ISSN 2070-0482, Mathematical Models and Computer Simulations, 2018, Vol. 10, No. 5, pp. 659–669. © Pleiades Publishing, Ltd., 2018.
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On Resolving Inverse Nonstationary Scattering Problems in a Two-Dimensional Homogeneous Layered Medium by the τ–p Radon Transform

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Received March 20, 2017

Abstract—We consider a two-dimensional nonstationary inverse scattering problem in a layered homogeneous acoustic medium. The data consist of a scattered wavefield from a surface point source registered on the boundary of the half-plane. We prove the uniqueness of the recovery of an acoustic impedance and velocity in a medium from the scattering data. An algorithm for solving an inverse two-dimensional scattering problem as a one-dimensional problem with the parameter based on the τ–p Radon transformation is constructed. Also, the numerical modeling results for the direct scattering problem and solutions of a pair of inverse scattering problems in a layered homogeneous acoustic medium are presented. The proposed algorithm is applicable to data processing in geophysical prospecting both for surface seismics and vertical seismic profiling.

Keywords: inverse nonstationary scattering problem, layered acoustic medium, Radon transformation, eikonal, acoustic impedance, surface seismics

DOI: 10.1134/S2070048218050022

1. INTRODUCTION

Integral transformations are used in various problems, including inverse geophysics problems, inverse problems of the mathematical sounding theory, inverse electroprospecting problems (see [1, 2]), transformations of time-series for processing records of seismic traces (see [3, 4]), problems to continue wavefields from the original ground downward (see [5–7]), and inverse problems of seismology and seismics (see [8, 9]). Inverse problems related to finding the physical parameters of layered media are especially important in practice. In such problems, using the Fourier and Fourier-Bessel integral transformations, we can reduce multidimensional problems to one-dimensional ones (see [1, 2, 8–11]).

Since [12–18], the Radon transformation has been used for the practical processing of seismic data. However, the Radon transform is not widespread in the surface and blast-hole seismics because there is no clear connection between the results of its application and the inverse scattering problem (see [19, 20]). Besides, the class of considered models is mainly reduced to the scalar wave equation with an unknown velocity (see [15–18]) but it does not match even basic models for acoustic and elastic media.

The purpose of the present paper is to use the τ–p Radon transform to solve the inverse scattering problem for acoustic waves in a two-dimensional layered-homogeneous medium. Such an inverse problem is quite topical in the practice of seismics in the case where the inverse dynamical seismic problem is solved s part of the acoustic model. However, it is obvious that it requires a separate mathematical investigation. It is important that posing the inverse problem given above allows us both to extend the class of considered models up to the equations of elasticity theory in porous media and increase the dimension of the space (see [21, 22]).

The main result of the present paper lies in the correspondence between two inverse scattering problems found by the Radon transform; one of these problems is to find properties of a layered half-plane by the field of the scattered waves under the assumption that the source evenly moves along the boundary line. We use the invariance property of the solution of the direct scattering problem with respect to the translation with respect to a phase variable; this allows us to reduce the original two-dimensional problem to a one-dimensional problem with a parameter. The value of this parameter is determined by the motion velocity of the virtual source and is controllable. In [23, 24], a similarly posed inverse problem is considered and it is shown that it is possible to find coefficients characterizing the layered half-plane by variation of the angle of incidence of a plane wave from the homogeneous part of the plane.

In [23, 24], the incident wave and the scattered wave are propagated at kinematic (sonic) speeds determined by the physical properties of the medium and the type of the wave (acoustic, electromagnetic, or elastic); however, the wave front moves along the boundary between the homogeneous half-plane and layered half-plane with the phase wave exceeding the acoustic speed in a neighborhood of the boundary.

In the case of a layered half-plane without a boundary with a material medium (considered in the present paper), the corresponding form of the excitation of the medium is a point source moving along the boundary at a supersonic speed and a wave forming the Mach cone (see [25]), which moves from the boundary to the layered medium. It is clear that it is impossible to implement such an excitation-observation scheme on a substantial base; therefore, we have to treat the source as a virtual one.

The most important particular practical result of the present paper (Theorem 1) means that, in posing the two-dimensional problem with a stationary source, the Radon transform allows us to use the scattering data to intermediately obtain the scattering data for the excitation scheme with a supersonic source moving along the boundary. This provides a possibility to reduce the inverse two-dimensional scattering problem to a one-dimensional problem with a parameter. Moreover, since the parameter p of the τ–p transformation has a visual geometrical interpretation (when posed as a two-dimensional problem, it determines the ray angle of departure from the source), it follows that the process of the solution based on the Radon transformation is easily controlled (both for the direct and inverse problem) by the ray-path method.

Note that the considered inverse problem is overdetermined because the wavefield registered at the half-plane boundary is a function of two variables (the time and the distance to the source), while two functions of one variable (the depth) are determined. The Radon transform allows us to use the excessiveness of the input data integrally, ensuring the practical stability of the algorithm. Also, it is important that solutions of real geophysical problems, obtained by means of the τ–p Radon transform (unlike the ones obtained by the Fourier transformation), do not leave the field of real numbers even when the input data contain errors.

2. INVERSE PROBLEMS IN TWO-DIMENSIONAL LAYERED-HOMOGENEOUS MEDIA WITH NONSTATIONARY SURFACE SOURCES

2.1. In a two-dimensional horizontally homogeneous acoustic medium, consider the propagation of compression waves $w(x,z,t)$ described by the wave equation

$$\rho(z)w_{tt} = (k(z)w_z)_z + k(z)w_{xx}, \quad -\infty < x,t < \infty, \quad z > 0, \tag{1}$$

with the following concordance conditions for $z = z_j$, $j = 1$–N, i.e., at discontinuity points of the first type of the piecewise-constant coefficients $\rho(z)$ and $k(z)$:

$$\begin{gathered} k(z_j + 0)w_z(x, z_j + 0, t) = k(z_j - 0)w_z(x, z_j - 0, t), \\ w(x, z_j + 0, t) = w(x, z_j - 0, t), \end{gathered} \tag{2}$$

where $\rho, k > 0$ are the density and elastic parameters of the medium and it is assumed that $z_{j+1} > z_j$. Such concordance conditions require the pressure and displacements in the acoustic medium to be continuous.

Introduce the propagation velocity $a = \sqrt{k/\rho}$ for the perturbations and the acoustic impedance $\varkappa = a\rho$ of the medium. The functions $a(z)$ and $\varkappa(z)$ are also piecewise-constant and their points of discontinuity are the points z_j introduced above. According to the geophysical tradition, z is the depth, the axis x coincides with the original ground $z = 0$, and t is the physical time.

Let a known source of oscillations, defined by the condition

$$k(0)w_z(x,0,t) = -\varphi(t - px), \quad -\infty < x,\ t < \infty,$$

where $\varphi \in C^1(-\infty,\infty)$, $\varphi(\tau) \equiv 0$ for $\tau < 0$, and $0 \le p < 1/a(0)$, move along the boundary line $z = 0$. At the beginning of our consideration, i.e., as $t \to -\infty$, it is assumed that the medium is at rest. It is obvious that one can look for a solution $w(x,z,t)$ of the original problem posed by (1)–(2) in the form

$$w(x,z,t) = v(z,t - px, p) \equiv v(z,\tau,p), \quad z \ge 0, \quad -\infty < \tau < \infty.$$

This yields the following boundary-value problem on the half-line $\{z \geq 0\}$ (the direct scattering problem) for the function $v(z,\tau,p)$:

$$v_{\tau\tau} = a^2(z)v_{zz} + p^2a^2(z)v_{\tau\tau}, \quad z > 0, \quad z \neq z_j, \quad -\infty < \tau < \infty, \tag{3}$$

$$[k(z)v_z(z,\tau,p)]_{z_j} = [v(z,\tau,p)]_{z_j} = 0, \quad -\infty < \tau < \infty, \tag{4}$$

$$k(0)v_z(0,\tau,p) = -\varphi(\tau), \quad -\infty < \tau < \infty, \tag{5}$$

$$\lim_{\tau\to-\infty} v(z,\tau,p) = 0, \quad z \geq 0, \tag{6}$$

where $[k]_{z_j}$ denotes the value of the jump of the function $k(z)$ at the point z_j.

That boundary-value problem without the initial-value conditions is a Fourier problem and the last condition plays the role of the radiation condition (see [26]). Consider solutions $v(z,\tau,p)$ such that $v(z,\tau,p) \equiv 0$ for negative τ provided that z is nonnegative. This is possible for nonnegative values of p such that $pa(z) \leq \varepsilon < 1$. Then the Fourier problem becomes an initial-boundary problem with the initial-value condition

$$v(z,\tau,p) \equiv 0, \quad z \geq 0, \quad \tau < 0. \tag{7}$$

Posing the initial-value conditions this way, we can consider generalized solutions of problem (3)–(5) and (7); also see [27, 28]. Therefore, assign $\varphi(\tau) = \delta(\tau)$, where δ is the Dirac delta-function.

2.2. For problem (3)–(5) and (7), pose the following inverse scattering problem: by the given trace of the solution

$$v(0,\tau,p) = f(\tau,p), \quad -\infty < \tau < \infty,$$

find functions $a(z)$ and $k(z)$ provided that p is known, fixed, and the amount of its values is finite and greater than one.

Transform (3)–(5) and (7) using standard reductions valid in the considered class of piecewise-constant coefficients $a(z)$ and $k(z)$. Introduce the following eikonal $\zeta(z)$ physically interpreted as the propagation time of the signal from the current point z to the boundary $\{z = 0\}$:

$$\zeta(z) = \int_0^z \frac{d\alpha}{a(\alpha)}, \quad \zeta_j = \zeta(z_j), \quad a(z(\zeta)) = c(\zeta), \quad \varkappa(z(\zeta)) = \sigma(\zeta),$$

where the function $z(\zeta)$ is inverse to the function $\zeta(z)$. Then $v(z(\zeta),\tau,p) = u(\zeta,\tau,p)$ and (3)–(5) and (7) takes the form

$$(1 - p^2c^2(\zeta))u_{\tau\tau} = u_{\zeta\zeta}, \quad \zeta > 0, \quad \zeta \neq \zeta_j, \quad -\infty < \tau < \infty, \tag{8}$$

$$[\sigma(\zeta)u_\zeta(\zeta,\tau,p)]_{\zeta_j} = [u(\zeta,\tau,p)]_{\zeta_j} = 0, \quad -\infty < \tau < \infty, \tag{9}$$

$$\sigma(0)u_\zeta(0,\tau,p) = -\delta(\tau), \quad -\infty < \tau < \infty, \tag{10}$$

$$u(\zeta,\tau,p) \equiv 0, \quad \zeta \geq 0, \quad \tau < 0, \tag{11}$$

which allows us to treat the inverse scattering problem as a problem to find the coefficients $c(\zeta)$ and $\sigma(\zeta)$ of (8) and (9) based on the scattering data:

$$u(0,\tau,p) = f(\tau,p), \quad \tau \geq 0. \tag{12}$$

3. τ–p RADON TRANSFORM AND RELATIONS BETWEEN INVERSE PROBLEMS FOR LAYERED HALF-PLANES AND HETEROGENEOUS LINES

3.1. It is easy to see that if the functions $c(\zeta)$ and $\sigma(\zeta)$ are found on a segment $[0,S]$, then the problem to find $a(z)$ and $k(z)$ on the segment $[0,H]$, where $H = \int_0^S c(\alpha)d\alpha$, by the field of scattered waves from a nonstationary source is completely solved. Let us show that these data can be obtained by mathematical modeling with respect to the field of a stationary source.

For the original equation given in (1), consider the following initial-boundary problem with a point boundary source:

$$k(0)w_z(x,0,t) = -\delta(x-x_0)\delta(t-t_0), \quad -\infty < x,\ t < \infty,$$
$$w(x,z,t) \equiv 0, \quad -\infty < x < \infty, \quad z \geq 0, \quad t < t_0. \tag{13}$$

It is obvious that the solution w of that problem is invariant with respect to translations with respect to x and t, i.e.,

$$w(x,z,t) = W(x-x_0, z, t-t_0),$$

where $W(x,z,t)$ is the solution of problem (1)–(13) for $x_0 = 0$ and $t_0 = 0$, i.e., $W(x,z,t)$ is the fundamental solution of the boundary-value problem.

For $n = 1,2,\ldots$, for Eq. (1), pose the auxiliary problem determined by the following initial-boundary conditions:

$$k(0)w_z(x,0,t) = -\delta(x-x_0)\varphi_n(t-t_0), \quad -\infty < x,\ t < \infty,$$
$$w(x,z,t) \equiv 0, \quad -\infty < x < \infty, \quad z \geq 0, \quad t < t_0,$$

where $\varphi_n \in C^1(-\infty,\infty)$ and $\varphi_n(\tau) \equiv 0$ provided that $\tau < 0$, while $\varphi_n(t-t_0)$ to $\delta(t-t_0)$ as $n \to \infty$. The solution w_n of this problem is invariant with respect to translations with respect to x and t as well, i.e., $w_n(x,z,t) = W_n(x-x_0, z, t-t_0)$.

Let $t_0 = px_0$. Then, taking into account the substitutions $x - x_0 = X$, $t - t_0 = T$, and $\tau = T - pX$, we obtain that

$$\int_{-\infty}^{\infty} \delta(x-x_0)\varphi_n(t-t_0)dx_0 = \int_{-\infty}^{\infty} \varphi_n(\tau + pX)\delta(X)dX = \varphi_n(\tau),$$

and

$$\int_{-\infty}^{\infty} W_n(x-x_0, z, t-t_0)dx_0 = \int_{-\infty}^{\infty} W_n(X, z, \tau + pX)dX = v_n(z,\tau,p),$$

which imply that $\varphi_n \to \delta$, $W_n \to W$, and $v_n \to \tilde{v}$ as $n \to \infty$ in the sense of generalized functions (i.e., weakly), where $\tilde{v}(z,\tau,p)$ is the $\tau - p$ Radon transform of the wavefield $W(X,z,T)$, i.e.,

$$\tilde{v}(z,\tau,p) = \int_{-\infty}^{\infty} W(X,z,\tau + pX)dX, \quad z \geq 0, \quad -\infty < \tau < \infty. \tag{14}$$

Remark 1. Solving the inverse problem numerically, we assume that the wavefield $W(X,0,T)$ is known only for the case where $X \in [-X_m, X_m]$ and $T \in [0, T_m]$. Thus, only τ and p such that $\tau \in [0,\tau_m]$, $p \in [0, p_m]$, and $\tau_m = T_m - p_m X_m$ are to be considered.

3.2. The following assertion is valid.

Theorem 1. The relation $\tilde{v} = v$ holds, where $\tilde{v}(z,\tau,p)$ is determined by the Radon transformation given by (14), while $v(z,\tau,p)$ is the solution of problem (3)–(5) and (7).

Proof. Let us prove that $\tilde{v}$ satisfies problem (3)–(5) and (7). For the derivatives contained in (1)–(2), we have the relations

$$\int_{-\infty}^{\infty} W_{tt}(x-x_0, z, t-t_0)dx_0 = \frac{\partial^2}{\partial\tau^2}\int_{-\infty}^{\infty} W(X,z,\tau + pX)dX = \tilde{v}_{\tau\tau},$$

$$\int_{-\infty}^{\infty} W_{zz}(x-x_0, z, t-t_0)dx_0 = \frac{\partial^2}{\partial z^2}\int_{-\infty}^{\infty} W(X,z,\tau + pX)dX = \tilde{v}_{zz},$$

$$\int_{-\infty}^{\infty} W_x(x-x_0, z, t-t_0)dx_0 = \int_{-\infty}^{\infty}\left(\frac{\partial}{\partial X} - p\frac{\partial}{\partial\tau}\right)W(X,z,\tau + pX)dX$$

$$= -p\int_{-\infty}^{\infty} W_\tau(X,z,\tau+pX)dX = -p\frac{\partial}{\partial\tau}\int_{-\infty}^{\infty} W(X,z,\tau+pX)dX = -p\tilde{v}_\tau,$$

and

$$\int_{-\infty}^{\infty} W_{xx}(x-x_0,z,t-t_0)dx_0 = p^2\frac{\partial^2}{\partial\tau^2}\int_{-\infty}^{\infty} W(X,z,\tau+pX)dX = p^2\tilde{v}_{\tau\tau},$$

which imply that $\tilde{v}(z,\tau,p)$ satisfies relations (3)–(4). The fact that the boundary-value and initial-value conditions posed by (5)–(6) are satisfied follows from the weak convergence properties considered above.

This completes the proof of the theorem.

Using the proved theorem, we can establish a one-to-one correspondence between the scattering problems for a nonstationary surface source and a stationary one. The latter problem is canonical for the land seismics; generally, it is treated as a two-dimensional problem. Reducing this problem to a one-dimensional one (by the τ–p Radon transform), we can construct an efficient method to solve the inverse scattering problem, treating it as a one-dimensional problem with a parameter.

4. ACOUSTIC IMPEDANCE AND VELOCITY IN A TWO-DIMENSIONAL LAYERED HOMOGENEOUS MEDIUM: UNIQUENESS

4.1. For further investigations of the inverse problem posed by (8)–(12), simplify it as follows. Introduce the following variable $Z = Z(\zeta, p)$ depending on the parameter p:

$$Z(\zeta,p) = \int_0^\zeta \sqrt{1-p^2c^2(\alpha)}d\alpha. \tag{15}$$

This reduces (8)–(11) to the form

$$U_{\tau\tau} = U_{ZZ}, \quad Z > 0, \quad Z \neq Z_j(p), \quad -\infty < \tau < \infty, \tag{16}$$

$$[\sigma(Z,p)U_Z(Z,\tau,p)]_{Z_j(p)} = [U(Z,\tau,p)]_{Z_j(p)} = 0, \quad -\infty < \tau < \infty, \tag{17}$$

$$\sigma(0,p)U_Z(0,\tau,p) = -\delta(\tau), \quad -\infty < \tau < \infty, \tag{18}$$

$$U(Z,\tau,p) \equiv 0, \quad Z \geq 0, \quad \tau < 0, \tag{19}$$

where $U(Z,\tau,p) = u(\zeta(Z,p),\tau,p)$, the function $\zeta(Z,p)$ is inverse to the function $Z(\zeta,p)$ provided that p is fixed, $Z_j(p) = Z(\zeta_j,p)$, and

$$\sigma(Z,p) = \sigma(\zeta(Z,p))\sqrt{1-p^2c^2(\zeta(Z,p))} \tag{20}$$

is another unknown piecewise-constant of the inverse problem, implicitly depending on $c(\zeta)$ and $\sigma(\zeta)$.

For the direct scattering problem posed by (16)–(19), the inverse one is posed as follows: to find $\sigma(Z,p)$ by the scattering data

$$U(0,\tau,p) = f(\tau,p), \quad \tau \geq 0. \tag{21}$$

Both problems have been studied well. From results of [29–34], it follows that $\sigma(Z,p)$ is uniquely determined by the additional information provided by (21). From the properties of the solution of the direct problem posed by (16)–(19), it follows that if the trace of the solution is well defined, then

$$U(0,+0,p) = f(0,p) = 1/\sigma(0,p).$$

This relation uniquely determines $c(0)$ and $\sigma(0)$. Then it uniquely determines $c(\zeta)$ and $\sigma(\zeta)$ for $\zeta \in [0,\zeta_1)$ provided that the function $f(\tau,p)$ is known for two different values of p.

However, if p is fixed, then the variable Z is determined by an unknown function $c(\zeta)$; therefore, the uniqueness of the recovering of $c(\zeta)$ and $\sigma(\zeta)$ is still an open question.

4.2. Further, we prove that $c(\zeta)$ and $\sigma(\zeta)$ are uniquely determined by the scattering data (21) provided by at least two values of the parameter p. To do this, consider the local posing of problem (16)–(19) and (21) with respect to τ.

Let the function $f(\tau, p)$ from (21) be given for $p = p_0,\ p_1,\ p_0 < p_1$, and, therefore, $\tau \in [0,\tau_0], [0,\tau_1]$, $\tau_0 > \tau_1$. Let there exist a solution $\sigma(Z, p)$ of problem (16)–(19) and (21) for p_0 and p_1 belonging to the corresponding segments. The following assertion is valid.

Theorem 2. Let the function $\sigma(Z, p)$ for $p = p_0,\ p_1$ and $Z \in [0,\tau_0], [0,\tau_1]$, respectively. Then $c(\zeta)$ and $\sigma(\zeta)$ are uniquely defined by the scattering data (21) for all $\zeta \in [0, S_2]$, where $S_2 = \min\{S_0, S_1\}$ and S_0 and S_1 are implicitly determined by the solution as follows:

$$\tau_0 = \int_0^{S_0} \sqrt{1 - p_0^2 c^2(\alpha)}\, d\alpha \quad \text{and} \quad \tau_1 = \int_0^{S_1} \sqrt{1 - p_1^2 c^2(\alpha)}\, d\alpha.$$

Proof. Since

$$Z(\zeta, p) = \int_0^{\zeta} \sqrt{1 - p^2 c^2(\alpha)}\, d\alpha,$$

it follows that a function $\zeta(Z, p)$ inverse to $Z(\zeta, p)$ exists for any fixed admissible p. Since $dZ(\zeta, p) = \sqrt{1 - p^2 c^2(\zeta)}\, d\zeta$, it follows that the following integral relation holds:

$$\int_0^{Z(\zeta,p_0)} \frac{d\alpha}{\sigma(\alpha, p_0)} = \int_0^{Z(\zeta,p_1)} \frac{d\alpha}{\sigma(\alpha, p_1)}. \tag{22}$$

By virtue of the monotonicity of integrals from (22) with respect to the upper limit, the last relation implies that a one-to-one correspondence between $Z(\zeta, p_1)$ and $Z(\zeta, p_0)$ takes place:

$$Z(\zeta, p_1) = F(Z(\zeta, p_0)),$$

where the function F is strictly monotonic, has an inverse function F^{-1}, and the inequality $F(Z) < Z$ holds.

Introduce the function $C(Z) = c(\zeta(Z, p_0))$ of an independent variable Z. The relation

$$\sigma(Z(\zeta, p), p) = \sigma(\zeta)\sqrt{1 - p^2 c^2(\zeta)},$$

which is valid for all admissible ζ and p implies that

$$\sigma(Z, p_0)\sqrt{1 - p_1^2 C^2(Z)} = \sigma(F(Z), p_1)\sqrt{1 - p_0^2 C^2(Z)}.$$

This and the known functions $\sigma(Z, p_0)$, $\sigma(Z, p_1)$, and $F(Z)$ uniquely determine the coefficient $C(Z)$ on the segment $[0, \tau_2]$, where $\tau_2 = \min\{\tau_0, F^{-1}(\tau_1)\}$, as follows:

$$C^2(Z) = \frac{\sigma^2(Z, p_0) - \sigma^2(F(Z), p_1)}{p_1^2\sigma^2(Z, p_0) - p_0^2\sigma^2(F(Z), p_1)}. \tag{23}$$

Now, recover the function $c(\zeta)$ by $C(Z)$ from (23). Since $c(\zeta) = C(Z(\zeta,\ p_0))$, it follows that $dZ = \sqrt{1 - p_0^2 C^2(Z)}\, d\zeta$, which implies that

$$\zeta = \zeta(Z, p_0) = \int_0^{Z} \frac{d\alpha}{\sqrt{1 - p_0^2 C^2(\alpha)}} \equiv \Phi(Z). \tag{24}$$

Finally, expressing Z via the function Φ^{-1} inverse to Φ, we obtain that

$$c(\zeta) = C(\Phi^{-1}(\zeta)), \quad 0 \le \zeta \le S_2.$$

The function $c(\zeta)$ defined in this way is unique. Assume that it is not true and there exist two different solutions $c(\zeta) \ne c_*(\zeta)$ of the inverse problem. Then the following functions $Z(\zeta, p_0) \ne Z_*(\zeta, p_0)$ are defined:

$$Z(\zeta, p_0) = \int_0^{\zeta} \sqrt{1 - p_0^2 c^2(\alpha)}\, d\alpha \quad \text{and} \quad Z_*(\zeta, p_0) = \int_0^{\zeta} \sqrt{1 - p_0^2 c_*^2(\alpha)}\, d\alpha.$$

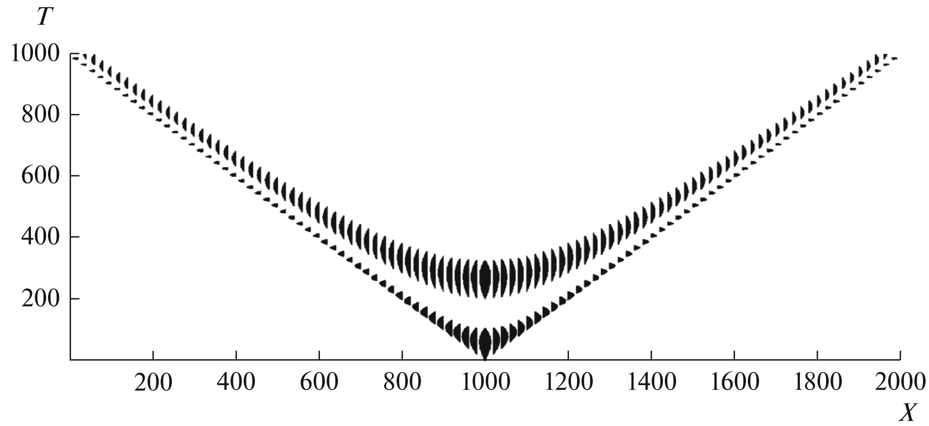

Fig. 1. Full field $W(X,z,T)$ of acoustic waves registered for $z = 0$. It corresponds to data of Fig. 2 and model in Fig. 3 and is represented in discrete form.

Obviously, their inverse functions are also different from each other: $\zeta(Z, p_0) \neq \zeta_*(Z, p_0)$. However, this is impossible because, by virtue of (23), which is valid for any $C(Z)$, relation (24) implies the relation $\zeta(Z, p_0) = \zeta_*(Z, p_0)$.

The unknown function $\sigma(\zeta)$ is uniquely determined on the segment $[0, S_2]$ by the following relation deduced from the previous consideration:

$$\sigma(\zeta) = \sigma(\Phi^{-1}(\zeta), p_0)/\sqrt{1 - p_0^2 c^2(\zeta)}.$$

This completes the proof of the theorem.

Remark 2. The constructed function $c(\zeta)$ allows us to obtain a solution of the original inverse problem in terms of the true vertical depth z. Indeed, taking into account that $dz = c(\zeta)d\zeta$, we consecutively find

$$z(\zeta) = \int_0^{\zeta} c(\alpha)d\alpha \equiv \Psi(\zeta), \quad a(z) = c(\Psi^{-1}(z)), \quad \text{and} \quad \varkappa(z) = \sigma(\Psi^{-1}(z)),$$

where the function Ψ^{-1} is inverse to Ψ.

5. INVERSE SCATTERING PROBLEMS IN A LAYERED-HOMOGENEOUS TWO-DIMENSIONAL MEDIUM: NUMERICAL SOLUTIONS

Following the geophysical interpretation of the data provided below, we move to the results of numerical simulations. In land seismics, the measuring equipment is rather densely located as a headland of geophones or an optical-fiber cable along a profile. This allows us to use the time records of the seismic traces to recover a wavefield on the original ground such that it is practically continuous and consists of a direct surface wave and a field of reflected waves.

Such a full wavefield is simulated in the present paper; in Fig. 1, it is presented in the discrete form (for better clearness): the wavefield $W(X,z,T)$ from a point source, registered for $X = X_n$, $n = 0, \pm 1, \ldots, \pm m$, and $z = 0$, is displayed after its partial deconvolution, i.e., the reduction of the actuating signal to a delta-shaped impulse by the solution of convolution-type equations. In seismics, such a wavefield is obtained as a result of the primary conversion of the oscillations of the medium along a profile of the original ground.

A homogeneous plane layer lying on a homogeneous underlying half-plane is taken as the model of the medium for the wavefield computation in Fig. 1. Recall that all vertical measurements are done in terms of the eikonal ζ; i.e., their dimension is time. For the selected model, the depth of the horizontal boundary of the discontinuity of the acoustic impedance is equal to 100.

Figure 2 displays the graphs of the solution $\sigma(Z, p)$ of the inverse problem (16)–(19) and (21) for $p = 0$ and $p = 0.6$. Following the proposed approach, we use the Radon transformation $f(\tau, p)$ of the wavefield $W(X, 0, T)$ for $p = 0$ and $p = 0.6$ as the input data. A random error of 10% is inserted in the solution; in

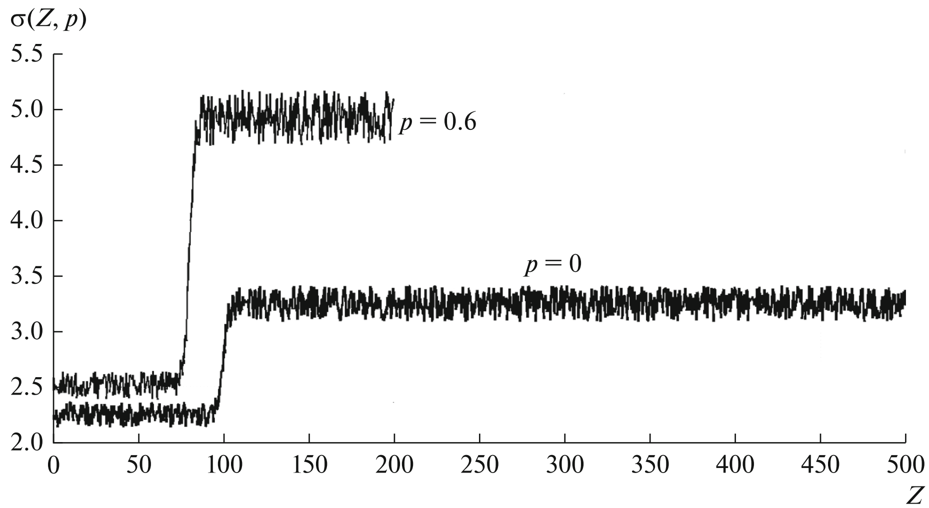

Fig. 2. Result of solving inverse scattering problem by data represented by Radon transformation $f(\tau, p)$ of wavefield in Fig. 1 for $p = 0$ and $p = 0.6$. Inserted error is 10%.

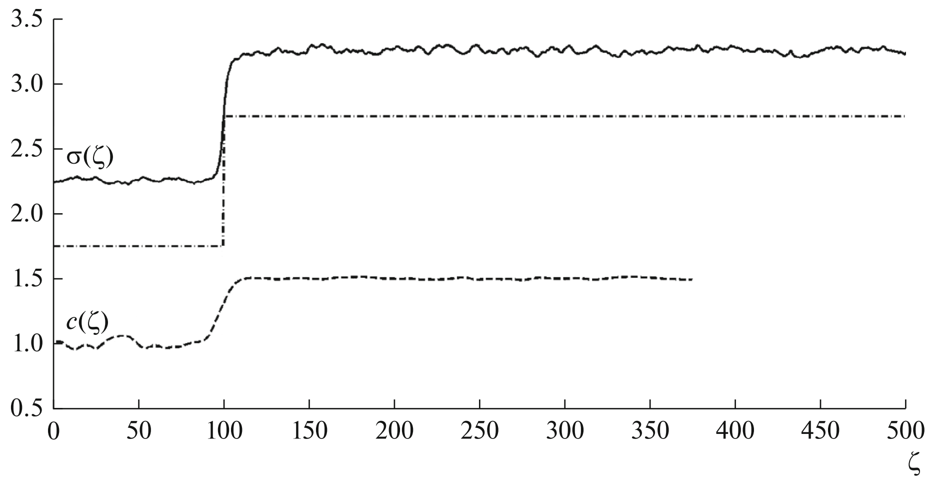

Fig. 3. Result of solving inverse problem for wavefield in Fig. 1 by data of Fig. 2: $c(\zeta)$ is dashed curve and $\sigma(\zeta)$ is dense curve. Original model of medium is $c(\zeta) = 1 + 0.5\,\mathrm{sgn}(\zeta - 100)$, where $\sigma(\zeta)$ is dashed-dotted curve shifted lower by 0.5.

practice, this corresponds to a high-level error of the oblique summing of real wave fields with respect to the variable X.

In Fig. 3, the results of solving an inverse problem are represented in the form of time sections. If the velocity section $c(\zeta)$ is known, then it is easy to pass to the depth sections, i.e., to the variable z. The dashed curve corresponds to the obtained values of $c(\zeta)$. The dense curve corresponds to the acoustic impedance $\sigma(\zeta)$. The original model of the medium for $\sigma(\zeta)$ is shown by the dashed-dotted curve translated lower by 0.5, while $c(\zeta) = 1 + \mathrm{sgn}(\varsigma - 100)$ corresponds to the original velocity model.

Since the data for solving the inverse problem contains a significant error, we use the variational form of the regularization method with a stabilizer in the space W_2^1 (see [35, 36]) to smooth the obtained results. The regularization parameter is selected a priori and depends on the error level. Resolving such a problem at the stage of processing the preliminary data is the standard procedure contained in standard data processing packages.

The simulation is performed under the assumption of a full deconvolution; i.e., we assume that the source is a delta function. A specific property of the obtained results is the recovery of the velocity $c(\zeta)$ for the lower values of the eikonal ζ compared with the acoustic impedance $\sigma(\zeta)$. This is an important corol-

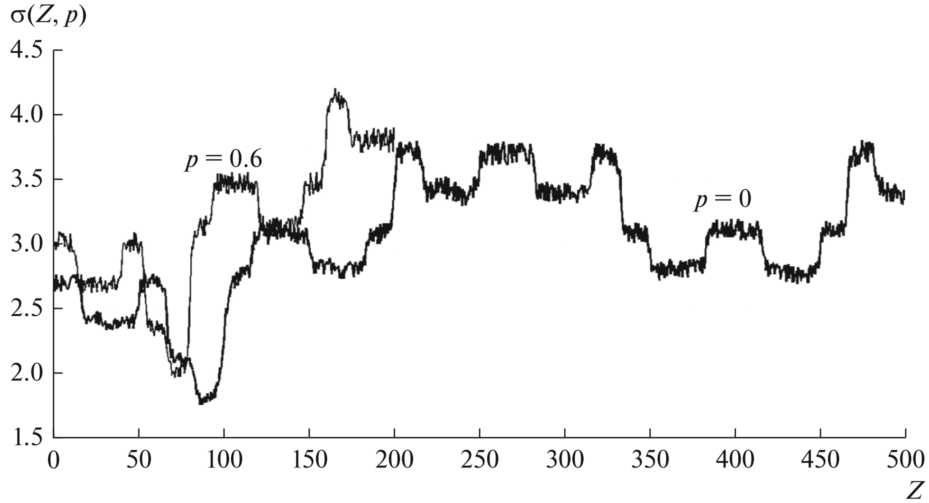

Fig. 4. Result of solving inverse problem by scattering data, presented by Radon transformation $f(\tau, p)$ for model in Fig. 5 for $p = 0$ and $p = 0.6$. Inserted error is 5%.

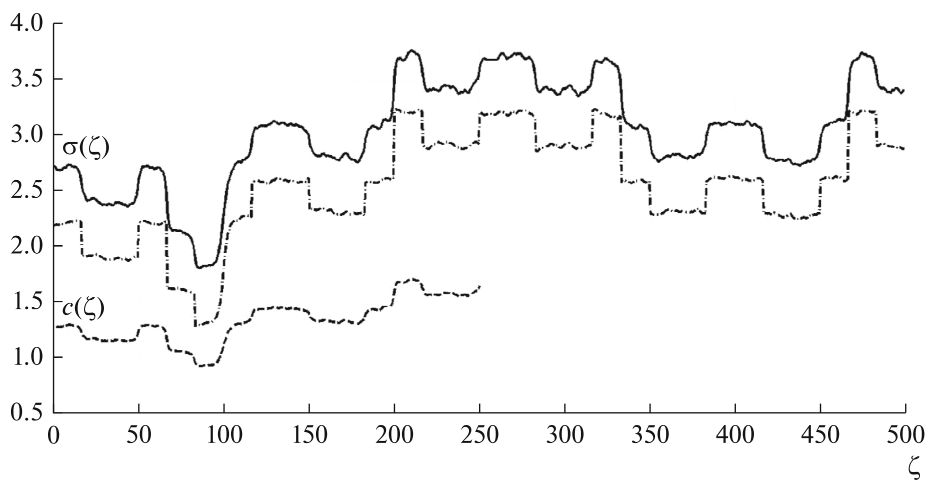

Fig. 5. Result of solving inverse scattering problem for multilayered medium: $c(\zeta)$ is dashed curve and $\sigma(\zeta)$ is dense curve. Original model is $c(\zeta) = 0.2 + 0.4\sigma(\zeta)$, where $\sigma(\zeta)$ is dashed-dotted curve shifted lower by 0.5.

lary of the fact that the acoustic impedance is recovered under vertical sounding even in the case where the velocity is not known, while the velocity is recovered only under the oblique sounding provided that the acoustic impedance is known. Note that the angle of departure of the ray with respect to the vertical of about 37° corresponds to the value $p = 0.6$ of the parameter. If p is less than 0.25–0.3, i.e., if the angle of departure of the ray is less than 14°–17°, then the velocity $c(\zeta)$ is recovered unstably.

It is important that if the inverse problem is considered in geometric optics, then the wavefield (see Fig. 1) determines only the width of the layer and the velocity above the conditional interface (see [37]), i.e., for $\zeta < 100$ in the considered case. If we apply the dynamical approach to solve the problem by the radon transform, then this parameter can be recovered significantly below the interface; i.e., the velocity is up to the eikonal value $\zeta = 373$ and the rigidity is up to the value $\zeta = 500$; actually, this corresponds to the oblique ($p = 0.6$) and vertical ($p = 0$) sounding of a layered medium by a plane incident wave.

Figures 4 and 5 provide the results of solving the inverse scattering problem for the model of a multi-layered medium typical for seismics with variations of the acoustic impedance and an error level of about 5% for the input data. The visible smoothness of the output curves is the result of the regularization related to the significant error level.

Concluding the review of the results presented above, we note that the resolving power of the proposed algorithm is sufficient in the practice of geophysical investigations and this algorithm is stable and can be used to solve scattering problems for a broad range of models of layered media.

ACKNOWLEDGMENTS

Sections 1–3 were completed at the faculty of Computational Mathematics and Cybernetics of Moscow State University, state reg. no. AAAA-A16-116021510092-2. The work in Sections 4 and 5 was supported by Russian Foundation for Basic Research, project no. 17-01-00525.

The author expresses thanks A.A. Tabakov for discussing the results of the present paper and for his longstanding and useful collaboration.

REFERENCES

1. A. N. Tikhonov, "On uniqueness of solution of the electroprospecting problem," Dokl. Akad. Nauk SSSR **69**, 797–800 (1949).
2. A. N. Tikhonov, "Mathematical basis of the theory of electromagnetic soundings, USSR Comput. Math. Math. Phys. **5**, 207–211 (1965).
3. A. N. Kolmogorov, "Sur l'interpolation et l'extrapolation des suites stationnaires," C. R. Acad. Sci. Paris **208**, 2043–2045 (1939).
4. J. F. Claerbout and A. G. Jonson, "Extrapolation of time dependent waveforms along their path of propagation," Geophys. J. R. Astron. Soc. **26**, 285–295 (1971).
5. P. S. Schultz, "Velocity estimation and downward continuation by wave front synthesis," Geophysics **43**, 691–714 (1978).
6. J. F. Claerbout, *Fundamentals of Geophysical Data Processing* (McGraw-Hill, New York, 1976).
7. G. A. McMechan, "Migration by extrapolation of time-dependent boundary value," Geophys. Prosp. **31**, 413–420 (1983).
8. V. G. Romanov, "Determining the parameters of a stratified piecewise constant medium for the unknown shape of an impulse source," Sib. Math. J. **48**, 1074–1084 (2007).
9. V. G. Romanov, "On the problem of determining the parameters of a layered elastic medium and an impulse source," Sib. Math. J. **49**, 919–943 (2008).
10. V. G. Romanov, *Inverse Problems of Mathematical Physics* (VSP, Netherlands, 1984; Nauka, Moscow, 1984).
11. M. M. Lavrent'ev, K. G. Reznitskaya, and V. G. Yakhno, *One-Dimensional Inverse Problems of Mathematical Physics* (Nauka, Novosibirsk, 1982; Am. Math. Soc, Providence, RI, 1986).
12. E. N. Bessonova, V. M. Fishman, V. Z. Ryaboy, and G. A. Sitnikova, "The tau method for inversion of travel times. 1. Deep seismic sounding data," Geophys. J. R. Astron. Soc. **36**, 377–398 (1974).
13. C. H. Chapman, "A new method for computing synthetic seismograms," Geophys. J. R. Astron. Soc. **54**, 481–518 (1978).
14. C. H. Chapman, "Generalized radon transform and slant stacks," Geophys. J. R. Astron. Soc. **66**, 445–453 (1981).
15. R. W. Clayton and G. A. McMechan, "Inversion of refraction data by wavefield continuation," Geophysics **46**, 860–868 (1981).
16. T. M. Brocher and R. A. Phinney, "Inversion of slant stacks using finite-length record sections," J. Geophys. Res. **86**, 7065–7072 (1981).
17. G. A. McMechan, "p-x imaging by localized slant stacks of T-x data," Geophys. J. R. Astron. Soc. **72**, 213–221 (1983).
18. B. Milkereit, W. D. Mooney, and W. M. Kohler, "Inversion of seismic refraction data in planar dipping structure," Geophys. J. Int. **82**, 81–103 (1985).
19. M. E. Kupps, A. J. Harding, and J. A. Orcutt, "A comparison of tau-p transform methods," Geophysics **55**, 1202–1215 (1990).
20. B. Zhou and S. A. Greenhalgh, "Linear and parabolic τ-p transforms revisited," Geophysics **59**, 1133–1149 (1994).
21. A. V. Baev, "On t-local solvability of inverse scattering problems in two dimensional layered media," Comput. Math. Math. Phys. **55**, 1033–1050 (2015).
22. A. V. Baev, "Imaging of layered media in inverse scattering problems for an acoustic wave equation," Math. Models Comput. Simul. **8**, 689–702 (2016).
23. A. S. Alekseev and A. G. Megrabov, "Direct and inverse problems of scattering of plane waves in heterogeneous layers," Mat. Probl. Geofiz., No. 3, 8–36 (1972).

24. A. G. Megrabov, "A method for recovering density and velocity in an inhomogeneous layer as a function of depth from a family of plane waves reflected from a layer at different angles," Mat. Probl. Geofiz., No. 5, 78–107 (1974).
25. W. Nowacki, *Teoria Sprezystosci* (PWN, Warsawa, 1970).
26. A. N. Tikhonov and A. A. Samarskii, *Equations of Mathematical Physics* (Dover, New York, 2013; Nauka, Moscow, 1972).
27. A. V. Baev and N. V. Kutsenko, "Solving the inverse generalized problem of vertical seismic profiling," Comput. Math. Model. **15**, 1–18 (2004).
28. A. V. Baev and N. V. Kutsenko, "Identification of a dissipation coefficient by a variational method," Comput. Math. Math. Phys. **46**, 1796–1807 (2006).
29. A. S. Blagoveshchenskii, "The local method of solution of the nonstationary inverse problem for an inhomogeneous string," Tr. MIAN im. V. A. Steklova **115**, 28–38 (1971).
30. A. V. Baev, "A method of solving the inverse scattering problem for the wave equation," Comput. Math. Math. Phys. **28** (1), 15–21 (1988).
31. A. V. Baev, "On the solution of the inverse problem of scattering of a plane wave by a layered-inhomogeneous mediumi," Sov. Phys. Dokl. **32**, 26–28 (1988).
32. A. V. Baev, "On the solution of the inverse boundary value problem for the wave equation with a discontinuous coefficient," Comput. Math. Math. Phys. **28** (6), 11–21 (1988).
33. A. V. Baev, "On local solvability of inverse scattering problems for the Klein-Gordon equation and the Dirac system," Math. Notes **96**, 286–289 (2014).
34. A. V. Baev, "Solution of an inverse scattering problem for the acoustic wave equation in three-dimensional media," Comput. Math. Math. Phys. **56** (12), 15–21 (2016).
35. A. N. Tikhonov, "Solution of incorrectly formulated problems and regularization methods," Sov. Math. Dokl. **4**, 1035–1038 (1963).
36. A. N. Tikhonov and V. Ya. Arsenin, *Solutions of Ill-Posed Problems* (Wiley, New York, 1977; Nauka, Moscow, 1986).
37. A. A. Kaufman, A. L. Levshin, and K. L. Larner, *Acoustic and Elastic Wave Fields in Geophysics II* (Elsevier, New York, 2002; Nedra, Moscow, 2003).

Translated by A. Muravnik