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Continuum Model and Method of Calculating for Dynamics of Inelastic Layered Medium[1]

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Abstract—Mathematical modeling of the processes of wave propagation in a layered medium with viscoplastic slip conditions at contact boundaries was carried out, also as passing of waves through a fluid-containing layered massif. An improved model of a layered medium with nonlinear viscoplastic slip conditions at interlayer boundaries was constructed. A numerical solution method for the equations of a layered medium with viscoplastic interlayers for a power slip condition was developed. An example was given of a numerical calculation of the passing of a transverse elastic wave through a layered massif possessing effective anisotropic viscoplastic properties. A two-dimensional problem of the reflection from the buried layered massif was numerically solved for a system of waves excited by a nonstationary surface source. A comparison was made between the dynamics of the surface points for the elastic solution and the solution, taking into account the influence of the buried layered massif, as well as the effect of the thickness of the layers. The proposed models can be useful in solving of the dynamic problems of seismic survey and interpretation of wave patterns obtained in its course.

Keywords: layered medium, slip condition, continuum model, layered viscous-plastic massif, explicit-implicit scheme, seismic survey

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1. INTRODUCTION

In this work, the asymptotic homogenization method [1, 2] is used to construct a refined model of a layered medium with nonlinear viscous-plastic slip conditions on the interlayer boundaries. A physical object with such properties is, for example, a fluid-containing layered elastic geological massif. It is assumed that in thin inter-layers between elastic layers there is a very viscous liquid (oil) or a viscous-plastic mass (sand soaked in oil). Such models can be useful in solving dynamic seismic exploration problems and interpreting wave propagation obtained in the process of its implementation.

A numerical scheme has been developed for solving a non-stationary system of equations for a layered medium with slippage, based on explicit approximation of the equations of motion and implicit approximation of the constitutive relations related to the type of slip conditions at the interlayer boundaries.

Examples are given of the calculation of the passage of an elastic wave through a fluid-containing layered massif. The shapes and amplitudes of transmitted and reflected waves are determined.

2. MODEL OF A LAYERED MEDIUM WITH A POWER LAW OF VISCOUS-PLASTIC SLIP AT THE INTERLAYER BOUNDARIES

We consider an infinite layered medium in the Cartesian rectangular coordinate system x_1, x_2, x_3. The axis x_3 is perpendicular to the plane-parallel interfaces of the layers. The coordinates of interlayer boundaries are $x_3 = x^{(s)} = s\varepsilon$, $s = 0, \pm 1, \pm 2, \ldots$, where the constant layer thickness $\varepsilon \ll 1$ is a small parameter with respect to the wavelength or other characteristic size of the disturbance propagating in the massif of a layered medium. It is assumed that between the elastic layers there are thin viscous or viscous-plastic layers

[1] The article was translated by the authors.

characterized by thickness $\delta \ll \varepsilon$, however, we neglect the thickness of these thin inter-layers and replace them with slip conditions at the compressed elastic layer boundaries:

$$\sigma_{33} < 0, \quad [u_3] = [\sigma_{\gamma 3}] = [\sigma_{33}] = 0,$$

where $[u_{\gamma,t}]/\varepsilon = \kappa\sigma_{\gamma 3}$ in the case of linear viscous slip condition or $[u_{\gamma,t}]/\varepsilon = \kappa\sigma_{\gamma 3}\left\langle F(\sigma_{\beta 3}\sigma_{\beta 3}/\tau_s^2 - 1)\right\rangle$ in the case of nonlinear viscous-plastic slip condition, $\kappa = \delta/(\varepsilon\eta)$, η – viscosity coefficient. Here the square brackets $[f] = f|_{x^{(s)}+0} - f|_{x^{(s)}-0}$ denote a magnitude jump of f at inter-layer boundary, $\langle F(y)\rangle = F(y)H(y)$ is a nonlinear non-zero function beyond the yield limit $\sigma_{\beta 3}\sigma_{\beta 3} = \tau_s^2$, $H(y)$ is the Heaviside function, $H(y) = 0$ for $y < 0$, $H(y) = 1$ for $y \geq 0$. The nonlinear viscous-plastic slip condition transforms into a linear viscous slip condition, if replaced $\langle F(y)\rangle$ by one. Greek subscripts β, γ take values 1 and 2, Latin subscripts – values 1,2,3, u_k are displacement components, $u_{k,t}$ are velocity components, σ_{ij} stress tensor components. For compactness of formulas, differentiation is indicated as follows: $\partial(\ldots)/\partial x_j = (\ldots)_{,j}$, $\partial(\ldots)/\partial t = (\ldots)_{,t}$, $\partial(\ldots)/\partial\xi = (\ldots)_{,\xi}$. The layers themselves are considered to be isotropic linearly elastic (for $x_3 \neq x^{(s)}$):

$$\sigma_{ij,j} - \rho u_{i,tt} = 0, \quad \sigma_{ij} = C_{ijkl}u_{k,l},$$

where ρ is a density, modulus of elasticity tensor has the form

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

We assume that the desired functions $u_k = u_k(x_l, \xi, t)$ are smooth in "slow" variables x_i and smooth in a "fast" variable $\xi = x_3/\varepsilon$, with the exception of points $\xi = x^{(s)}/\varepsilon$, where they can tolerate discontinuities of the first kind. Also these functions are 1-periodic [1]:

$$[[u_i]] = u_i|_{\xi^{(s)}+1/2} - u_i|_{\xi^{(s)}-1/2} = 0.$$

The displacements of the medium will be represented as an asymptotic series in powers of the small parameter ε:

$$u_i = w_i(x_k, t) + \varepsilon u_i^{(1)}(x_k, \xi, t) + \varepsilon^2 u_i^{(2)}(x_k, \xi, t) + \varepsilon^3 u_i^{(3)}(x_k, \xi, t) + \ldots .$$

For the function of the "fast" variable ξ we introduce the operation "averaging" $\langle f\rangle$: $\langle f\rangle = \int_{-1/2}^{1/2} f d\xi$. Displacement coefficients must satisfy the condition $\left\langle u_k^{(n)}\right\rangle = 0$.

The decomposition of the components of the stress tensor corresponds to the decomposition of the displacement components:

$$\sigma_{ij} = \sigma_{ij}^{(0)} + \varepsilon\sigma_{ij}^{(1)} + \varepsilon^2\sigma_{ij}^{(2)} + \ldots ,$$

where $\sigma_{ij}^{(n)} = C_{ijkl}u_{k,l}^{(n)} + C_{ijk3}u_{k,\xi}^{(n+1)}$. All stress approximations are 1-periodic functions of ξ. In particular, the following conditions are valid $[\sigma_{i3}^{(n)}] = 0$, $[[\sigma_{i3}^{(n)}]] = 0$. Easy to see that $\left\langle \sigma_{i3,\xi}^{(n)}\right\rangle = 0$. The procedure for obtaining an asymptotic system of equations in the case of linear conditions for jumps of tangential displacements (slips) on the interlayer boundaries is described in [3, 4].

Similarly, a refined second order theory is obtained if in the asymptotic system of equations we keep the terms of the order ε^2 and apply the averaging operation over the periodicity cell $\langle\ \rangle$:

$$C_{ijkl}w_{k,jl} + C_{ijk3}\left\langle u_{k,\xi}^{(1)}\right\rangle_{,j} + \varepsilon C_{ijk3}\left\langle u_{k,\xi}^{(2)}\right\rangle_{,j} + \varepsilon^2 C_{ijk3}\left\langle u_{k,\xi}^{(3)}\right\rangle_{,j} = \rho w_{i,tt}.$$

The first approximation of the displacements will be as follows:

$$u_k^{(1)} = \varphi_k(\xi - \operatorname{sgn}\xi/2), \quad \varphi_3 = 0.$$

The second approximation of the displacements looks like this:

$$u_k^{(2)} = \psi_k\,(\xi^2 - \xi\operatorname{sgn}\xi + 1/6)/2,$$

$$\psi_\gamma = -\varphi_{\gamma,3}, \quad \psi_3 = -\lambda\varphi_{\beta,\beta}/(\lambda + 2\mu).$$

The solution for the third displacement approximation is

$$u_k^{(3)} = \chi_k(\xi^3/6 - \xi^2\mathrm{sgn}\xi/4 + \xi/12) + \Omega_k(\xi - \mathrm{sgn}\xi/2),$$

$$\chi_\gamma = \varphi_{\gamma,33} - \varphi_{\gamma,\beta\beta} - 2(\lambda + \mu)\varphi_{\beta,\beta\gamma}/(\lambda + 2\mu) + \rho\varphi_{\gamma,tt}/\mu,$$

$$\chi_3 = 2(\lambda + \mu)\varphi_{\beta,\beta 3}/(\lambda + 2\mu), \quad \Omega_3 = 0.$$

Functions $\varphi_\gamma, \Omega_\gamma$ are determined from the slip conditions at jumps of tangential velocities. Using these results, the refined system of equations can be obtained as follows:

$$\rho v_{\gamma,t} = s_{\gamma j,j} + \varepsilon^2\mu\Omega_{\gamma,3}, \quad \rho v_{3,t} = s_{3j,j} + \varepsilon^2\mu\Omega_{\beta,\beta},$$

$$\tau_{ij,t} = \lambda\delta_{ij}v_{k.k} + \mu(v_{i,j} + v_{j,i}), \quad \varphi_{\gamma,t} = -\kappa s_{\gamma 3}\langle F(\Delta)\rangle,$$

$$\Omega_{\gamma,t} = -\kappa\mu\left((g_\gamma + \Omega_\gamma)\langle F(\Delta)\rangle + 2s_{\gamma 3}s_{\beta 3}(g_\beta + \Omega_\beta)\langle F'(\Delta)\rangle/\tau_s^2\right),$$

$$s_{ij} = \tau_{ij} + \mu(\varphi_i\delta_{j3} + \varphi_j\delta_{i3}), \quad \Delta = s_{\beta 3}s_{\beta 3}/\tau_s^2 - 1,$$

$$g_\gamma = \left(\rho\varphi_{\gamma,tt}/\mu - \varphi_{\gamma,\beta\beta} - (3\lambda + 2\mu)\varphi_{\beta,\beta\gamma}/(\lambda + 2\mu)\right)/12.$$

In this non-stationary system, the following notation is entered: $s_{ij} = \sigma_{ij}^{(0)}$, $v_k = w_{k,t}$. For additional functions φ_γ and Ω_γ, that have the meaning of distributed slips of the first and third orders in ε, nonlinear differential equations are obtained.

Note that if in the resulting refined system we put $\varepsilon = 0$ and $\Omega_\gamma = 0$, then it goes into a first-order semi-linear system for describing the dynamics of a layered medium with nonlinear slip conditions on contact boundaries, which follows from the slip theory in its discrete variant [5].

3. DYNAMIC SYSTEMS OF EQUATIONS

3.1. One-dimensional Non-stationary System of Equations

We formulate a one-dimensional non-stationary system of equations for describing transverse non-stationary perturbations parallel to the direction of the layers.

We introduce the notation for independent variables and desired functions for a given type of perturbation: $z = x_3$, $\tau = s_{13}$, $u = v_1$, $\varphi = \varphi_1$, $\Omega = \Omega_1$, $g = g_1$.

The system of equations in this case takes the form

$$\rho u_{,t} = \tau_{,z} + \varepsilon^2\mu\Omega_{,z}, \quad \tau_{,t} = \mu u_{,z} + \mu\varphi_{,t},$$

$$\varphi_{,t} = -(\tau/t_0)\langle F(\Delta)\rangle/\mu, \quad \Omega_{,t} = -(g + \Omega)(\tau^2/\tau_s^2)\left(\langle F(\Delta)\rangle + 2\left\langle F_\Delta'(\Delta)\right\rangle\right)/t_0,$$

$$g = c_g\varphi_{,tt}, \quad c_g = \rho/\mu/12, \quad \Delta = \tau^2/\tau_s^2 - 1, \quad t_0 = 1/(\kappa\mu).$$

where $t_0 = 1/(\kappa\mu)$ is a parameter that represents characteristic time for shear stresses relaxation to yield limit τ_s.

For function $F(\Delta)$ a power approximation is often taken:

$$F(\Delta) = \Delta^{1+q}, \quad q > 0.$$

Given this approximation, nonlinear equations for τ, φ and Ω take the form

$$\tau_{,t} = \mu u_{,z} - (\tau/t_0)\left\langle\Delta^{1+q}\right\rangle, \quad \varphi_{,t} = -(\tau/t_0)\left\langle\Delta^{1+q}\right\rangle/\mu,$$

$$\Omega_{,t} = -(g + \Omega)\left((3 + 2q)\tau^2/\tau_s^2 - 1\right)\left\langle\Delta^q\right\rangle/t_0.$$

3.2. Two-dimensional Non-stationary System of Equations

Here we formulate a two-dimensional non-stationary system of equations for describing transverse non-stationary perturbations parallel to the direction of the layers.

In this case, we introduce the following notation for independent variables and desired functions for a given type of perturbation:

$$x = x_1,\ \ z = x_3,\ \ \sigma = s_{11},\ \ \tau = s_{13},\ \ s = s_{33},\ \ u = v_1,\ \ v = v_3,\ \ \varphi = \varphi_1,\ \ \Omega = \Omega_1,\ \ g = g_1.$$

The system of equations in this case takes the form

$$\rho u_{,t} = \sigma_{,x} + \tau_{,z} + \varepsilon^2\mu\Omega_{,z},\ \ \rho v_{,t} = \tau_{,x} + s_{,z} + \varepsilon^2\mu\Omega_{,x},$$

$$\sigma_{,t} = (\lambda + 2\mu)u_{,x} + \lambda v_{,z},\ \ s_{,t} = \lambda u_{,x} + (\lambda + 2\mu)v_{,z},$$

$$\tau_{,t} = \mu v_{,x} + \mu u_{,z} + \mu\varphi_{,t},\ \ \varphi_{,t} = -(\tau/t_0)\langle F(\Delta)\rangle/\mu,$$

$$\Omega_{,t} = -(g + \Omega)(\tau^2/\tau_s^2)(\langle F(\Delta)\rangle + 2\langle F'(\Delta)\rangle)/t_0,$$

$$g = (\rho\varphi_{,tt}/\mu - 4(\lambda + \mu)\varphi_{,xx}/(\lambda + 2\mu))/12,\ \ \Delta = \tau^2/\tau_s^2 - 1,\ \ t_0 = 1/(\kappa\mu).$$

The essential difference between the two-dimensional system of equations and the one-dimensional one is the presence of the second spatial derivative in the equation for the additional function g, which is included in the expression for the function Ω, responsible for the distributed slip.

4. EXPLICITLY-IMPLICIT SCHEME FOR SOLVING NON-STATIONARY SYSTEMS OF EQUATIONS FOR A LAYERED MEDIUM WITH VISCOUS-PLASTIC SLIPPAGE

For the numerical solution of dynamic problems of the propagation of perturbations in a layered medium, we construct a difference scheme for a nonlinear one-dimensional unsteady system taking into account the small value of the real relaxation time t_0.

Standard explicit difference schemes will be unstable in this case, since the time step Δt will exceed the characteristic value of the relaxation time t_0, $\Delta t > t_0$.

Therefore, at first, we approximate the equation of motion for velocity u, by some known explicit scheme, omitting the term $\varepsilon^2\mu\Omega_{,z}$ (splitting method).

Using the values u^{n+1} on the upper time layer, we approximate, using an implicit difference scheme, equations for τ, φ, Ω, containing a small parameter t_0 in the denominator of nonlinear free terms [5]. Hereinafter, the superscripts $n + 1$ and n denote the values in the upper and lower layers in time with a difference time approximation with step Δt.

The obtained nonlinear difference equations can be solved analytically by the perturbation method taking into account the presence of a small parameter in the system of equations and the values of the functions $\tau^{n+1}, \varphi^{n+1}$ and Ω^{n+1} can also be found in the upper time layer. Using the values found, we will calculate the full equation for the speed with the term $\varepsilon^2\mu\Omega_{,z}$.

Let us describe the implementation of this numerical method on the example of simple implicit approximations of the 1st order.

Nonlinear difference equation for shear stress τ^{n+1} on the upper layer over time is

$$\tau^{n+1} = \tau_e^{n+1} - \tau^{n+1}\left\langle(\tau^{n+1}/\tau_s)^2 - 1\right\rangle^{1+q}\Big/\gamma,$$

where the small parameter γ is $\gamma = t_0/\Delta t \ll 1$, and $\tau_e^{n+1} = \tau^n + \mu u_{,z}^{n+1}\Delta t$ is the shear stress on the upper time layer after "elastic" time step. Once again, we note that we do not specify or write out the scheme for calculating the "elastic" step, in particular, the spatial approximations of the terms of the equation, since the choice of well-proven schemes is great [6, 7].

When calculating three-dimensional problems with complex geometry using non-uniform, unstructured grids, finite-volume or finite-element methods can be applied to calculate the "elastic" step [5–7].

For some values of q an exact solution of a nonlinear equation may be constructed. However, for generality, we will seek a solution to the equation as an expansion in powers of the small parameter γ [7], limited to one member (the power exponent ν and the coefficient T are to be determined):

$$\tau^{n+1} = \tau_s(1 + \gamma^{\nu} T + o(\gamma^{\nu}))\mathrm{sgn}\,\tau_e^{n+1}.$$

The solution for τ^{n+1} (correction formula) with the selected accuracy will be

$$\tau^{n+1} = \tau_s\left(1 + \gamma^{1/(1+q)}\left(\left|\tau_e^{n+1}/\tau_s\right| - 1\right)^{1/(1+q)}\Big/2\right)\mathrm{sgn}\,\tau_e^{n+1} \text{ for } \left|\tau_e^{n+1}/\tau_s\right| - 1 \geq 0,$$

$$\tau^{n+1} = \tau_e^{n+1} \text{ for } \left|\tau_e^{n+1}/\tau_s\right| - 1 < 0.$$

A nonlinear difference equation for 1st order slips φ on the upper layer in time looks like this:

$$\varphi^{n+1} = \varphi^n - (\tau^{n+1}/\mu)\left\langle(\tau^{n+1}/\tau_s)^2 - 1\right\rangle^{1+q}\Big/\gamma.$$

The solution with the chosen accuracy is

$$\varphi^{n+1} = \varphi^n - (\tau_s/\mu)\left\langle\left|\tau_e^{n+1}/\tau_s\right| - 1\right\rangle\left(1 + \gamma^{1/(1+q)}\left(\left|\tau_e^{n+1}/\tau_s\right| - 1\right)^{1/(1+q)}\Big/2\right)\mathrm{sgn}\,\tau_e^{n+1}.$$

In the limit $\gamma \to 0$, we obtain the more compact formula:

$$\varphi^{n+1} = \varphi^n - (\tau_s/\mu)\left\langle\left|\tau_e^{n+1}/\tau_s\right| - 1\right\rangle\mathrm{sgn}\,\tau_e^{n+1}.$$

The nonlinear difference equation for 3rd order slips Ω on the upper layer in time is

$$\Omega^{n+1} = \Omega^n - (g^{n+1} + \Omega^{n+1})\left((3 + 2q)(\tau^{n+1}/\tau_s)^2 - 1\right)\left\langle(\tau^{n+1}/\tau_s)^2 - 1\right\rangle^q\Big/\gamma.$$

After the calculations we get

$$\Omega^{n+1} = \Omega^n - (g^{n+1} + \Omega^{n+1})\left((3 + 2q)\left(1 + \gamma^{1/(1+q)}\left(\left|\tau_e^{n+1}/\tau_s\right| - 1\right)^{1/(1+q)}\right)\left\langle\left|\tau_e^{n+1}/\tau_s\right| - 1\right\rangle^{q/(1+q)}\Big/\gamma^{1/(1+q)},$$

where

$$g^{n+1} = -c_g(\tau_s/\mu)(G^{n+1} - G^n)\big/\Delta t^2,$$

$$G^n = \left\langle\left|\tau_e^n/\tau_s\right| - 1\right\rangle\left(1 + \gamma^{1/(1+q)}\left(\left|\tau_e^{n+1}/\tau_s\right| - 1\right)^{1/(1+q)}\Big/2\right)\mathrm{sgn}\,\tau_e^n.$$

In the limit $\gamma \to 0$ we get a less cumbersome calculation formula:

$$\Omega^{n+1} = c_g(\tau_s/\mu)\left(\left\langle\left|\tau_e^{n+1}/\tau_s\right| - 1\right\rangle\mathrm{sgn}\,\tau_e^{n+1} - \left\langle\left|\tau_e^n/\tau_s\right| - 1\right\rangle\mathrm{sgn}\,\tau_e^n\right)\Delta t^2.$$

The found values Ω^{n+1} are used to calculate the new values of velocity u^{n+1} with account of the term $\varepsilon^2\mu\Omega_{,z}$.

5. EXAMPLES OF CALCULATIONS FOR THE PASSAGE OF AN ELASTIC WAVE THROUGH A FLUID-CONTAINING LAYERED PACKET

The resulting model was used to study wave processes in geological massifs with fluid-containing layered structure. Note that numerical calculations of wave processes in layered and block media with simplified (linear) connections between structural elements were carried out in [9–11]. However, for small sizes of layers or blocks, such direct modeling becomes difficult, especially when taking into account nonlinear contact slip conditions. The urgency of applying effective (homogenized) models is increasing.

The numerical solution of the obtained system of equations was constructed by the method of finite volumes of the second order to calculate the "elastic" step and further use of the above explicitly-implicit scheme taking into account small viscosity parameters for time derivatives of desired functions φ and Ω [5].

An example of a numerical solution of the problem of passing a transverse wave (dotted line) through a layered packet $0.5 < x_3 < 0.7$ with viscoplastic interlayers located in an elastic massif $0.0 < x_3 < 1.0$, is shown in Figs. 1–4. Material characteristics: density $\rho = 1.0$, Lame modules $\lambda = \mu = 0.333$, yield limit $\tau_s = 0.00001$, relaxation parameter $\gamma = 0.1$, layer thickness $\varepsilon = 0$–0.10, amplitude of impulse velocity $U_0 = 0.000035$, pulse duration $T = 0.20$. The dimensionless values of the parameters are given, and the

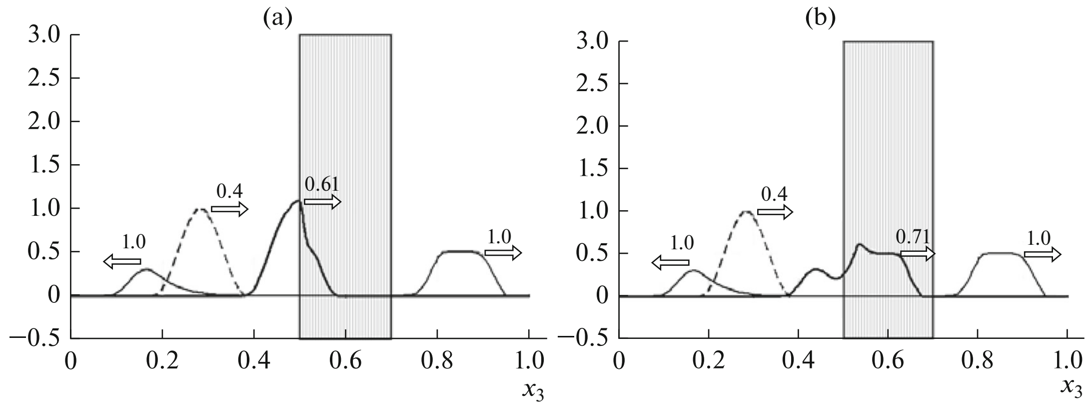

Fig. 1. Forms of incident, transmitted through a layered packet and reflected waves at $\varepsilon = 0$. Velocity U/U_0, wave enters the layered packet – a). Velocity U/U_0, wave completely in the layered packet – b).

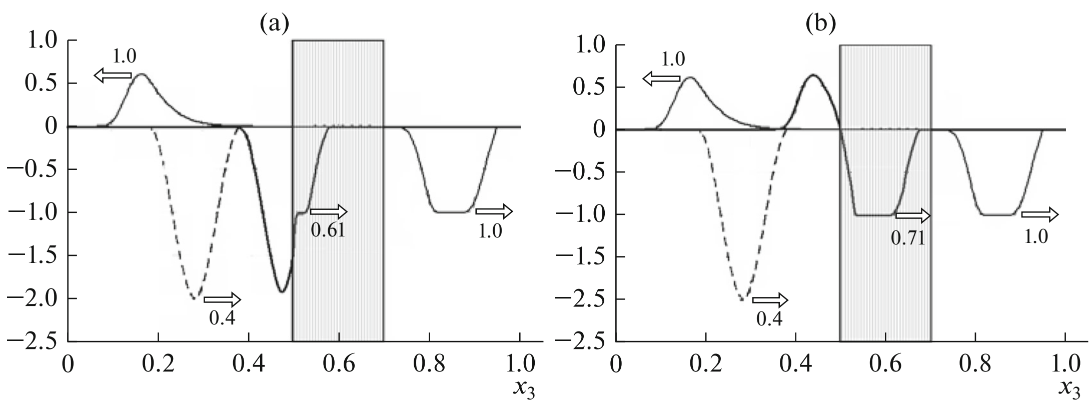

Fig. 2. Forms of incident, transmitted through a layered packet and reflected waves at $\varepsilon = 0$. Velocity τ/τ_s, wave enters the layered packet – a). Velocity τ/τ_s, wave completely in the layered packet – b).

stresses are referred to $\lambda + 2\mu$, spatial coordinates to the size of solution domain, $H_{\max}$, material velocities to the velocity of elastic longitudinal wave $a = \sqrt{(\lambda + 2\mu)/\rho}$. The parameters of the time dimension are assigned to the value $H_{\min}/a$. The incident elastic shear wave, shown in Figs. 1–4 by a dotted line, in the absence of a layered packet would propagate with a constant shape and amplitude. Figures 1–4 for different points in time show the forms of transmitted and reflected waves passing through a layered packet for the case $\varepsilon = 0$ (Figs. 1, 2 without the correction term in the equation for speed) and for the case $\varepsilon = 0.1$ (Fig. 3, 4 with the correction term $\varepsilon^2\mu\Omega_{,z}$). Figure 1, 3 shows profiles of relative transverse velocities U/U_0, and Figs. 2, 4 shows profiles of relative shear stresses τ/τ_s at different points in time as the initial impulse propagates. The values of the time instants are indicated next to the wave profile.

It can be seen from these figures that the presence of a layered packet leads to a strong transformation of the initial profile of the incident elastic wave. Accounting for the influence of an additional term in the system of refined equations related to the slip function Ω, strongly influences the profile of a transmitted wave inside a nonlinear layered packet, but only slightly affects the shape of the transmitted and reflected waves.

Also, calculations were carried out for a two-dimensional problem of the normal action on an elastic half-space with a recessed laminated packet having the viscoplastic properties described above. An impulse was applied to the half-space, concentrated in a fairly narrow zone on the surface and acting for a short time (Fig. 5). Geometrical parameters of the computational domain: the maximum depth $H_{\max} = 1.00$, the maximum horizon $X_{\max} = 1.00$, depth of the layered package $H_1 = 0.25$, the thickness of the lay-

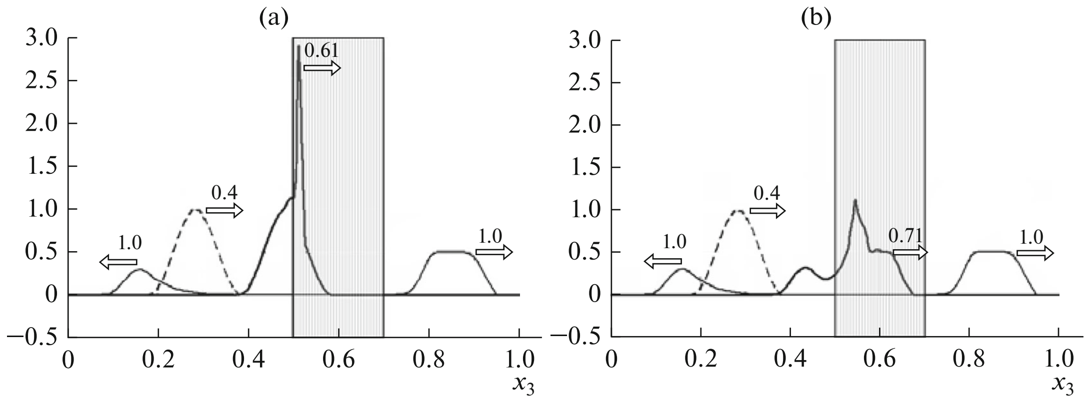

Fig. 3. Forms of incident, transmitted through a layered packet and reflected waves at ε = 0. Velocity U/U_0, wave enters the layered packet – a). Velocity U/U_0, wave completely in the layered packet – b).

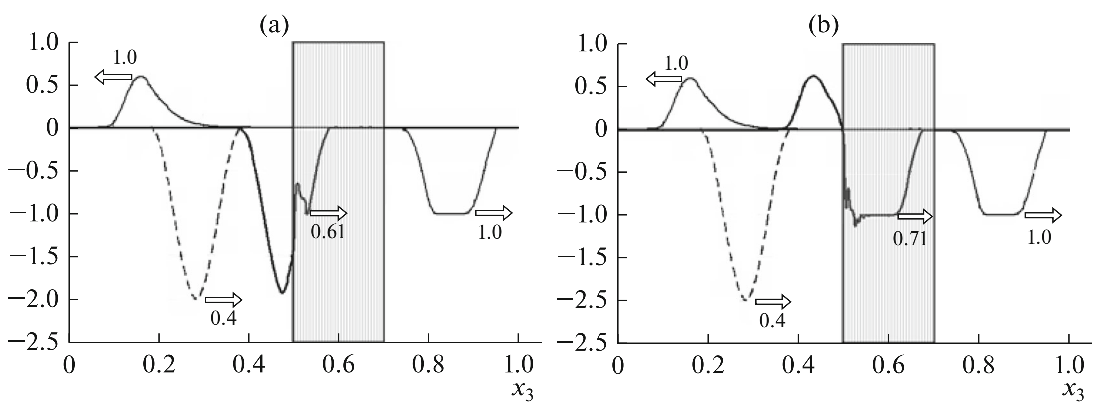

Fig. 4. Forms of incident, transmitted through a layered packet and reflected waves at ε = 0.1. a) stress τ/τ_s, wave enters the layered packet; b) stress τ/τ_s, wave completely in the layered packet.

ered package $\Delta H = 0.10$, external influence (displacement pulse) is applied in the interval $0 \le x \le d$, $d = 0.02$. In this case, on the boundary section $z = 0$, when $x \le d$, $t \le T_0$ the vertical velocity $V(t) = V_0 \sin(2\pi t/T_0)$ is set, where the signal amplitude is $V_0 = 0.0035$, the period (duration) is $T_0 = 0.10$. For $z = 0$ the end-to-end boundary condition $\tau = 0$ is satisfied; for $z > d$ the condition $\sigma = 0$ is satisfied. For $t > T_0$ the boundary condition for the velocity on the segment $0 \le x \le d$ changes to the free surface condition $\sigma = 0$. The boundary condition on the axis of symmetry $x = 0$ has the form: $\tau = 0$, $u = 0$. At the outer boundaries of the computational domain, the conditions of "transmission" are applied, and the normal derivatives of the desired functions vanish. The number of grid nodes in the calculation is 200 × 200.

A system of elastic cylindrical waves diverges from the surface, schematically repeating the wave pattern of the well-known solution of the plane Lamb problem. Upon reaching the layered packet, internal slips on the interlayer boundaries begin, described by additional distributed functions φ and Ω. These slips "relieve" the shear stresses inside the layer and change the wave pattern. Comparison of numerical solutions for an elastic system, an inelastic system with ε = 0 (taking into account first-order slip with a nonzero function φ and zero function Ω) and an inelastic system with non-zero системы ε = 0.05 (all additional slips φ and Ω are taken into account) at time $t = 0.5$ is shown in Figs. 6–8 for normalized shear stress (Fig. 6a, b), radial velocity (Fig. 7a, b) and vertical speed (Fig. 8a, b). In Fig. 6b, the formation of a wave reflected from a layered packet is noticeable in a restricted area between the z direction, where there is no tangential stress due to the symmetry of the problem, and distant sections of the layered packet, where the

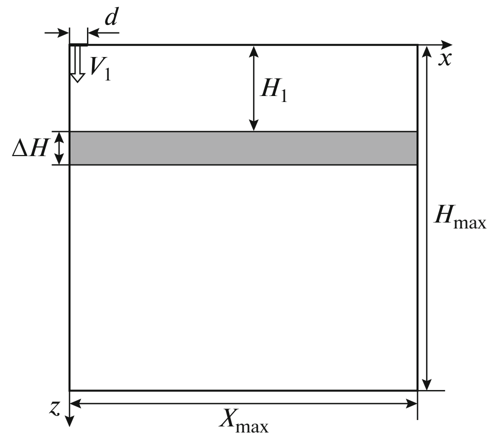

Fig. 5. Geometry of solution domain.

tangential stresses are small due to the radial damping of the quasi-cylindrical wave from the surface source.

Figures 9–11 shows the waveforms of signals that come to different points of the surface $z = 0$, $x = 0.2$, $x = 0.4$, $x = 0.6$ with waves reflected from the recessed layered packet. The coordinates of the surface points are given next to each plot of the normalized speed for the reflected signal. Figure 9 shows the normalized values of the vertical velocity in the signal reflected from a layered packet with layers thick $\varepsilon = 0.05$.

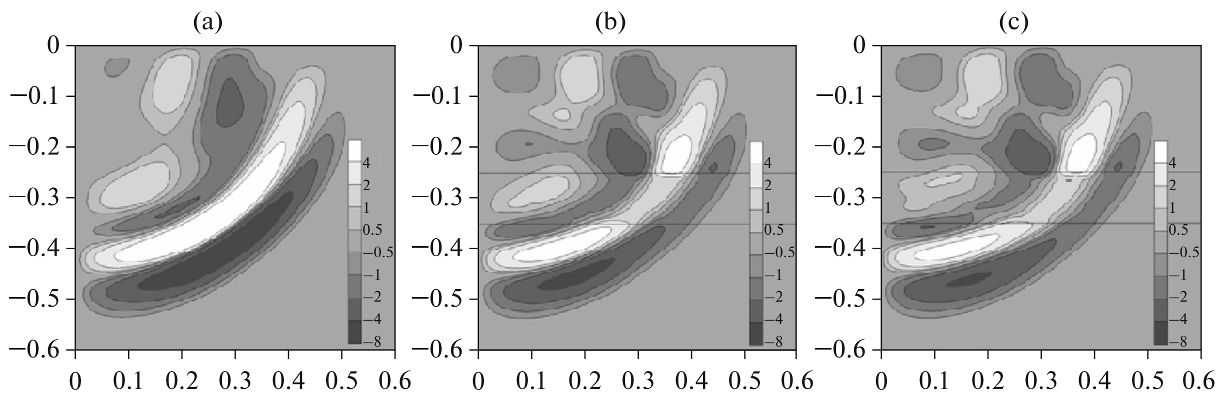

Fig. 6. Shear stress τ/τ_s, $t = 0.5$: a) no layer; b) nonlinear layer $\varepsilon = 0$; c) nonlinear layer $\varepsilon = 0.05$.

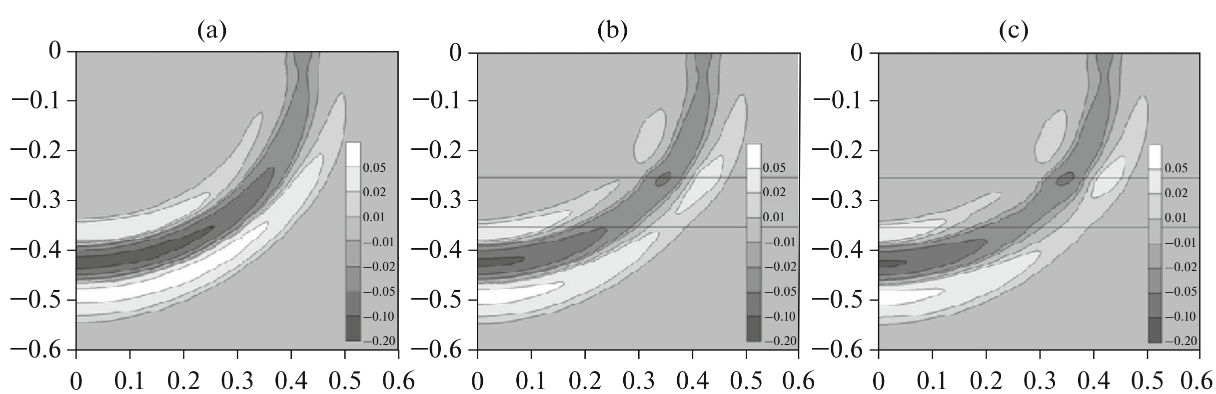

Fig. 7. Radial velocity V_r/V_0, $t = 0.5$: a) no layer; b) nonlinear layer $\varepsilon = 0$; c) nonlinear layer $\varepsilon = 0.05$.

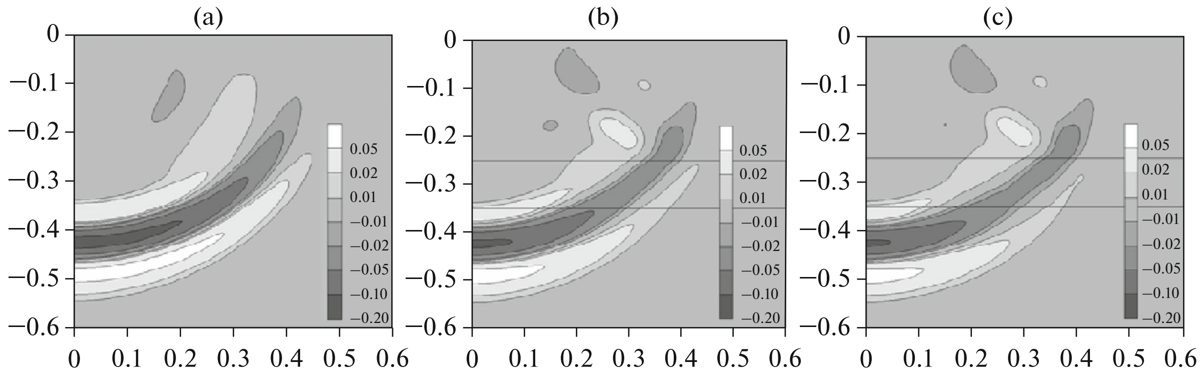

Fig. 8. Vertical velocity V_r/V_0, $t = 0.5$: a) no layer; b) nonlinear layer ε = 0; c) nonlinear layer ε = 0.05.

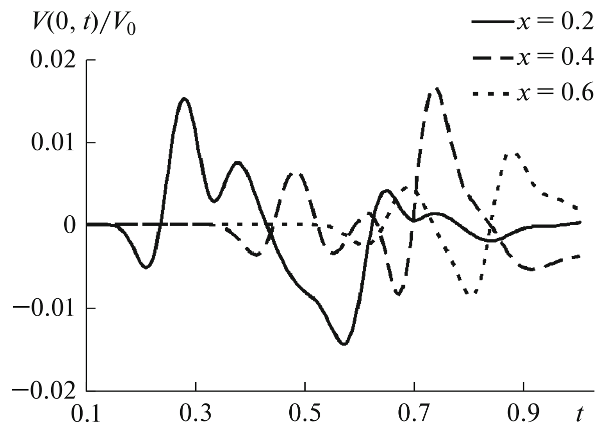

Fig. 9. Reflected signal on the surface at ε = 0.05.

Figure 10 shows the effect of the presence of a layered packet on the reflected signal on the surface, i.e. velocity differences in reflected signals in the presence and absence of a layered packet for different values ε = 0.05, (Fig. 10a), ε = 0.10 (Fig. 10b).

Figure 11 shows the effect of layer thickness on the reflected signal on the surface, i.e. velocity differences in the reflected signals on the surface in the presence of a layered packet for the values ε = 0.05 and ε = 0 (Fig. 11a), as well as for ε = 0.10 and ε = 0 (Fig. 11b).

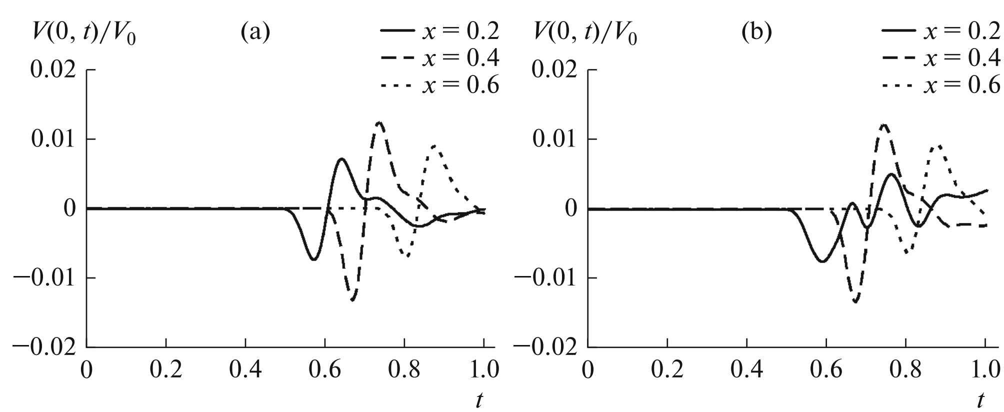

Fig. 10. The effect of layered packet on reflected signal on the surface: a) difference $(V_{\varepsilon=0.05} - V_{el})/V_0$; b) difference $(V_{\varepsilon=0.10} - V_{el})/V_0$.

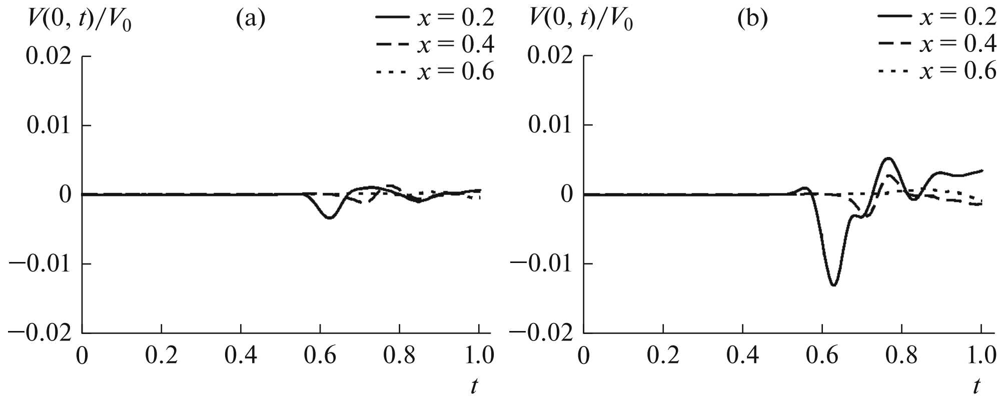

Fig. 11. The effect of layer thickness ε on reflected signal on the surface: a) difference $(V_{\varepsilon=0.05} - V_{\varepsilon=0})/V_0$; b) difference $(V_{\varepsilon=0.10} - V_{\varepsilon=0})/V_0$.

It can be seen from the above graphs that the inclusion of terms of order ε^2 in the constitutive equations of the model leads to a rather significant change in the relative amplitudes of the reflected waves during their passage through the layered fluid-containing packet. Reflected and surface-reaching signals of a specific shape and directivity can serve as an indicator and a means of detecting buried fluid-containing layered packets in a massif of elastic medium.

6. CONCLUSION

Using the method of asymptotic homogenization, a refined model of a layered medium with nonlinear viscous-plastic slippage conditions on the interlayer boundaries is constructed. The model equations take into account terms up to the 2nd order in the parameter of the layer thickness. A numerical difference scheme for solving the obtained nonlinear system of equations based on explicit approximation of the equations of motion and implicit approximation of the remaining equations containing a small parameter in the denominators of free nonlinear terms is proposed. Nonlinear implicit difference equations are solved analytically using the method of expansion in a small parameter. The scheme works for a wide class of nonlinear functions describing viscous-plastic slip at the interlayer boundaries.

An example is given of a numerical calculation of the transverse elastic wave passing through a layered packet with effective anisotropic viscoplastic properties. The two-dimensional problem of the reflection of a system of waves, excited by a surface localized nonstationary source, from this layered packet was also solved numerically. A comparison was made of the dynamics of the velocities of surface points for a purely elastic solution and a solution with allowance for the effect of the recessed laminated packet, as well as the effect of the parameter of the layer thickness. Such models can be useful in solving dynamic problems of seismic exploration and interpretation of wave patterns obtained in the process of its implementation.

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