the medium's motion is equivalent to the intersection of the electron trajectories in its Lagrangian description.

For a one-dimensional plane non-linear plasma wave, the limiting amplitude of the electric field up to which the wave can exist and when approaching which the electron density perturbations become infinitely large was determined in [3]. However, it was shown in [4, 5] that oscillations can also be broken with field amplitudes lower than the limiting value but not long after their excitation. Such oscillations are conveniently called *long-lived* and their breaking time is inversely proportional to the third power of the electric field, which leads to the fast increase in the breaking time on a decrease in the amplitude of the oscillations. In the case of one-dimensional plane geometry, the breaking of long-lived oscillations considered in [5] is related to the amplitude dependence of frequency due to relativistic effects. For cylindrical and spherical oscillations, the breaking is due to the contribution of electron nonlinearities to the frequency shift and explained by the intersection of the electron trajectories [4].

The results given above were significantly refined in [6, 7]. In particular, it was found that breaking long-lived oscillations is associated with the formation and sharp growth in time of the maximum electron density localized in space, which is located outside the axis of the symmetry of the problem. In addition, it was shown that nonlinear cylindrical oscillations are broken almost one-and-a-half times faster than it was predicted in [4] because of the intersection of the trajectories of the neighboring particles rather than the trajectories of particles separated from each other by a radius that is twice the amplitude. In this work, the results obtained in [6] for breaking cylindrical oscillations are summarized taking into account the electron-ion collisions.

It should be noted that attempts were made earlier to take into account the effect of collisions on electronic oscillations (see, e.g., [8, 9]). However, the results obtained there relied on the non-relativistic plasma model in the case of the plane geometry and, moreover, the plasma resistance was taken into account simultaneously with the viscosity. As a result, the breaking effect was outside the attention of

researchers. For the first time, it was possible to trace the effect of electron collisions on breaking plane plasma oscillations in [10], where the origin of the breaking was the allowance for only the relativistic factor.

The article has the following structure. First, a detailed statement of the problem is given in the Eulerian and Lagrangian variables, including the initial and boundary conditions necessary for describing the time evolution of localized plasma oscillations. Then the results of the analytical consideration of the formulated problem of the perturbation method [11] in the weak nonlinearity regime are set forth. The following is a description of the numerical simulation of the cylindrical oscillations of the plasma, the particle method, according to the so-called leapfrog scheme [12]. The results of the calculations demonstrate that the breaking of plasma oscillations is associated with the formation of an electron density maximum outside the axis of oscillations, which increases in magnitude with time and becomes infinite after several periods. Accounting for electron-ion collisions leads to the fact that the time of the breaking of plasma oscillations increases with the increasing frequency of collisions. In addition, it was found that, for each initial amplitude of the electric field, there is a certain threshold value of the collision frequency, which when exceeded does not lead to a density singularity. In conclusion, the results of the studies are systematized.

2. STATEMENT OF THE PROBLEM

2.1. Basic equations. We consider the plasma as a cold relativistic electron liquid ignoring the recombination effects and the motion of ions. Then the system of hydrodynamic equations describing it together with the Maxwell equations in the vector form appears in the form

$$\frac{\partial n}{\partial t} + \operatorname{div}(n\mathbf{v}) = 0,$$

$$\frac{\partial \mathbf{p}}{\partial t} + (\mathbf{v}\nabla)\mathbf{p} = e\mathbf{E} + \frac{e}{c}[\mathbf{v}\times\mathbf{B}] - \nu_{ei}\mathbf{p}, \quad \mathbf{v} = \frac{\mathbf{p}}{m\gamma}, \quad \gamma = \sqrt{1 + \frac{|\mathbf{p}|^2}{m^2c^2}}, \tag{1}$$

$$\frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} = -\frac{4\pi}{c}en\mathbf{v} + \operatorname{rot}\mathbf{B}, \quad \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} = -\operatorname{rot}\mathbf{E},$$

where e, m are the electron charge and mass (an electron charge has a negative sign: $e < 0$); c is the speed of light; $n, \mathbf{p}$, and $\mathbf{v}$ are the electron concentration, momentum, and velocity; γ is the Lorentz factor; and $\mathbf{E}$ and $\mathbf{B}$ are electric and magnetic field vectors.

The system of equations (1) is one of the simplest plasma models, which is often called the hydrodynamic equations of the cold plasma; it is well known and described in sufficient detail in textbooks and monographs (see, e.g., [13−16]).

We draw attention to the presence in the equation for the momentum of the term $\nu_{ei}\mathbf{p}$ describing electron-ion collisions. Accounting for this effect can be interpreted as the friction force between particles; in the non-relativistic case (see, e.g., [13], p. 44), the formula of the following form is used:

$$\mathbf{g}_\alpha = -\nu_{\alpha\beta}(\mathbf{v}_\alpha - \mathbf{v}_\beta),$$

where $\nu_{\alpha\beta}$ is the effective frequency of collisions of charged particles of type α with particles of type β when $\alpha \neq \beta$. A detailed description of the formulas for plasma transfer coefficients is presented in [17].

2.2. Statement in Euler variables. From the basic equations of the plasma model under consideration (1), we obtain a system whose solutions have the axial (cylindrical) symmetry.

We will denote independent variables in a cylindrical coordinate system in the usual manner by (r, φ, z). Let us assume the following points:

—the solution is determined only by the r-components of the vector functions $\mathbf{p}, \mathbf{v}$, and $\mathbf{E}$;

—the dependence on variables φ and z in these functions is absent, i.e., $\partial/\partial\varphi = \partial/\partial z = 0$.

Then the non-trivial equations follow from system (1):

$$\frac{\partial n}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(rnv_r) = 0, \quad \frac{\partial p_r}{\partial t} + v_r\frac{\partial p_r}{\partial r} = eE_r - \nu_{ei}\,p_r,$$

$$\gamma = \sqrt{1 + \frac{p_r^2}{m^2c^2}}, \quad v_r = \frac{p_r}{m\gamma}, \quad \frac{\partial}{\partial t}E_r = -4\pi env_r. \tag{2}$$

We introduce dimensionless quantities

$$\rho = k_p r, \quad \theta = \omega_p t, \quad N = \frac{n}{n_0}, \quad V = \frac{v_r}{c}, \quad P = \frac{p_r}{mc}, \quad E = -\frac{eE_r}{mc\omega_p}, \quad \nu = \frac{\nu_{ei}}{\omega_p},$$

where $\omega_p = \sqrt{4\pi e^2 n_0/m}$ is the plasma frequency, n_0 is the unperturbed electron density, and $k_p = \omega_p/c$. In the new variables, system (2) takes the form

$$\frac{\partial}{\partial\theta} N + \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho N V) = 0, \quad \frac{\partial}{\partial\theta} P + E + V\frac{\partial}{\partial\rho} P + \nu P = 0,$$

$$\gamma = \sqrt{1 + P^2}, \quad V = \frac{P}{\gamma}, \quad \frac{\partial}{\partial\theta} E = NV. \tag{3}$$

From the first and last equations (3), it follows that

$$\frac{\partial}{\partial\theta}\left[N + \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E) \right] = 0.$$

This relation is true both in the absence of the plasma oscillations ($N \equiv 1, E \equiv 0$) and their presence. Therefore, from here we have a simpler expression for the electron density

$$N = 1 - \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E). \tag{4}$$

Now, excluding the density N and factor γ from system (3), we arrive at the equations describing the free cylindrical one-dimensional relativistic oscillatory motions of electrons in a cold plasma taking into account collisions:

$$\frac{\partial}{\partial\theta} P + E + V\frac{\partial}{\partial\rho} P + \nu P = 0, \quad \frac{\partial}{\partial\theta} E - V + V\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E) = 0, \quad V = \frac{P}{\sqrt{1 + P^2}}. \tag{5}$$

Further, we assume that inequality $\nu \ll 1$ meaning that the frequency of electron-ion collisions ν_{ei} significantly is less than the plasma frequency ω_p.

We consider the electron oscillations in the vicinity of the axis $\rho = 0$. We assume that the velocity and momentum of electrons at the initial moment of time ($\theta = 0$) are zero,

$$P(\rho, \theta = 0) = 0, \quad V(\rho, \theta = 0) = 0, \tag{6}$$

and we assume that the oscillations are excited at the initial time by the following electric field:

$$E(\rho, \theta = 0) = \left(\frac{a_*}{\rho_*}\right)^2 \rho \exp\left(-2\frac{\rho^2}{\rho_*^2}\right), \tag{7}$$

where the parameters ρ_* and a_* characterize the scale of the localization region and the maximum value $E_{\max} = a_*^2/(\rho_* 2\sqrt{e}) \approx 0.3\, a_*^2/\rho_*$ of the electric field (7), respectively. The form of function (7) is chosen from the considerations that such oscillations can be excited in the rarefied plasma ($\omega_l \gg \omega_p$) by a laser pulse with frequency ω_l when it is focused with a spherical lens, and when the focal spot has the shape of a circle.

If the electric field of laser radiation has the axial symmetry and the Gaussian distribution over the spatial coordinates and time

$$E_L(\rho, z, t) = E_{0L} \exp\left[-\frac{\rho^2}{\rho_*^2} - \frac{\omega_p^2}{\tau_*^2}\left(t - \frac{z}{c}\right)^2 \right] \cos\left[\omega_l\left(t - \frac{z}{c}\right) \right], \tag{8}$$

where $\tau_* = \omega_p \tau_p$ and $\rho_* = k_p R_p$ are dimensionless values of the duration τ_p and the radius of the focal spot R_p of the laser pulse, then at some point z remote from the trailing edge of the pulse at a distance larger

than the length of the plasma wave ($k_p |z| \gg 1$), the following relation holds true relating quantity a_* to the laser pulse parameters [6]

$$a_*^2 = a_0^2 \sqrt{\pi/2} \tau_* \exp(-\tau_*^2/8), \tag{9}$$

where $a_0 = eE_{0L}/(m\omega_l c)$ is the normalized amplitude of the laser field. Under the optimal wake wave excitation ($\tau_* = 2$), when its amplitude is maximum, the relation (9) takes the form $a_*^2 = \sqrt{2\pi/e} a_0^2 \approx 1.52 a_0^2$.

Note that at large distances from the axis $\rho = 0$, due to the initial condition (7), the plasma oscillations are not excited. Therefore, we assume that the following conditions are met:

$$P(\rho \to \infty, \theta) = 0, \quad V(\rho \to \infty, \theta) = 0, \quad E(\rho \to \infty, \theta) = 0. \tag{10}$$

Thus, taking into account the specificity of the cylindrical coordinate system ($\rho \geq 0$), we will study in the first quadrant $\{(\rho, \theta): 0 < \rho < \infty, \theta > 0\}$ the solutions of system (5) determined by the initial and boundary conditions (6), (7), and (10).

2.3. Statement in Lagrangian variables. The quasi-linear system of equations (5) is written in Eulerian variables; its form in Lagrangian variables will be useful in the future:

$$\frac{d}{d\theta} P(\rho_0, \theta) + E(\rho_0, \theta) + \nu P(\rho_0, \theta) = 0,$$
$$\frac{d}{d\theta} E(\rho_0, \theta) + \frac{V(\rho_0, \theta)E(\rho_0, \theta)}{\rho} = V(\rho_0, \theta), \tag{11}$$

where $d/d\theta = \partial/\partial\theta + V\partial/\partial\rho$ is the total time derivative.

We recall that the function $R(\rho_0, \theta)$ determines the displacement of a particle with a Lagrangian coordinate ρ_0, so that

$$\rho(\rho_0, \theta) = \rho_0 + R(\rho_0, \theta), \tag{12}$$

satisfies the equation

$$\frac{d}{d\theta} R(\rho_0, \theta) = V(\rho_0, \theta). \tag{13}$$

Expressing velocity V in terms of the displacement R in accordance with formula (13), we write the second equation (11) in the form

$$(\rho_0 + R)\frac{d}{d\theta} E + E\frac{d}{d\theta} R = (\rho_0 + R)\frac{d}{d\theta} R. \tag{14}$$

Equation (14) has the first integral

$$(\rho_0 + R) E = \frac{1}{2}(\rho_0 + R)^2 + C,$$

where the constant C is determined from the condition of equality of the electric field to zero in the absence of the particle displacement. Then from this relation we find the expression for the electric field (see also [4, 6])

$$E(\rho_0, \theta) = \frac{1}{2} \frac{[\rho_0 + R(\rho_0, \theta)]^2 - \rho_0^2}{\rho_0 + R(\rho_0, \theta)}, \tag{15}$$

and the basic system of equations in Lagrangian variables (11) takes the form

$$\frac{d}{d\theta} P(\rho_0, \theta) + E(\rho_0, \theta) + \nu P(\rho_0, \theta) = 0,$$
$$\frac{d}{d\theta} R(\rho_0, \theta) = \frac{P(\rho_0, \theta)}{\sqrt{1 + P^2(\rho_0, \theta)}} \equiv V(\rho_0, \theta). \tag{16}$$

We note that relation (15) is very useful for determining the Lagrangian coordinate particle ρ_0 and the initial condition $R(\rho_0, 0)$ by the given distribution (7): $E_0(\rho) = E(\rho, \theta = 0)$. The algorithm is as follows: for some ρ from the equation (see relationship (15) at $\theta = 0$)

$$\frac{1}{2}\frac{\rho^2 - \rho_0^2}{\rho} = E_0(\rho)$$

the quantity is determined from the explicit formula

$$\rho_0 = \sqrt{\rho^2 - 2\rho E_0(\rho)}, \tag{17}$$

and then from (12) the sought initial displacement at point ρ_0 is found:

$$R(\rho_0, 0) = \rho - \rho_0. \tag{18}$$

In summary, the trajectories of all particles, each of which is identified by the Lagrangian coordinate ρ_0, can be determined by the independent integration of systems of ordinary differential equations of form (15) and (16). To this end, two initial conditions are required: $R(\rho_0, 0)$ and $P(\rho_0, 0)$. From (6), we have $P(\rho_0, 0) = 0$. To determine $R(\rho_0, 0)$, first set the position of particle ρ in the initial time, then the Lagrangian coordinate is determined from formula (17), and the initial displacement is calculated by formula (18). Knowledge of the Lagrangian coordinate ρ_0 and the displacement function $R(\rho_0, \theta)$ unambiguously characterizes the particle trajectory by formula (12).

3. THEORY IN THE WEAK NONLINEARITY REGIME

The presented system of equations (15) and (16) makes it possible to analytically (by the perturbation method [11]) analyze the trajectories of particles in the case of a weak nonlinearity. The electron density in the plasma oscillations is determined in terms of the particle displacement function by the following formula:

$$N\left(\rho_0 + R(\rho_0, \theta), \theta\right) = \frac{1}{\left\{1 + \dfrac{\partial}{\partial \rho_0} R(\rho_0, \theta)\right\}\left\{1 + \dfrac{R(\rho_0, \theta)}{\rho_0}\right\}}, \tag{19}$$

as follows from relations (4) and (15).

In the weak nonlinearity regime, i.e., when it is possible to use the expansion $P \approx V(1 + V^2/2)$, from the system of equations (16) follows one equation for the small diplacements $|R(\rho_0, \theta)| \ll \min\{1, \rho_0\}$ from the initial position

$$\left(\frac{d^2}{d\theta^2} + \nu\frac{d}{d\theta} + 1\right)R(\rho_0, \theta) + \frac{1}{2}\left(\frac{d}{d\theta} + \nu\right)\left[\frac{d}{d\theta}R(\rho_0, \theta)\right]^3 - \frac{R^2(\rho_0, \theta)}{2\rho_0} + \frac{R^3(\rho_0, \theta)}{2\rho_0^2} = 0. \tag{20}$$

The solution of Eq. (20) will be sought in the form

$$R(\rho_0, \theta) = R_0(\rho_0, \theta) + \frac{1}{2}\left\{R_1(\rho_0, \theta)\exp\left[-i\Psi(\rho_0, \theta)\right] + R_2(\rho_0, \theta)\exp\left[-2i\Psi(\rho_0, \theta)\right] + c.c.\right\}, \tag{21}$$

where $R_n(\rho_0, \theta), n = 0, 1, 2$, are the amplitudes of electron displacements at the zero, first, and second harmonics, which slowly vary in time over the oscillation period, $\Psi(\rho_0, \theta) = \int_0^\theta d\theta'\Omega(\rho_0, \theta')$, $\Omega(\rho_0, \theta)$ is the dimensionless oscillation frequency (the frequency divided by the plasma frequency ω_p), and $c.c.$ denotes conjugate terms. We apply the procedure for separating terms on time scales in Eq. (20). In this case, based on representation (21), the amplitudes of the zero and second harmonics of the electron displacements can be expressed in terms of the amplitude of the first harmonic:

$$R_0(\rho_0, \theta) = \frac{\left|R_1(\rho_0, \theta)\right|^2}{4\rho_0}, \quad R_2(\rho_0, \theta) = -\frac{\left|R_1(\rho_0, \theta)\right|^2}{12\rho_0}. \tag{22}$$

The equation for the amplitude of the displacement $R_1(\rho_0, \theta)$ in the main frequency has the form

$$i\left[\frac{d}{d\theta}R_1(\rho_0,\theta) + \frac{\nu}{2}R_1(\rho_0,\theta)\right] + \left[\Omega(\rho_0,\theta) - 1 + \frac{3}{16}\left|R_1(\rho_0,\theta)\right|^2 - \frac{1}{12\rho_0^2}\left|R_1(\rho_0,\theta)\right|^2\right]R_1(\rho_0,\theta) = 0. \tag{23}$$

We recall that the equation obtained is applicable when under the conditions $\nu \ll 1$ and $\left|R(\rho_0,\theta)\right| \ll \min\{1,\rho_0\}$.

We assume that the equality

$$\Omega(\rho_0,\theta) = 1 - \frac{3}{16}\left|R_1(\rho_0,\theta)\right|^2 + \frac{1}{12\rho_0^2}\left|R_1(\rho_0,\theta)\right|^2, \tag{24}$$

which determines the nonlinear correction to the frequency of cylindrical plasma oscillations, holds [6]. Then the solution of Eq. (23) can be written in the form

$$R_1(\rho_0,\theta) = R_1(\rho_0,\theta = 0)\exp\left(-\frac{1}{2}\nu\theta\right). \tag{25}$$

Taking into account relations (24) and (25), according to (21), the electron displacement at the plasma frequency is determined by the following formula:

$$R(\rho_0,\theta) = R_0(\rho_0)\exp\left(-\frac{1}{2}\nu\theta\right)\cos\left[\psi(\rho_0,\theta)\right], \tag{26}$$

where $R_0(\rho_0) = R(\rho_0,\theta = 0)$ is the particle displacement at the initial time and the phase $\Psi(\rho_0,\theta)$ has the form

$$\Psi(\rho_0,\theta) = \theta + \left(\frac{1}{12\rho_0^2} - \frac{3}{16}\right)R_0^2(\rho_0)\frac{1 - \exp(-\nu\theta)}{\nu}. \tag{27}$$

From formula (27) in the limit at $\nu \to 0$, we have the well-known result

$$\Psi(\rho_0,\theta) = \left[1 + \left(\frac{1}{12\rho_0^2} - \frac{3}{16}\right)R_0^2(\rho_0)\right]\theta$$

for the phase of nonlinear cylindrical oscillations of the collisionless plasma [6].

It follows from formula (19) that the density singularity occurs when the following condition holds:

$$1 + \frac{\partial}{\partial\rho_0}R(\rho_0,\theta) = 0. \tag{28}$$

Using formulas (26) and (27), from Eq. (28) we obtain the equation for determining the time at which the plasma oscillations are broken:

$$1 - G(\theta)\left\{\frac{R_0(\rho_0)}{12}\frac{\partial}{\partial\rho_0}\left[\frac{R_0^2(\rho_0)}{\rho_0^2}\right] - \frac{3}{8}R_0^2(\rho_0)\frac{\partial}{\partial\rho_0}R_0(\rho_0)\right\}\sin\left[\Psi(\rho_0,\theta)\right] = 0, \tag{29}$$

where

$$G(\theta) = \frac{1 - \exp(-\nu\theta)}{\nu}\exp\left(-\frac{1}{2}\nu\theta\right). \tag{30}$$

Let quantity ν be fixed, then the function $G(\theta)$ at low θ values increases with time, it reaches the maximum $G_{\max} = 2/(3\sqrt{3}\nu)$ at $\theta_{\max} = (1/\nu)\ln 3$, and then decreases and tends to zero at $\theta \to \infty$.

In analyzing Eq. (29), we take into account that the initial particle displacements in the weak nonlinearity regime in accordance with formula (7) are

$$R_0(\rho_0) = \left(\frac{a_*}{\rho_*}\right)^2\rho_0\exp\left(-2\frac{\rho_0^2}{\rho_*^2}\right). \tag{31}$$

In this case, under the condition $\left|R(\rho_0,\theta)\right| \ll \min\{1,\rho_0\}$, Eq. (29) takes the form

$$1 - G(\theta)F(\rho_0) = 0, \tag{32}$$

where

$$F(\rho_0) = \frac{3}{8}\frac{a_*^6}{\rho_*^4}F_1(\rho_0) + \frac{2}{3}\frac{a_*^6}{\rho_*^6}F_2(\rho_0), \qquad (33)$$

and the functions $F_1(\rho_0)$ and $F_2(\rho_0)$ are determined by the formulas

$$F_1(\rho_0) = \frac{\rho_0^2}{\rho_*^2}\left|1 - 4\frac{\rho_0^2}{\rho_*^2}\right|\exp\left(-6\frac{\rho_0^2}{\rho_*^2}\right), \quad F_2(\rho_0) = \frac{\rho_0^2}{\rho_*^2}\exp\left(-6\frac{\rho_0^2}{\rho_*^2}\right). \qquad (34)$$

In relations (32)−(34), the variable ρ_0 is indicated as the main argument, because the quantities a_* and ρ_* here appear as external parameters.

Equation (32) is basic for analyzing the effects of electron-ion collisions on the breaking time of oscillations. As quantity ν in Eq. (32) tends to zero, we obtain the relation of the form

$$1 - \theta F(\rho_0) = 0.$$

In particular, it follows from it that the shortest time of the occurrence of a density singularity is implemented under the conditions when $F(\rho_0)$ as a function of the variable ρ_0 reaches the maximum value. We introduce the notation for the time of the breaking of oscillations disregarding collisions

$$\theta_{wb}^{(0)} = 1\Big/\max_{\rho_0} F(\rho_0).$$

Then the basic equation (32) can be represented in a convenient form:

$$[1 - \exp(-\nu\theta)]\exp\left(-\frac{1}{2}\nu\theta\right) = \nu\theta_{wb}^{(0)} \equiv b. \qquad (35)$$

Equation (35) is cubic with respect to the unknown $\exp(-\nu\theta/2)$ and has physically meaningful real solutions $0 < \nu\theta < 1$ under the condition that parameter b satisfies the inequalities

$$0 < b \le \frac{2}{3\sqrt{3}} \approx 0.385. \qquad (36)$$

We note that an approximate solution of Eq. (35) for the time of the breaking under the condition $0 < \nu\theta_{wb}^{(0)} < 1$ has the form

$$\theta_{wb} \approx \theta_{wb}^{(0)}\left[1 + \nu\theta_{wb}^{(0)}\right]. \qquad (37)$$

It follows from formula (37) that allowance for collisions leads to the fact that the time of breaking increases. In this case, the breaking effect of the plasma oscillations takes place only under condition (36). In this case, the density singularity appears as a result of the increase in the off-axis density maximum.

When the dimensionless frequency of electron collisions ν exceeds the threshold value of $2\big/(3\sqrt{3}\,\theta_{wb}^{(0)})$, and the inequality inverse to (36) holds

$$b \equiv \nu\theta_{wb}^{(0)} > \frac{2}{3\sqrt{3}} \approx 0.385, \qquad (38)$$

the density singularity does not arise, which is associated with damping of the plasma oscillations. We note that at $\nu\theta_{wb}^{(0)} \to 2\big/(3\sqrt{3})$ the time of the breaking is $\theta_{wb} \approx 2.854\theta_{wb}^{(0)}$ as follows form the analysis of Eq. (35).

Now we turn to the consideration of important particular cases of the breaking of oscillations. Let the inequality $\rho_* \gg 1$ meaning that the plasma oscillations localized in a fairly large area of space be fulfilled. In this case, the correction to the frequency of oscillations is determined primarily by the relativistic effects and from (32) the equation given below follows:

$$1 - \frac{3}{8}\frac{a_*^6}{\rho_*^4}G(\theta)F_1(\rho_0) = 0. \qquad (39)$$

We have from Eq. (39) that the smallest time of the appearance of the density singularity is implemented under the conditions when $F_1(\rho_0)$ as a function of the variable ρ_0 reaches the maximum value.

From the explicit formula (34) it follows that the maximum of the function $F_1(\rho_0)$ is $F_{1,\max} = 1/(18\sqrt{e})$ and it is located at the distance $\rho_0 = \rho_*/(2\sqrt{3})$ from the axis of symmetry. With allowance for this circumstance, Eq. (39) can be written as

$$[1 - \exp(-\nu\theta)]\exp\left(-\frac{1}{2}\nu\theta\right) = \nu\theta_{1,wb}^{(0)}, \tag{40}$$

where $\theta_{1,wb}^{(0)}$ is the time of the breaking of weakly nonlinear oscillations of the collisionless ($\nu = 0$) plasma, which is determined by the following relation:

$$\theta_{1,wb}^{(0)} = \frac{8}{3}\frac{\rho_*^4}{a_*^6}\frac{1}{F_{1,\max}} = 48\sqrt{e}\frac{\rho_*^4}{a_*^6}. \tag{41}$$

Now let the plasma oscillations be excited in a narrow region of the space, when the condition $\rho_* \ll 1$ is fulfilled. In this case, another equation follows from Eq. (32):

$$1 - \frac{2}{3}\frac{a_*^6}{\rho_*^6}G(\theta)F_2(\rho_0) = 0. \tag{42}$$

Here also the smallest time of the appearance of the density singularity is implemented under the conditions when $F_2(\rho_0)$ as a function of variable ρ_0 reaches the maximum value. We have from the explicit formula (34) that the maximum of the function $F_2(\rho_0)$ is $F_{2,\max} = 1/(6e)$ and it is at the distance $\rho_0 = \rho_*/\sqrt{6}$ from the axis of symmetry. With allowance for this circumstance, Eq. (42) can be written in the form analogous to (40),

$$[1 - \exp(-\nu\theta)]\exp\left(-\frac{1}{2}\nu\theta\right) = \nu\theta_{2,wb}^{(0)}. \tag{43}$$

Only in this case is the time of the breaking of weakly nonlinear oscillations of the collisionless ($\nu = 0$) plasma $\theta_{2,wb}^{(0)}$ determined by the relation [6]

$$\theta_{2,wb}^{(0)} = \frac{3}{2}\frac{\rho_*^6}{a_*^6}\frac{1}{F_{2,\max}} = 9e\frac{\rho_*^6}{a_*^6}. \tag{44}$$

4. NUMERICAL SIMULATION

We recall that the main feature of the statement of the problem in Eulerian variables is the density singularity at the time of the oscillations break, i.e., the effect of the formation of unbounded derivatives at the finiteness of the solution itself (gradient catastrophe, see, e.g., [18, 19]). Therefore, for the calculations, it is preferable to choose Lagrangian variables, in which the smooth trajectories of the particles are to be determined. This means that first of all, we should discuss the numerical integration of equations (15) and (16) with the initial conditions (6) and (7).

However, it should be noted that additionally, in order to control the dynamics of the oscillations, the calculations were carried out according to a splitting scheme constructed by the finite difference method based on Eulerian variables [7]. In this case, to obtain the same breaking quantitative characteristics, the space-time steps had to be reduced by factors of 10−20, which fully corresponds to the increase in the calculation's labor intensity in Eulerian variables. In addition, the numerical integration of Cauchy problems in Lagrangian variables was reproduced by the fourth-order classical Runge−Kutta method [20]. Here, in full accordance with the theory, the complexity of the calculation increased by about four times compared to the leapfrog scheme traditionally used to integrate the equations of motion of particles (see, e.g., [12]).

For the sake of completeness, we present the calculation formulas of the second-order accuracy method, the main purpose of which is the numerical analysis of the effect of electron-ion collisions on their relativistic breaking process.

4.1. Algorithm based on Lagrangian variables. In the initial time $\theta = 0$, let the particle with number k be characterized by the initial position of the radius $\rho_0(k)$ and the initial displacement $R(k,0)$, where $1 \le k \le M$ and M is the total number of particles. The initial position of all particles form the electric field,

which has form (7). However, the deviation of particles at the initial time creates at the point with the coordinate $\rho_k = \rho_0(k) + R(k,0)$ the electric field in accordance with formula (15). Comparing expressions (7) and (15), we can determine the desired $\rho_0(k)$ and $R(k,0)$ values. For this, we set the initial spatial grid $\rho_k = kh$, where h is the radial discretization parameter characterizing the proximity of the neighboring particles. In the grid nodes by formula (7) we calculate the values of the electric field $E_0(\rho_k)$. This electric field is formed by the displacement of particles; i.e., based on (15) we have the equations for determining the initial positions $\rho_0(k)$:

$$E(\rho_k, 0) = \frac{1}{2} \frac{\rho_k^2 - \rho_0^2(k)}{\rho_k},$$

and then, recalling that $\rho_k = \rho_0(k) + R(k,0)$, from the found initial positions of the particles, we find their initial deviations $R(k,0)$ from these initial positions (see (17), (18)). Thus, for calculating the trajectory of each particle, the initial $\rho_0(k)$ and $R(k,0)$ data are obtained, to which the condition of immobility of the particles at the initial time from (6), i.e., $P(k,0) = 0$, should be added.

Equations (16), as noted above, are ordinary differential equations. Therefore, they can be integrated in a numerically normal way, e.g., according to the second-order accuracy scheme traditional for the equations of motion (the so-called leapfrog scheme). Let τ be the time discretization parameter, i.e., $\theta_j = j\tau$, $j \geq 0$, then the calculation formulas will have the following form:

$$\frac{P(k, \theta_{j+1/2}) - P(k, \theta_{j-1/2})}{\tau} = -R(k, \theta_j) - \nu \frac{P(k, \theta_{j+1/2}) + P(k, \theta_{j-1/2})}{2},$$

$$\frac{R(k, \theta_{j+1}) - R(k, \theta_j)}{\tau} = \frac{P(k, \theta_{j+1/2})}{\sqrt{1 + P^2(k, \theta_{j+1/2})}}.$$

At any time θ_j the variable Euler grid can be calculated from the formula already given

$$\rho_k = \rho_0(k) + R(k, \theta_j), \quad 1 \leq k \leq M,$$

in the nodes of which, in accordance with Eq. (15), the values of the electric field are determined. This was used to represent the electron density for illustrative purposes: in the calculations, the formula used was a second-order accuracy numerical differentiation in the middle of the subsegments

$$N\left(\frac{\rho_{k+1} + \rho_k}{2}, \theta_j\right) = 1 - \left[\frac{E(\rho_{k+1}, \theta_j) - E(\rho_k, \theta_j)}{\rho_{k+1} - \rho_k} + \frac{E(\rho_{k+1}, \theta_j) + E(\rho_k, \theta_j)}{\rho_{k+1} + \rho_k}\right].$$

Here it is convenient to assume that at any moment of time a particle with the number $k = 0$, in which the displacement is always absent, is located on the axis; i.e., its trajectory simply coincides with the axis and on this trajectory the electric field is always zero.

4.2. The choice of parameters for calculations. At least 8640 M particles were used in the calculations; however, in this case the values of the sampling parameters τ (time step) and h (initial distance between particles) are more important: $h = \tau = 1/3200$, since the formulas used have the second order of accuracy with respect to the indicated values. The correctness of the calculations was regularly monitored by additional calculations with double the number of particles, i.e., at half the steps in time and space. The updated time values and the coordinates of the breaking of oscillations differed in the 4th–5th significant digit.

In order to illustrate the theoretical results obtained in the previous section, two variants of breaking were chosen. The first one, we call it relativistic ($\rho_* \gg 1$), is determined by the parameters in the initial condition (7): $a_* = 2.3725$ and $\rho_* = 3.9$ and the second one (we call it non-relativistic) is determined from [6] with parameters $a_* = 0.365$ and $\rho_* = 0.6$. The fundamental difference between these two sets of parameters is that a simultaneous multiple decrease in the a_* and ρ_* values in the non-relativistic case barely changes the time at which the oscillations break, and with allowance for relativism the change of this value is quite significant. This effect is based on the property of the electron density invariance in the non-relativistic case described in [7]. For example, for option $a_* = 0.365$ and $\rho_* = 0.6$, the coordinates of the breaking were calculated in three ways (leapfrog scheme, splitting scheme, and fourth-order Runge–Kutta method):
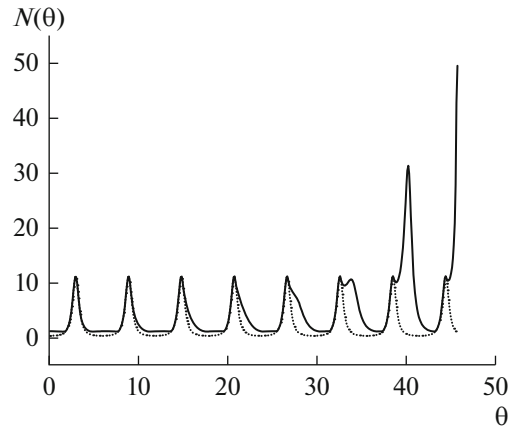
Fig. 1. Dynamics of collisionless plasma density ($\nu\theta_{2,wb}^{(0)} = 0$): solid line is maximum over region, dotted line is symmetry axis.

$$\theta_{2,wb}^{(0)} \approx 45.684375, \quad \rho_{2,wb}^{(0)} \approx 0.03430446.$$

The simultaneous halving of a_* and ρ_* led to the change in the time of the breaking in the fourth significant digit of 45.725 (after rounding the three match)); another decrease by 50% gives 45.737. Taking this into consideration, this option may be used as a non-relativistic one, i.e., for illustration of the case $\rho_* \ll 1$.

In turn, for the relativistic version with parameters $a_* = 2.3725$ and $\rho_* = 3.9$ the coordinates of the breaking were also calculated in three ways:

$$\theta_{1,wb}^{(0)} \approx 27.784062, \quad \rho_{1,wb}^{(0)} \approx 0.16893305.$$

This variant was chosen so that in the first periods of the oscillations the extremes of the electron density amplitudes in both variants (relativistic and non-relativistic) would be close to each other. By formulas (41) and (44), a similar approach leads to a significant reduction in the time of the breaking for relativistic oscillations.

4.3. Results of numerical experiments. The fundamentally important theoretical result established in Section 3 is that the value of the upper boundary of the product $\nu\theta_{wb}^{(0)} \equiv b \le 2/(3\sqrt{3}) \approx 0.385$; if this value is exceeded, the breaking of oscillations should no longer occur. The above limitation is obtained in the weak nonlinearity regime but it is universal; i.e., it does not depend on whether the effects of relativism are taken into account.

The systematic numerical calculations showed that in the relativistic case the breaking stops at $\nu\theta_{1,wb}^{(0)} \approx 0.4148$, and in the non-relativistic case it stops at $\nu\theta_{2,wb}^{(0)} \approx 0.3647$. If in the presented numbers the last digits are reduced by one, then the calculations demonstrate the breaking effect. With allowance for the choice of two illustrative sets of parameters that can be considered as limiting for the general case, described by relations (32) and (35), the approximate values obtained should be considered completely consistent with the theory. This means that the simultaneous numerical and analytical studies of the effect of electron-ion collisions make it possible to formulate the following hypothesis: the boundary of the damping of nonlinear cylindrical plasma oscillations with allowance for collisions is determined by the product $\nu\theta_{wb}^{(0)} \equiv b \in [0.3647, 0.4148]$, where $\theta_{wb}^{(0)}$ is the time of the breaking of oscillations in the collisionless case. The theoretical value of the boundary $b \approx 0.385$ obtained for the weak nonlinearity regime with a high degree of accuracy coincides with the middle of the segment given above.

To demonstrate the dynamics of the changes in the electron density at the breaking of oscillations, we turn to the figures.

Figure 1 shows two characteristic plots illustrating, as a function of time, the electron density (concentration) on the axis of symmetry and its maximum value over the entire region. The plots are for the collisionless case, where $\nu\theta_{2,wb}^{(0)} = 0$. The density function has a pronounced periodic character on the axis of symmetry. However, the maximum density in the region locally coincides with the regular maxima on the
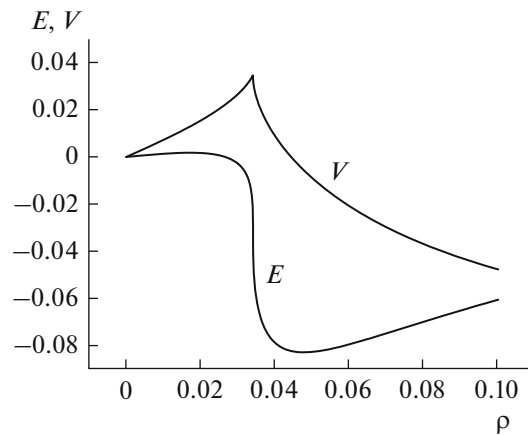
Fig. 2. Spatial distribution of velocity and electric field at time of breaking of oscillations without collisions ($\nu\theta_{2,wb}^{(0)} = 0$).

axis, and then (in this case, in the sixth period of oscillations), an off-axis maximum is formed (approximately at $\theta = 33.9$). This maximum increases fast from period-to-period: at time $\theta \approx 40.1$ it almost doubles in value and, finally, in the next period, at $\theta_{2,wb}^{(0)} \approx 45.7$—the maximum electron density goes to infinity; i.e., the oscillations break. The spatial distributions of the velocity and electric field functions corresponding to the time of the breaking of the oscillations are shown in Fig. 2.

In accordance with formula (4), the density singularity is determined by the stepped form of the electric field function in the vicinity of $\rho_{2,wb}^{(0)} = 0.0343$. A more detailed description of the scenario of the breaking of cylindrical oscillations in the case of the collisionless plasma is given in [6].

Let us turn to illustrations accounting collisions. If the cause of the breaking is relativistic effects then the effect of geometry, i.e., oscillations, plane or cylindrical, is almost negligible. The illustrations of the effect of electron-ion collisions on the relativistic breaking in the plane case are given in [10]; therefore, below we focus on the breaking effect arising from the geometric (cylindrical) nonlinearity of the equations.

From the theoretical analysis of the weak nonlinearity regime, it follows that when electron collisions are taken into account, the time of the appearance of the density singularity increases (see formula (37)). Figure 3 shows the plasma density dynamics with rare collisions ($\nu\theta_{2,wb}^{(0)} = 0.25$), and parameters a_* and ρ_* correspond to the breaking occurring in the non-relativistic case. In this calculation, the electron density singularity occurs only at the 11th period $\left(\theta_{2,wb} \approx 64.676\right)$ and not in the 8th period, as was the case in the collisionless case ($\theta_{2,wb}^{(0)} \approx 45.684$).

In full accordance with the theory discussed in Section 3, the calculations show that the density singularity with the given initial parameters occurs only in the case of relatively rare collisions, when the inequality $\nu\theta_{2,wb}^{(0)} < 0.3647$ holds. When the equality $\nu\theta_{2,wb}^{(0)} = 0.3646$ holds, the time of the appearance of the density singularity is $\theta_{2,wb} \approx 126.96$ and it is about 2.78 times longer than the time of the breaking in the collisionless plasma. This result is also in good agreement with the theory, where the factor of the increase in the time of the breaking is predicted to be approximately 2.85.

When the condition $\nu\theta_{2,wb}^{(0)} \geq 0.3647$ holds the density singularity no longer occurs. Such a scenario of the evolution of the electron density is presented in Fig. 4 for the variant $\nu\theta_{2,wb}^{(0)} = 0.45$. In this case, the off-axis maximum after its formation initially increases but in the future it decreases due to the strong damping of oscillations. Also here the exponential decay of the amplitude of the regular maxima of the electron density located on the axis of symmetry of the problem should be noted.

Note that the qualitative behavior of the electron density when considering collisions of particles does not depend on the causes of the breaking effect of the oscillations, as follows from the analytical consideration in Section 3.
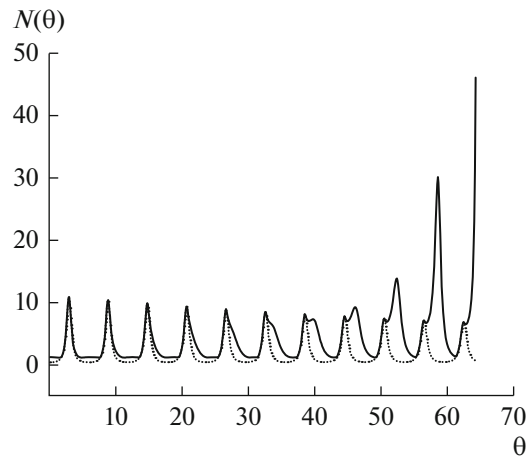
Fig. 3. Plasma density dynamics with rare collisions ($\nu\theta_{2,wb}^{(0)} = 0.25$): solid line is maximum over region, dotted line is symmetry axis.
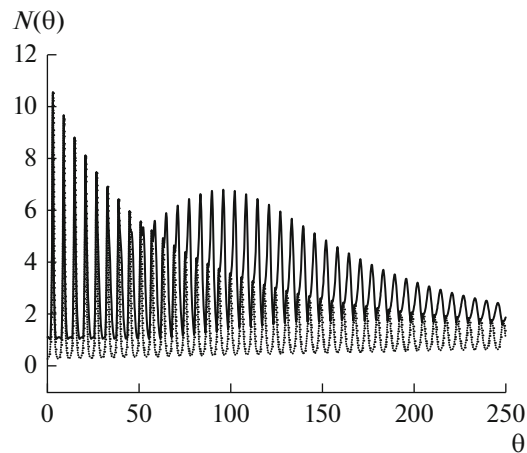

Fig. 4. Plasma density dynamics with frequent collisions ($\nu\theta_{2,wb}^{(0)} = 0.45$): solid line is maximum over region, dotted line is symmetry axis.

5. CONCLUSIONS

The effect of electron–ion collisions of electrons on the breaking of cylindrical nonlinear plasma oscillations was studied numerically and analytically in this work. When there are no collisions in the plasma the plasma oscillations break due to the formation of an electron density maximum outside the axis, which increases with time and after several periods of oscillations tends to infinity.

An expression for the particle displacement as a function of time and initial coordinate is derived analytically from the equations of motion of the perturbation method in the weak nonlinearity regime. Based on the condition that the electron density becomes infinity, the time of breaking is found and it is shown that with increasing collision frequency this time increases. It is established numerically and analytically that there is some threshold value for the collision frequency above which the density singularity does not occur.

The calculations show that at collision frequencies above the threshold value the maximum density outside the axis of oscillations after its formation increases for some time, reaches an extremum, and then falls due to the damping of the oscillations. The results of the analytical consideration are in good agreement with the numerical experiments and can be useful when discussing various physical effects associated with plasma oscillations and waves.

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