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On the Issue of the Gravitational Instability of the Solar Protoplanetary Disk

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Abstract—The gravitational instability of a homogeneous isotropic infinite gravitating gaseous medium is investigated in order to study the physical processes that take place during the formation of the solar planetary system. The analytical and numerical solutions of the motion equations of such a medium are considered in two approximations: cold gas and gas at a finite temperature. The real solutions describing the behavior of both wave density disturbances of a homogeneous medium and single disturbances are obtained. Waves of gravitational instability whose amplitude grows exponentially and whose highs and lows, as well as their nodal points, retain their positions in space follow the basic laws of Jean's model. The authors interpret this wave of instability as an analogue of protoplanetary rings, which can be formed in protoplanetary disks. According to the numerical calculation results, the reaction of a homogeneous gravitating medium to the single initial perturbation of its density is significantly different from the laws of Jean's model. The instability localized in single initial perturbations extends to the region $\lambda < \lambda_J$, although in this case the growth of the perturbation density is considerably less than for $\lambda > \lambda_J$. It is discovered that the gravitational instabilities in the region $\lambda > \lambda_J$ suppress sound. It is shown that, without taking into account the rotation of the Sun's protoplanetary disk medium, its critical density in the event of a large-scale gravitational instability is about four orders of magnitude smaller than the critical density in accordance with the theory of planet formation by the accumulation of solids and particles.

Keywords: homogeneous isotropic gaseous medium, gravitational instability, dispersion equation, sound wave, wave of gravitational instability

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INTRODUCTION

This work is dedicated to the issue of the gravitational instability of the solar protoplanetary disk aimed at developing an adequate model for the formation of the solar system. In the first approximation, the gravitational instability of a homogeneous isotropic infinite gravitating gaseous medium was studied. The research presented in this article is a continuation of the research [1–9] and includes the development of appropriate algorithms for solving the formulated problems.

Currently, the commonly accepted theory of the formation of the solar planetary system is the Schmidt–Safronov theory [10, 11], which is the theory of the planets' formation by the accumulation of solids and particles. At the same time, this theory faces significant difficulties when trying to explain some phenomena.

One of the key issues in the protoplanetary disk evolution in the model [10, 11] is the formation of rather large bodies (planetary embryos) able to keep growing using their own gravitational field. In the scientific literature, the possibility of such aggregation of small solid bodies was questioned more than once [12–14]. The experimental and theoretical studies in recent years [15] showed that the probability of particles sticking together tends to zero when they are larger than $\gg$10 cm. Thus, it should not be expected that the accumulation of particles in collisions can lead to the formation of large bodies of about 1–10 m [16, 17].

The most complex issue to explain in the model [10, 11] is the issue of planetary distances (the Titius–Bode rule). Since the processes of accumulating the solid bodies formed in a protoplanetary disk are of a stochastic character the theory that on distances at which planets must be formed in a protoplanetary disk cannot possibly be calculated in this model [11]. If the substance distribution in a protoplanetary disk is preset, this theory can only estimate the number of planets formed and their relative distances.

It is well known that the planets of the solar system are divided into two groups in terms of physical characteristics: inner planets, which are terrestrial planets (Mercury, Venus, Earth, and Mars) and outer planets, which are gas giants (Jupiter, Saturn, Uranus, and Neptune). The solid body accumulation model [10, 11] assumes the existence of two phases in the formation of the outer planets of the solar system. In the first phase, a solid body embryo (core) of a planet is formed through the accumulation of a solid body. Im the second phase, the gas component is accreted onto the solid body embryo formed. There is another hypothesis about the formation of the outer planets of the solar system, which is more realistic, assuming that each planet was formed from a corresponding protoplanet (in this case, a protoplanet is a gas–dust cloud, which, in the process of gravitational compression, is transformed into a planet). This hypothesis has the highest evidentiary force for Jupiter and Saturn [9].

As a result of the study of the opportunities to overcome failures of the Schmidt–Safronov theory, other models for the formation of the solar system emerged. The alternative models [18–24] to the model of the solid body accumulation, as a rule, assume the formation of protoplanets with their subsequent conversion into the corresponding planets.

For the choice of an adequate model for the formation of planets of the solar system, the works by E.M. Galimov [23–25] are important. Based on the results of the geochemical investigation of the planet compositions, E.M. Galimov formulated the hypothesis that a planet body is formed from a gas–dust cloud and, for the case of the Earth–Moon system, showed that this hypothesis is the best one for explaining the known geochemical facts and limitations.

Note that all the models for the formation of the solar planetary system are based on the fundamental property of a gravitating medium, namely, its gravitational instability, while the difference between the models consists only in the spatial dimensions on which the gravitational instability starts developing. However, this circumstance leads to significant differences in the formation of the solar planetary system.

In the solid body accumulation model [10, 11], the gravitational instability at the initial phase of the protoplanetary disk evolution, i.e., actually, in a gaseous medium, is excluded. The gravitational instability in this model occurs only at the phase of the formation of a dust layer in a disk and leads to the formation of planetesimals with typical sizes, e.g., for the Earth zone, $\approx 4 \times 10^7$ cm [11].

In the Fridman–Polyachenko model [18, 19], which is based on estimates, it is assumed that the solar protoplanetary disk at the initial phase of its evolution was also gravitationally unstable. In these conditions, ring-shaped density disturbances that increase in time (protoplanetary rings) are formed in a protoplanetary disk, which are "located in it in accordance with the Titius–Bode law" [19]. Thus, the wavelength of the ring-shaped gravitational instabilities corresponds to the distance between planets, and the typical radial size (the width of ring-shaped gas–dust clouds) corresponds to the planet zones. For example, the width of a ring-shaped gas–dust cloud in the Earth region is $\approx 6 \times 10^{12}$ cm. Subsequently, in this model, protoplanets are formed from protoplanetary rings.

The Eneev–Kozlov model [20–22], as the solid body accumulation model, assumes the existence of a gravitational instability with respect to the disturbances of a rather small wavelength in a protoplanetary disk, which leads to the formation of planetesimals. However, unlike the solid body accumulation model, T.M. Eneev and N.N. Kozlov assume that planetesimals are not solid bodies "but more or less rarefied gas–dust clouds." In this model, the following result is of interest: there is a ring compression of the protoplanetary disk substance whose mass is "negligibly small compared to the mass of the central body (the Sun)" [13, 20–22]. The evolution of each of the ring regions is completed by the formation of a gas–dust protoplanet. In the terminology accepted in this work, the protoplanetary rings are the ring compression of a substance. In other words, a protoplanetary disk in the Eneev–Kozlov model proved gravitationally unstable to disturbances with a wavelength approximately equal to the distance between planets.

In the model presented in [2–4, 7–9], it is shown that a protoplanetary disk at the initial phase of its evolution could be gravitationally unstable with respect to large-scale disturbances. This leads to the formation of protoplanetary (toroidal) rings and, further, protoplanets. Within this model, it was also shown that, for the solar protoplanetary disk, in the axisymmetric approximation, there are analytical solutions in the form of protoplanetary (toroidal) rings that correspond to the planet zones [3, 4]. According to the model [2–4, 7–9], large-scale gravitational instabilities can also occur in a dust layer if, due to some reason, they could not occur at the initial phase of the protoplanetary disk evolution.

Hence, the in which the solar planetary system is formed depends significantly on the size of the initiating gravitational instabilities. Thus, to develop an adequate model for the formation of the solar planetary system, we should consider whether a large-scale gravitational instability could occur in the solar protoplanetary disk.

Taking the crucial importance of this question into consideration for the development of the model, it is advisable to consider in detail the initiation and development of the gravitational instability in a gravitating medium. The main principles of the gravitational instability change insignificantly depending on the size and geometry of a gravitating medium. Thus, at the current stage, the simplest gravitating system is considered. Such a system is a medium that is infinite in all directions, homogeneous, and isotropic; its stability was studied in the classic work by Jeans [26]. On the assumption of the homogeneity and isotropy of such a medium, Jeans assumed that the gravitational force at any point of this medium is zero and the system is stationary. Note that, in fact, these assumptions made by Jeans are incorrect since a homogeneous infinite gravitating medium must be nonstationary [27]. Thus, strictly speaking, the problem of the gravitational instability of an infinite gravitating medium must be solved for a nonstationary case [28, 29]. However, the main qualitative conclusions made by Jeans prove to be correct for the detailed consideration of the instability of a contracting or expanding homogeneous medium. Due to this, in this study of a homogeneous gravitating medium, Jean's traditional approach is followed.

GRAVITATIONAL INSTABILITY OF A HOMOGENEOUS ISOTROPIC MEDIUM: ANALYTICAL SOLUTIONS

A system of gas dynamics equations taking gravity into consideration is written in the following form [19, 30–32]:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{U}) = 0, \quad \rho\frac{\partial \mathbf{U}}{\partial t} + \rho(\mathbf{U}\nabla)\mathbf{U} = -\operatorname{grad} p + \rho \operatorname{grad} \varphi, \tag{1}$$

$$\frac{\partial e}{\partial t} + \operatorname{div}[(p+e)\mathbf{U}] = \rho \mathbf{U} \cdot \operatorname{grad} \varphi, \quad p = p(\rho, \varepsilon), \tag{2}$$

and inside the medium, there is the Poisson equation

$$\nabla^2 \varphi \equiv \Delta \varphi = -4\pi G \rho, \tag{3}$$

where $\rho(\mathbf{r},t)$ is the gaseous medium density; $p(\mathbf{r},t)$ is the gaseous medium pressure; $\mathbf{U}(\mathbf{r},t)$ is the speed vector; $p = p(\rho,\varepsilon)$ is the equation of state of a gaseous medium; $e = \rho\left(\varepsilon + \frac{1}{2}\mathbf{U}\cdot\mathbf{U}\right)$ is the energy of a volume unit; $\varepsilon(\mathbf{r},t)$ is the internal energy of a mass unit; $\varphi(\mathbf{r},t)$ is the gravitational potential; and G is the gravitational constant.

Assume that a gaseous medium is homogeneous and in equilibrium in the initial state. Then, as is well known [19], the system of equations (1)–(3) for the disturbed state by linearization can be reduced to one equation:

$$\frac{\partial^2 \rho_\Delta}{\partial t^2} = \operatorname{div}(\operatorname{grad} p_\Delta) + \omega_J^2 \rho_\Delta, \tag{4}$$

where $\omega_J^2 = 4\pi G \rho_0$ is the square of Jean's frequency; ρ_0 is the density of the initial state of a medium; and ρ_Δ and p_Δ are the amplitude of the density and pressure disturbances, respectively.

Assume that a homogeneous isotropic infinite gaseous medium is "cold" (T → 0 K). Then, the medium pressure can be neglected and Eq. (4) is converted into the following equation:

$$\frac{\partial^2 \rho_\Delta}{\partial t^2} = \omega_J^2 \rho_\Delta. \tag{5}$$

One of the partial solutions of this equation is the following solution:

$$\rho_\Delta = \rho_{\Delta,0}(\mathbf{r}) \exp(\omega_J t), \tag{6}$$

where $\rho_{\Delta,0}(\mathbf{r})$ is the initial density disturbance of a medium.

Solution (6) describes the major instability of a gravitating medium, which was first introduced by Jeans.

Assume that a homogeneous isotropic infinite gaseous medium is at a finite temperature ($T \neq 0$ K). Hereinafter, to make the presentation the results simpler, the behavior of a homogeneous medium under its plane initial density disturbances is considered. For this purpose, instead of the **r** vector, the x coordinate is introduced, and the initial disturbance density changes along its direction. Then, Eq. (4) is converted into the following equation

$$\frac{\partial^2 \rho_\Delta}{\partial t^2} = \frac{\partial^2 p_\Delta}{\partial x^2} + \omega_J^2 \rho_\Delta. \tag{7}$$

It is known that [1, 11, 33, 34] in the initial phase of the protoplanetary disk's evolution, its medium with a sufficient degree of accuracy can be regarded as a one-component ideal gas. Then, in the approximation of an ideal gas, Eq. (7) is transformed to the following form:

$$\frac{\partial^2 \rho_\Delta}{\partial t^2} = C_0^2 \frac{\partial^2 \rho_\Delta}{\partial x^2} + \omega_J^2 \rho_\Delta, \tag{8}$$

where $C_0 = (\partial p/\partial \rho|_S)_0^{1/2}$ is the speed of sound in an unperturbed gaseous medium.

For the subsequent calculation, the following dimensionless variables are introduced: $\rho_{\Delta,\mathrm{dl}} = \rho_\Delta/\rho_0$, $t_{\mathrm{dl}} = \omega_J t$, and $\omega_{\mathrm{dl}} = \omega/\omega_J$. An infinite homogeneous medium has no typical length parameter of a finite value. As this work is focused on the solar protoplanetary disk, thus, without losing consistency, we introduce the length parameter for nondimensionalization equal to the outer radius of the solar protoplanetary disk r_{ex}. In these conditions, $x_{\mathrm{dl}} = x/r_{\mathrm{ex}}$, $\lambda_{\mathrm{dl}} = \lambda/r_{\mathrm{ex}}$, and $C_0^2/(\omega_J^2 r_{\mathrm{ex}}^2)$ is a dimensionless value. With consideration of the introduced dimensionless variables, Eq. (8) is written in the following form:

$$\frac{\partial^2 \rho_{\Delta,\mathrm{dl}}}{\partial t_{\mathrm{dl}}^2} = \frac{C_0^2}{\omega_J^2 r_{\mathrm{ex}}^2} \frac{\partial^2 \rho_{\Delta,\mathrm{dl}}}{\partial x_{\mathrm{dl}}^2} + \rho_{\Delta,\mathrm{dl}}. \tag{9}$$

Hereinafter, the dl index in the notation of variables is mostly omitted.

Find the solutions for Eq. (9) in the form of a sinusoidal wave by presetting its initial solution (the density disturbance at $t = 0$) in the segment $0 \le x \le 1$ of an infinite homogeneous space in the form of a wave with the number of peaks equal to the number of planets in the solar system (eight planets and the asteroid belt). Such a preset initial disturbance permits the subsequent admissible comparison with some parameters of the solar planetary system. It can be shown that, under such initial conditions, Eq. (9) has the following real partial solutions [35].

A wave of gravitational instability,

$$\rho_\Delta = C_1 \exp(\omega t)[\cos((2\pi x/\lambda) + C_2)], \tag{10}$$

where $\lambda = 1/9$; $C_2 = \pi$; and $\omega^2 = -(\lambda_J^2/\lambda^2) + 1$ is the dispersion equation; $\lambda_J = 2\pi(C_0/(\omega_J r_{\mathrm{ex}}))$ is Jeans' dimensionless wavelength; $\omega^2 > 0$ if $\lambda > \lambda_J$; $\omega^2 = 0$ if $\lambda = \lambda_J$; and if $\lambda < \lambda_J$ then $\omega^2 < 0$, and a real partial solution does not exist.

A stationary wave,

$$\rho_\Delta = C_1[\cos((2\pi x/\lambda) + C_2)], \tag{11}$$

where $\lambda = 1/9$, $C_2 = \pi$, and $\lambda = \lambda_J$.

A standing wave,

$$\rho_\Delta(x,t) = C_1 \cos \omega t \cos((2\pi x/\lambda) + C_2), \tag{12}$$

where $\lambda = 1/9$; $C_2 = \pi$; $\omega^2 = (\lambda_J^2/\lambda^2) - 1$ is the dispersion equation; $\omega^2 > 0$ if $\lambda < \lambda_J$; $\omega^2 = 0$ if $\lambda = \lambda_J$; and if $\lambda > \lambda_J$ then $\omega^2 < 0$, and a real partial solution does not exist.

A running wave or heavy sound,

$$\rho_\Delta = C_1 \cos(\omega t + (2\pi x/\lambda) + C_2), \tag{13}$$

where $\lambda = 1/9$; $C_2 = \pi$; $\omega^2 = (\lambda_J^2/\lambda^2) - 1$ is the dispersion equation; $\omega^2 > 0$ if $\lambda < \lambda_J$; $\omega^2 = 0$ if $\lambda = \lambda_J$; and if $\lambda > \lambda_J$ then $\omega^2 < 0$ and a real partial solution does not exist.

Hence, the gravitational instability in the form of a wave process in a homogeneous isotropic medium is possible only for $\lambda > \lambda_J$, as follows from the Jeans model. Jeans' critical wavelength (λ_J) separates the ranges of stable ($\lambda < \lambda_J$) and unstable ($\lambda > \lambda_J$) disturbances of an isotropic medium.

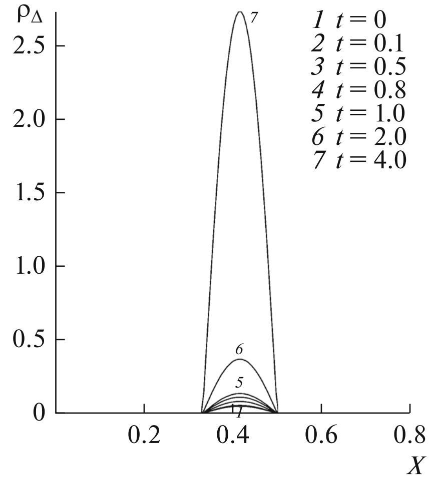

Fig. 1. Gravitational instability, sinusoidal half-wave, $T \to 0$ K.

NUMERICAL EQUATION SOLVING TECHNIQUE FOR A HOMOGENEOUS ISOTROPIC GRAVITATING MEDIUM

Omitting the indices of variables, convert Eq. (9) into the following form:

$$\frac{\partial^2 \rho}{\partial t^2} = C^2 \frac{\partial}{\partial x}\left(\frac{\partial \rho}{\partial x}\right) + \rho, \tag{14}$$

where $C^2 = C_0^2/(\omega_J^2 r_{\text{ex}}^2)$.

For performing the numerical calculations, the region along the x axis is divided into counting intervals by points with integer indices. The ρ function is defined in the center of the interval (x_i, x_{i+1}) and denoted by the index $i + 1/2$. The transition from the layer n to the layer $n + 1$ with a constant time step τ is performed in accordance with the standard difference scheme:

$$\rho_{i+1/2}^{n+1} = 2\rho_{i+1/2}^{n} - \rho_{i+1/2}^{n-1} + \frac{C^2\tau^2}{x_{i+1} - x_i}\left[\frac{\rho_{i+3/2}^{n} - \rho_{i+1/2}^{n}}{x_{i+3/2} - x_{i+1/2}} - \frac{\rho_{i+1/2}^{n} - \rho_{i-1/2}^{n}}{x_{i+1/2} - x_{i-1/2}}\right] + \tau^2\rho_{i+1/2}^{n}. \tag{15}$$

For each calculated case, by introducing the initial and boundary conditions written in finite differences, a difference scheme is developed, in accordance with which the calculations are performed.

NUMERICAL CALCULATION RESULTS FOR THE GRAVITATIONAL INSTABILITY OF A HOMOGENEOUS ISOTROPIC COLD MEDIUM

The motion of a homogeneous isotropic infinite cold ($T \to 0$ K) gaseous medium is described by Eq. (5). This equation was numerically solved in dimensionless variables in the segment $0 \le x \le 1$ under different initial conditions using the technique described in Chapter 3. The motion of a homogeneous medium under the initial plane density disturbance of a rectangular shape was studied. Such a disturbance, unlike the wave disturbance, is localized in space along the x axis; hereinafter, it is referred to as a single disturbance. In these calculations, the density disturbances increase in time in accordance with the exponential law ~exp(ωt), where ω =1, regardless of the spatial dimensions of the initial disturbance. The divergence between the numerical and analytical solutions in these calculations does not exceed 0.025%. In the case $T \to 0$ K, the initial rectangular shape of the density disturbance under the increase in its amplitude in time is kept. If the shape of a single initial disturbance is changed the disturbance is also not blurred along the x axis and the density increase in time at each point of this disturbance is also proportional to ~exp(ωt), where ω =1.

For example, in Fig. 1, the shape of the initial density disturbance is a positive sinusoidal half-wave in the form of $\rho_\Delta(t = 0) = 0.05\cos(6\pi x - \pi/2)$ preset in the segment $1/3 \le x \le 0.5$, outside which $\rho_\Delta(t = 0) = 0$. The preset initial condition is $\partial\rho_\Delta/\partial t|_{t=0} = \omega\rho_\Delta(t = 0)$ and the boundary conditions are $\rho_\Delta(x = 0, t)$

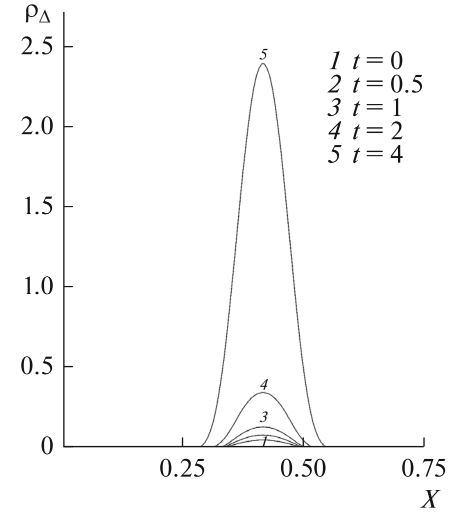

Fig. 2. Gravitational instability, sinusoidal half-wave, $T \neq 0$ K, $\lambda > \lambda_J$.

$= 0$, $\rho_\Delta(x = 1, t) = 0$. The divergence between the numerical and analytical solutions in these calculations does not exceed 0.025%.

The calculations of the motion of a medium under its initial density disturbance in the form of a sinusoidal wave were performed. The initial wave disturbance was preset with a number of peaks in the segment $0 \le x \le 1$ equal to the number of planets in the solar system (eight planets and the asteroid belt). The numerical calculations showed that, as a result of the instability, a wave of gravitational instability is formed. The initial disturbance wavelength can be arbitrary. Waves of gravitational instability in a homogeneous medium are of a similar nature as waves of gravitational instability in a protoplanetary disk [18, 19] from which protoplanetary rings are formed.

The results of this chapter are detailed in [35].

NUMERICAL CALCULATION RESULTS FOR THE GRAVITATIONAL INSTABILITY OF A HOMOGENEOUS ISOTROPIC MEDIUM AT A FINITE TEMPERATURE

Equation (9) was numerically solved for single initial density disturbances of a homogeneous medium at a finite temperature ($T \neq 0$ K) using the technique described in Chapter 3. From the calculation results under the initial medium disturbance of a rectangular shape ($\lambda \approx 2L = 1/3$, where L is the size of a single disturbance and $\lambda > \lambda_J$), it follows that, unlike in the case $T \to 0$ K, at a finite temperature ($T \neq 0$ K), against the background increase in the initial disturbance caused by the gravitational instability, the density disturbances propagate into the ambient medium and the wave propagates inside the region of the initial disturbance. Until the moment of time when rarefaction waves propagating from the opposite ends of the initial disturbance meet, the amplitude of the initial density disturbance increases in accordance with the law ~$\exp(\omega t)$, where $\omega = 1$. After the interaction of the rarefaction waves, the law of the increase in the disturbance amplitude changes. It can be explained by the fact that ω in the exponential law starts decreasing in time. The numerical calculations showed that, unlike a cold medium, the disturbance density for its different points increases in accordance with different laws.

The behavior of the initial density disturbance of a rectangular shape for $\lambda < \lambda_J$ was numerically calculated. The analysis of the calculation results shows that the gravitational instability under these conditions occurs not only in the range of $\lambda > \lambda_J$ but also in the range of $\lambda < \lambda_J$. The gravitational instability development scenario in these regions is the same, and it is described when considering the case $\lambda > \lambda_J$.

The behavior of a homogeneous isotropic medium under the initial density disturbance in the form of a positive sinusoidal half-wave, which is described by the formula

$$\rho_\Delta(t = 0) = 0.05\cos(6\pi x - \pi/2) \quad (16)$$

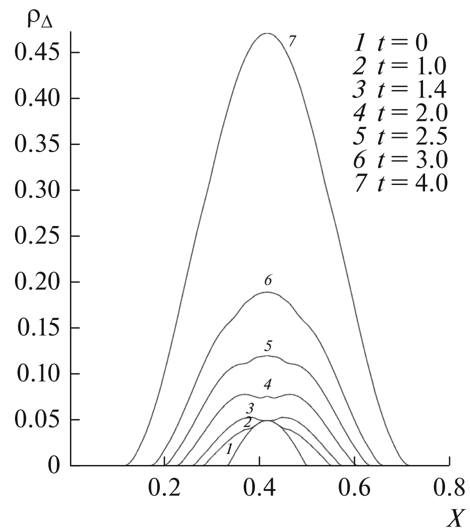

Fig. 3. Gravitational instability, sinusoidal half-wave, $\lambda = \lambda_J$.

in the segment $1/3 \leq x \leq 0.5$, outside which $\rho_\Delta\,(t = 0) = 0$, for the case $\lambda > \lambda_J$, is shown in Fig. 2. For solving the Cauchy problem, the initial condition $\partial\rho_\Delta\,/\partial t|_{t=0} = \omega\rho_\Delta(t = 0)$ and the boundary conditions $\rho_\Delta(x = 0, t) = 0$, $\rho_\Delta(x = 1, t) = 0$ are also preset. From the graphs in Fig. 2, it follows that, in a medium under the preset initial and boundary conditions, the gravitational instability occurs. The density disturbance curves at both sides of the peak depending on the distance are strictly monotonic at all times. In a similar formulation, the calculation, which is presented in Fig. 1, at a temperature of zero of the medium was performed. The comparison of Figs. 1 and 2 demonstrates that, starting from the same moment of time, the disturbance amplitude at a finite temperature increases slower than in the case of a cold medium.

The numerical calculations were performed for the cases $\lambda = \lambda_J$ and $\lambda < \lambda_J$. The initial density disturbance was also preset using formula (16) in the segment $1/3 \leq x \leq 0.5$, outside which $\rho_\Delta(t = 0) = 0$. The initial condition $\partial\rho_\Delta/\partial t_{t=0} = 0$ and the boundary conditions $\rho_\Delta(x = 0, t) = 0$, $\rho_\Delta(x = 1, t) = 0$ were also preset. The calculations show (Fig. 3) that the nature of the curves $\rho_\Delta(x, t = \text{const})$ for $\lambda = \lambda_J$ changes significantly compared to the case $\lambda > \lambda_J$. In some segment of time, the density curves on both sides of the central point, depending on the distance, are of a nonmonotonic nature. At longer times, the disturbance amplitude increases and the curves are gradually smoothed out. At the moment of time $t = 10$, the disturbance amplitude is lower by a factor of about six than in the range of $\lambda > \lambda_J$. The changes in the initial density disturbance (formula (16)) for $\lambda < \lambda_J$, depending on the distance, for different moments of time are shown in Fig. 4.

The behavior of the density disturbance in the case $\lambda < \lambda_J$, as can be seen from Fig. 4, differs from both situations $\lambda > \lambda_J$ and $\lambda = \lambda_J$. In the initial phase, approximately by moment of time $t = 0.5$, the initial disturbance is divided into two disturbances, which start propagating in opposite directions. This qualitatively coincides with the solution of the telegraph equation [35, 36]. However, starting from a certain moment of time ($t > 1$), the amplitude in the center starts increasing. This can be explained by the fact that, in Eq. (9), the second derivative $\partial^2\rho_\Delta/\partial x^2$ becomes low and the second term becomes predominant in the right-hand side of this equation. In this case, the main term in the solution is the expression of form (10) with an exponential cofactor by time. Thus, the amplitude of the density disturbance starts increasing and, by the time $t \approx 2.5$, its value exceeds the amplitude of the initial disturbance.

Further, consider the calculation results for the motion of a homogeneous medium, when the initial disturbances are preset in the form of a sinusoidal wave. In Fig. 5, the gravitational instability of a medium is shown, which is observed when the initial density disturbance is a sinusoidal wave of the following form:

$$\rho_\Delta = 0.01\cos(18\pi x + \pi), \tag{17}$$

preset in the segment $0 \leq x \leq 1$ for the case $\lambda > \lambda_J$. The initial condition $\partial\rho_\Delta/\partial t|_{t=0} = \omega\rho_\Delta(t = 0)$ and the boundary conditions $\partial\rho_\Delta(x,t)/\partial x|_{x=0} = 0$ and $\partial\rho_\Delta(x,t)/\partial x|_{x=1} = 0$ are also preset. The initial disturbance

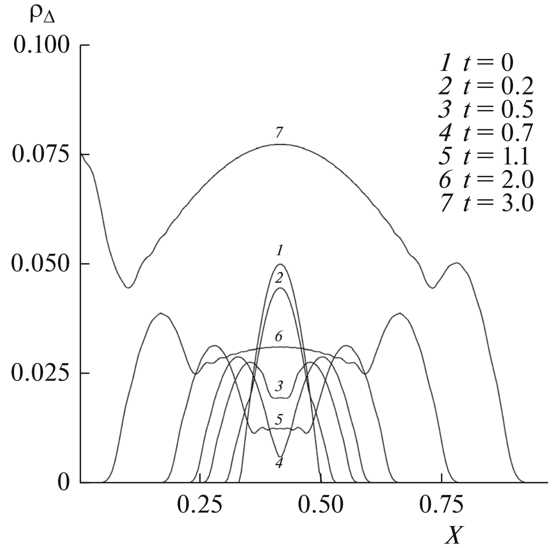

Fig. 4. Gravitational instability, sinusoidal half-wave, $\lambda < \lambda_J$.

wave is preset with nine peaks corresponding to the number of planets in the solar system, taking the asteroid belt into consideration, which are located at a distance equal to the protoplanetary disk width, i.e., in the segment $0 \leq x \leq 1$.

As can be seen from Fig. 5, the medium's instability in this case occurs in the form of a wave of gravitational instability. The increase in the wave's amplitude corresponds to the exponential law. If the wavelength λ approximates Jeans' critical wavelength λ_J the amplitude of a wave of gravitational instability drops sharply. For $\lambda = \lambda_J$, the amplitude of the initial density disturbance wave is constant in time and the phase speed is zero. This wave, unlike other types of waves, can be called a stationary wave. In the case $\lambda < \lambda_J$, under the initial and boundary conditions corresponding to the case $\lambda > \lambda_J$, a standing wave is formed in a homogeneous medium. A homogeneous medium is gravitationally unstable in the range $\lambda \leq \lambda_J$.

In the research mentioned above (Sections 2, 4, and 5), the wave density disturbances of a homogeneous gravitating medium with a zero phase speed were obtained. Based on Eq. (9), it was of interest to investigate the behavior of the wave disturbances in the form of a running wave (13). The numerical calculations were performed using the following initial and boundary conditions. The initial density disturbance remained unchanged and had form (17). The initial condition (the time derivative) was preset in the form $\left.\frac{\partial \rho_\Delta}{\partial t}\right|_{t=0} = -0.01\omega \sin\left(\frac{2\pi}{\lambda}x + \pi\right)$. The boundary conditions were preset in accordance with (13):

$$\begin{aligned} \frac{\partial \rho_\Delta}{\partial x}(x = 0, t) &= 0.01\frac{2\pi}{\lambda}\sin(\omega t), \\ \frac{\partial \rho_\Delta}{\partial x}(x = 1, t) &= 0.01\frac{2\pi}{\lambda}\sin(\omega t), \end{aligned} \tag{18}$$

or

$$\begin{aligned} \rho_\Delta(x = 0, t) &= 0.01\cos(\omega t + \pi), \\ \rho_\Delta(x = 1, t) &= 0.01\cos(\omega t + 2\pi/\lambda + \pi). \end{aligned} \tag{19}$$

The best variant of the boundary conditions, which gave a high level of calculation accuracy, was a modified variant of (19), namely, a periodic variant:

$$\begin{aligned} \rho_\Delta(t, x = x_{-1/2}) &= \rho_\Delta(t, x = x_{N-1/2}), \\ \rho_\Delta(t, x = x_{1/2}) &= \rho_\Delta(t, x = x_{N+1/2}). \end{aligned} \tag{20}$$

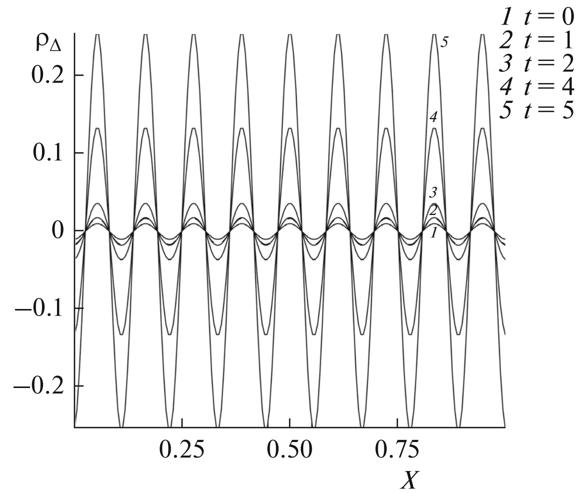

Fig. 5. Gravitational instability of homogeneous medium in form of wave for $\lambda > \lambda_J$, $T \neq 0$ K.

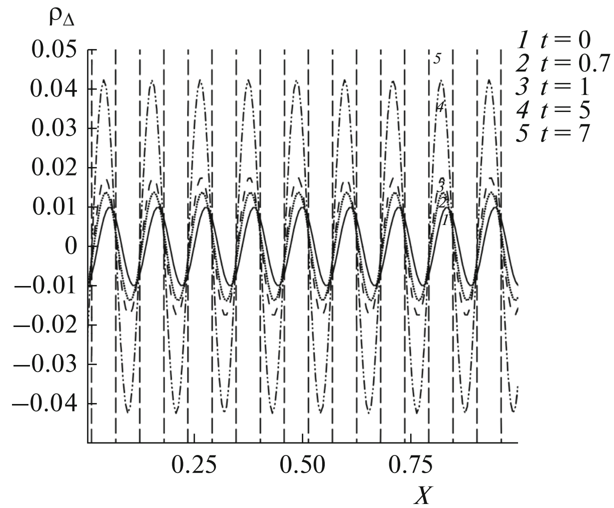

Fig. 6. Waves of gravitational instability ($\lambda > \lambda_J$) under initial and boundary conditions corresponding to running wave, $T \neq 0$ K.

As a result of the calculations, the numerical solution of Eq. (9) was found in the form of a running wave for $\lambda < \lambda_J$, which describes a heavy sound in a gravitating gaseous medium. This numerical solution at the moment of time $t = 10$ coincides with the analytical solution in the phase within 1%.

Are there waves of gravitational instability in the form of a running wave in the region $\lambda > \lambda_J$? To answer this question, the numerical calculations were performed in the region $\lambda > \lambda_J$ with the initial and boundary conditions corresponding to a running wave (formulas (18–20)). The results of these calculations are shown in Fig. 6. In can be seen that, in the initial moments of time ($t \leq 1$), when the gravitational instability only starts developing, the density disturbance wave starts moving. However, under the increase in the amplitude of the wave disturbance, the wave's motion stops and the phase speed of a wave of gravitational instability becomes zero. Thus, in the range $\lambda > \lambda_J$, regardless of the initial and boundary conditions, a wave of gravitational instability occurs with a phase speed of zero.

The results of Chapter 5 are detailed in the preprint [35].

ANALYSIS AND DISCUSSION OF THE RESEARCH RESULTS

1. The results of the numerical calculations agree with the basic concepts concerning the gravitational instability of a cold ($T \to 0$ K) homogeneous isotropic infinite gaseous medium. This medium is gravita-

tionally unstable against any types of density disturbances, including both disturbances in the form of waves and single initial disturbances. The initiation of gravitational instabilities in such a medium does not depend on the wavelength in wave disturbances or the spatial dimensions of the single initial disturbances. The typical spatial size and shape of the disturbance remain constant in time. The density disturbance at each point increases in time in accordance with the exponential law: $\sim\exp(\omega_J t)$, where $\omega_J = \sqrt{4\pi G\rho_0}$. These characteristics of the behavior of gravitational instabilities are explained by the fact that the temperature impeding the formation of gravitational instabilities in this case tends to zero.

In a protoplanetary disk, situations can occur when the dust component is predominant over the gas one. In this case, the medium behaves like cold gas [18, 19]. Thus, the results of the research of a cold homogeneous isotropic medium confirm the possibility of the formation of solid bodies of different sizes, including small ones, from the dust component.

2. Unlike a cold medium, a homogeneous isotropic infinite gaseous medium at finite temperature ($T \neq 0$ K) can be gravitationally unstable against certain types of its density disturbance. In the wave (harmonic) initial density disturbances, a gaseous medium follows the laws of Jeans' model. When $\lambda < \lambda_J$, a medium is gravitationally unstable. Under $\lambda > \lambda_J$, waves of gravitational instability occur in the medium (Section 2, formula (10)). These waves are of a similar nature as the ring-shaped density disturbances of the solar protoplanetary disk (protoplanetary rings) that increase in time.

According to the results of the performed numerical calculations, the reaction of a homogeneous gravitating medium ($T \neq 0$ K) to single initial density disturbances differs significantly from the laws of Jeans' model. The gravitational instability of single disturbances also expands to the region $\lambda < \lambda_J$, although the increase in the density disturbance in this case is significantly lower than for $\lambda > \lambda_J$. This can be explained by the fact that Jeans' theory is based on the initial density disturbances of a medium in the form of a harmonic wave extending throughout a homogeneous medium. Under single initial density disturbances, the problem of a medium's motion differs significantly from the problem in Jeans' formulation. This is effectively a different boundary problem whose solution does not coincide with Jeans' solution, which leads to the results presented in this work.

Another distinctive feature of a medium at a finite temperature, along with the increase in the initial single disturbance density, is the increase in the typical spatial size of the disturbance itself and the change in its shape. These facts are explained by the influence of the temperature, which impedes the occurrence of a gravitational instability.

3. The propagation of sound density oscillations through a homogeneous gravitating medium at a finite temperature in the wave ranges $\lambda < \lambda_J$ and $\lambda > \lambda_J$ was studied. It was shown that, in the wave range $\lambda < \lambda_J$, standing waves (12) can form and waves in the form of a running wave can propagate under the corresponding initial and boundary conditions. A running wave is actualized in the form of a heavy sound wave (13). The distinctive feature of a heavy sound wave is the decrease in the phase speed of this wave when λ increases. Under $\lambda = \lambda_J$, the phase speed of a heavy sound wave becomes zero and the wave degenerates into a stationary wave (11). A standing wave in the presented research is a standing heavy sound wave. Under preset λ and λ_J, the frequency of a standing heavy sound wave is equal to the frequency of a running heavy sound wave. As λ reaches the λ_J value, a standing wave also degenerates into a stationary wave. In the wavelength range of $\lambda > \lambda_J$, waves in the form of standing and running heavy sound waves cannot exist. In this region, when a medium is disturbed, a wave of gravitational instability (10) occurs. The phase speed of an established wave of gravitational instability is zero and does not depend on the boundary or initial conditions in the numerical calculations. This interesting fact is explained by the gravitational sound suppression by the forming gas masses of the gravitational instabilities.

4. Currently, it is, probably, too early to talk about a completely developed model for the formation of a solar planetary system. Along with the theories about the formation of a planetary system—such as the formation of planets by the accumulation of solid bodies and particles [11]; the formation of protoplanets by the collision and aggregation of gas–dust clouds [13]; or the formation of planets by the occurrence of a disturbance wave on the scale of a whole disk leading to the formation of protoplanetary rings (and, subsequently, planets) as a result of a large-scale gravitational instability disk [2–4, 7–9]—there are hypotheses of the formation of some planets of the solar system, e.g., Jupiter, in the form of a separate protoplanet [15]. Thus, in order to consider possible scenarios of the formation of the planets of the solar system, we need to know the conditions of the initiation of gravitational instabilities of different types and their dynamics. The research results for a homogeneous isotropic medium are presented in this work. Note the following result of this study: it was shown that the instability of single initial disturbances also expands to the region $\lambda < \lambda_J$. Thus, the boundaries of the possible formation of single gravitational instabilities in a

homogeneous medium are expanded. If such a phenomenon also occurs in a protoplanetary disk this can significantly increase the probability of the formation of a single protoplanetary ring and, as a consequence, lead to the formation of a separate protoplanet from this ring.

5. In [7], the critical density of a protoplanetary disk was introduced as the minimum density under which a gravitational instability can occur taking into consideration the possible disturbances within the whole wavelengths range. This definition is also considered correct for a homogeneous gaseous medium. If a protoplanetary disk medium has the density distribution of the dust component that is close to homogeneous and the dust weight content does not exceed several percent, then the averaged parameters of such a medium are described sufficiently accurately by the equation of state of ideal gas [1, 11, 33]. In this research, a homogeneous gaseous medium is also an ideal gas. Then, the configuration of the initial wave density disturbances of a medium, which was studied in this work, helps estimate the critical density of the solar protoplanetary disk without taking into consideration the stabilizing effect of the rotational motion of a disk on its gravitational instability. Indeed, the gravitational instability of a homogeneous isotropic gaseous medium was studied under such initial wave density disturbances when nine wavelengths, which was the number of planets of the solar system including the asteroid belt, could fit in the distance equal to the width of the protoplanetary disk. A wave of gravitational instability is observed in a homogeneous medium in the form of a harmonic wave; thus, this estimation can give only an averaged value of the critical density of the solar protoplanetary disk, which is, apparently, of interest even in this approximation. The critical density ($\rho_{0,\mathrm{cr}}$) was estimated using Jeans' expression for the critical wavelength, from which it follows that [7]

$$\rho_{0,\mathrm{cr}} = \pi\gamma RT \big/ \mu G\lambda_J^2, \qquad (21)$$

where R is the gas constant, erg K^{-1} mol^{-1}; T is the temperature of the gaseous medium of a protoplanetary disk, K; μ is the molecular weight of a disk gaseous medium, g/mol; γ is the C_p/C_v ratio for a disk gaseous medium (the adiabatic index); and G is the gravitational constant, $\mathrm{cm}^3/(\mathrm{g\ s}^2)$.

If the weight proportion of dust particles is about 1.5% of a substance of the solar composition, the molecular weight for such a medium is 2.53 g/mol and the adiabatic index is 1.43 [33]. According to the current ideas, the temperature of a gaseous medium, e.g., for the Earth's zone in the initial phase of the protoplanetary disk evolution was $T \sim 300$ K and the density was $\sim 3 \times 10^{-9}$ g/cm^3 [11]. The average temperature of a disk in its equatorial plane calculated by averaging the data on the temperatures of all the zones is ≈ 150 K. In the calculations made within the present study, nine wavelengths (the number of planets in the solar system, including the asteroid belt) could fit in the distance equal to the protoplanetary disk width. Thus, $\lambda = \lambda/r_{\mathrm{ex}} = 1/9$. For the critical density estimation, it is assumed that

$$\lambda > \lambda_J, \quad \lambda_J \approx r_{\mathrm{ex}}/9 \approx 6.57 \times 10^{13}\ \mathrm{cm}, \qquad (22)$$

$T \approx 150$ K, $\gamma = 1.43$, $\mu = 2.53$ g/mol, $G = 6.673 \times 10^{-8}$ cm^3/(g s^2), $r_{\mathrm{ex}} = 0.591 \times 10^{15}$ cm, and $R = 8.31434 \times 10^7$ erg K^{-1} mol^{-1}.

Substituting the values of the variables and the parameters in (21), find the critical density:

$$\rho_{0,\mathrm{cr}} \approx 7.7 \times 10^{-11} \mathrm{g/cm}^3. \qquad (23)$$

This critical density value is about an order of magnitude higher than in [7]. This is explained by the fact that, in this work, the estimated λ_J value is about a quarter of the λ_J value in [7] taken equal to the halfwidth of the solar protoplanetary disk.

A comparison of the critical density of a disk in accordance with the model of large-scale instabilities [2–4] with the results of the theory of the planet formation by the accumulation of solid bodies and particles [10, 11] is of crucial importance. The critical density (23) under which a large-scale gravitational instability occurs is about four orders of magnitude lower than the critical density ($\rho_{\mathrm{cr}} \approx 3 \times 10^{-7}$ g/cm^3) calculated in accordance with the theory [10, 11] and two orders of magnitude lower than the density ($\sim 3 \times 10^{-9}$ g/cm^3) of a gaseous medium in the Earth zone [11].

Hence, based on the presented estimation of the critical density, it can be claimed that when considering the gravitational instability of the solar protoplanetary disk, the possible development of large-scale instabilities as instabilities initiated under the lowest critical density of all possible ones should be considered. The wavelength of large-scale instabilities can be comparable to the distance between planets.

CONCLUSIONS

The work is focused on the gravitational instability of an infinite homogeneous isotropic gaseous medium. Both the gravitational instabilities in the form of waves and single gravitational instabilities were studied using the numerical modeling.

The waves of gravitational instability in this medium are described by Jeans' theory [26]. The formation of the gravitational instabilities of this type in a homogeneous isotropic medium makes it possible to assume that such structures can also occur in gravitating media of a more complex configuration. In fact, e.g., the emergence of protoplanetary rings in protoplanetary disks was predicted [3, 7–9, 18, 37, 38], which can be interpreted as the result of the gravitational instability in the form of a wave in a protoplanetary disk [18, 19].

In the research of the dynamics of single initial density disturbances, it was established that single instabilities that occur in a homogeneous medium at a finite temperature cannot be described by Jeans' model [26] because the region of unstable initial disturbances expands to the region $\lambda < \lambda_J$. The typical spatial dimensions of single-density disturbances do not change in time in a cold homogeneous medium and increase in time in a medium at a finite temperature.

The propagation of sound density oscillations through a homogeneous gravitating medium at a finite temperature was studied. The sound suppression by gravitational instabilities in the region $\lambda > \lambda_J$ was established.

Based on the calculation results, the critical density of the solar protoplanetary disk was estimated. In the considered approximation (without consideration of the rotation), it was shown that the critical density (23) of the solar protoplanetary disk, under which a large-scale gravitational instability occurs, is about four orders of magnitude lower than the critical density ($\rho_{cr} \approx 3\times10^{-7}$ g/cm^3) calculated in accordance with the theory [10, 11].

In the process of the work, the numerical modeling methods were also elaborated with the purpose of using them for gravitating media of more complex configurations.

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