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Application of the Richardson Method in the Case of an Unknown Lower Bound of the Problem Spectrum

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Abstract—An algorithm is presented, which enables us to use the iterative Richardson method for solving a system of linear algebraic equations with the matrix corresponding to a sign-definite self-adjoint operator, in the absence of information about the lower boundary of the spectrum of the problem. The algorithm is based on the simultaneous operation of two competing processes, the effectiveness of which is constantly analyzed. The elements of linear algebra concerning the spectral estimates, which are necessary to understand the details of the Richardson method with the Chebyshev set of parameters, are presented. The method is explained on the example of a one-dimensional equation of the elliptic type.

Keywords: system of linear algebraic equations, matrix inversion, iterative methods, Richardson method

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1. INTRODUCTION

During the numerical solution of systems of linear algebraic equations (SLAEs), problems related to the absence of information on the boundaries of the spectrum of a difference operator, which can significantly decrease the effectiveness of the algorithm, are often encountered. Various approaches to solving this problem are possible. In particular, if the aim of the problem is to solve the corresponding elliptic equation, then we can use the fact that in discretizing the differential operator, the spectrum of the obtained matrix operator in SLAEs corresponds to the spectrum of the differential operator only in the lower part. Thus, an effective approach is to suppress the error arising in the lower boundary of the spectrum and restrict the error in its upper part by a milder nonincreasing condition. Local iteration (LI) schemes using Chebyshev parameters are based on this principle [1].

There exist alternative methods based on estimating the spectral radius boundaries for the matrix, for example, methods based on the generalized Perron complement and properties of matrix norms [2]. Also methods with preconditioners are effective [3, 4]. However, finding an easily invertible matrix close to the initial one is in most cases a nontrivial problem.

Note that the methods of the conjugate gradient type, not requiring a priori information on the spectrum boundary, can turn out to be ineffective. This happens when solving nonlinear applied problems of mathematical physics with coefficients possessing jumps on many orders and when using very large grids with billions of cells, etc. As a result, methods of this class show slow convergence or even a loss of exactness, i.e., computational instability. Sometimes empirical tricks can correct the situation, starting with the simplest ones, for instance, an approximation to a solution which has started to diverge is taken as the initial approximation in a new iteration cycle. In this paper we consider the problem of solving SLAEs under the assumption of an equivalent matrix spectrum, without distinguishing any of its part parts, and we apply the Richardson method, which is one of the well-known explicit two-layer iteration methods for solving SLAEs with a sign-definite self-adjoint operator. An advantage of the method is the simplicity of

its realization and the possibility of effective parallelizing. Applying the Chebyshev set of iterative parameters substantially increases the convergence speed of the method.

Before presenting the substance of our approach, let us note that the proposed algorithm of competing processes is constructed similarly to the three-layer semi-iterative Chebyshev method [5, 6]. However, the realization of this method from [6], when compared with our method, is related to increasing the volumes of machine memory and computations per iteration.

Our method is tested on the model problem for a one-dimensional elliptic equation. In this case the low boundary of the matrix spectrum is known exactly. This enables us to see how well the algorithm approximates the value in terms of the convergence of the iterative process.

2. ITERATIVE METHOD OF SOLVING A SYSTEM OF LINEAR ALGEBRAIC EQUATIONS

It is required to solve an algebraic system of equations

$$A\mathbf{y} = \mathbf{f}, \tag{1}$$

where A is a self-adjoint positive definite matrix, $\mathbf{y}$ is the vector of unknowns, $\mathbf{f}$ is the right-hand side. We consider a two-layer iteration method

$$B\frac{y^{k+1} - y^k}{\tau_k} + Ay^k = \mathbf{f}, \quad k = 1,\ldots,s, \tag{2}$$

where s is the number of iterations with a preconditioner, B is a self-adjoint positive definite operator, τ_k are certain factors, and the chosen operator B is also easily invertible. In (2), y^k is the unknowns vector computed at the kth iteration.

Multiplying (2) by $\tau_k B^{-1}$ on the left and introducing the variables $\delta y^k \equiv y^k - y$ and $x^k = B^{1/2}\delta y^k$, we obtain

$$x^{k+1} = (I - \tau_k C)x^k, \tag{3}$$

where I is the unit matrix. The operators A, B, and $C = B^{-1/2}AB^{-1/2}$ are self-adjoint positive definite. For $C = C^* > 0$, there exists a complete set of orthogonal eigenvectors forming a basis and having positive eigenvalues.

We assume that the spectrum of the matrix C is unknown, but it lies in a certain range

$$0 < \gamma_1 \le \lambda_n \le \gamma_2, \tag{4}$$

where $\gamma_1 \le \min_n \lambda_n$ and $\gamma_2 \ge \max_n \lambda_n$. Obviously, the condition $\gamma_1 B \le A \le \gamma_2 B$ implies $\gamma_1 I \le C \le \gamma_2 I$.

Taking into account the variability of the parameter τ_k, we have [5]

$$x^s = P_s(C)x^0, \quad P_s(C) = \prod_{k=1}^{s}(I - \tau_k C).$$

An estimate for norms yields

$$\|x^s\| \le \|P_s(C)\| \cdot \|x^0\|.$$

The norm of the resolving operator polynomial $P_s(C)$ with a self-adjoint positive definite operator argument $C = C^*$, $\gamma_1 I \le C \le \gamma_2 I$, is estimated through the usual polynomial $P_s(t)$ as follows [7]:

$$\|P_s(C)\| \le \max_{t\in[\gamma_1,\gamma_2]} |P_s(t)|, \quad P_s(t) = \prod_{k=1}^{s}(1 - \tau_k t).$$

The parameters $\tau_1,\ldots,\tau_s$ are found from the minimum condition for $\|P_s(C)\|$, i.e., $\min_{\tau_j}\{\max_{t\in[\gamma_1,\gamma_2]}|P_s(t)|\}$.

The solution of the minimization problem of the norm $\|P_s(C)\|$ is the Chebyshev polynomial normalized at the point $x_0 = -1/\rho_0$ [8]:

$$T_s(x)/T_s(x_0), \quad |x| < 1.$$

The maximum of the deviation from zero is

$$\frac{1}{|T_s(x_0)|} = \frac{2\rho_1^s}{1+\rho_1^{2s}} = q_s, \quad \rho_0 = \frac{1-\xi}{1+\xi}, \quad \xi = \frac{\gamma_1}{\gamma_2}, \quad \rho_1 = \frac{1-\sqrt{\xi}}{1+\sqrt{\xi}}. \tag{5}$$

The coincidence of the required polynomial with the extremal Chebyshev one (at the roots $x = x_k$)

$$P_s(t) = \prod_{k=1}^{s}[(1 - 0.5\tau_k(\gamma_2 + \gamma_1)) - 0.5\tau_k(\gamma_2 - \gamma_1)x] = T_s(x)/T_s(x_0) \tag{6}$$

yields the Chebyshev parameters for the Richardson method (2)

$$\tau_k = \frac{\tau_0}{1+\rho_0 x_k}, \quad x_k = \cos\left(\frac{2k-1}{2s}\pi\right), \quad k = 1,2,\ldots,s, \quad \tau_0 = \frac{2}{\gamma_1+\gamma_2}. \tag{7}$$

In the left-hand side of (6) the argument is t, and in the right hand side, it is x, which is admissible, since they are connected by the linear relation

$$t = 0.5[(\gamma_2 - \gamma_1)x + (\gamma_2 + \gamma_1)].$$

The computational stability of the scheme is achieved by a special ordering of the set of these parameters [5, 6, 9–11]. For the exposition below, it is important that the blocks of the optimal resulting decrease of the iterative error (with preincreases) are ordered in powers of four, eight, etc., iterations.

Since the lower bound γ_1 for the spectrum can be unknown, let us examine separately the case when it is increased. In other words, the variable t varies from γ_1 to 0 and the variable x decreases from -1 to $-1/\rho_0$, without reaching the end values of the segment. The ratio $T_s(x)/T_s(x_0)$ increases from $2\rho_1^s/(1+\rho_1^{2s})$ to 1 strictly monotonically. Since for $x \in (-1/\rho_0, -1)$, we have [8]

$$[T_s(x)]'_x = \frac{s}{2\sqrt{x^2-1}}\left[\left(x+\sqrt{x^2-1}\right)^s - \left(x-\sqrt{x^2-1}\right)^s\right],$$

and the assumption $[T_s(x)]'_x = 0$ (the necessary condition for the extremum) takes us out of the interval $x \in (-1/\rho_0, -1)$. Thus, the transition operator of the Richardson method also acts as a compressing one for $0 < t < \gamma_1$. Hence, we have proved the following theorem:

Theorem 1. *For an increased estimate of the lower boundary of the spectrum* $\gamma_1 > \min\lambda_n > 0$, *the iteration process* (2) *converges.*

3. FINDING A GRID SOLUTION FOR THE EXAMPLE OF A ONE-DIMENSIONAL EQUATION OF THE ELLIPTIC TYPE

For the subsequent analysis, let us use an elliptic type equation of the form

$$-\partial^2 y/\partial x^2 = f(x), \quad y(a) = y_a, \quad y(b) = y_b, \tag{8}$$

where $x \in [a,b]$ is the space coordinate, $y(x)$ is the unknown quantity, $f(x)$ is the right-hand side, and y_a and y_b are the known values of the unknown quantity at the boundary of the definition domain.

Let us define a uniform difference net over the space with the step $h = (b-a)/N$, where N is the number of cells. Approximating (8) in the standard way, we obtain a system of algebraic equations of form (1) with the positive definite matrix A. For solving the system, let us use the iterative process (2), where for simplicity, we set $B = I$:

$$\mathbf{y}^{k+1} = \mathbf{y}^k + \tau_k(\mathbf{f} - A\mathbf{y}^k). \tag{9}$$

As the zero iteration, we can take any vector

$$\mathbf{y}^0 = (y_1,\ldots,y_N)^T,$$

for example, the zero vector. Applying the iterative process (9) gives a consecutive set of vectors $\mathbf{y}^k$. The iteration process stops if, for example, the condition $\|\mathbf{y}^k - \mathbf{y}^{k-1}\|_C \equiv \max_i |y_i^k - y_i^{k-1}| < \varepsilon$ holds, where ε is the given precision of the solution and k is the number of iterations.

Let us obtain a numerical solution of Eq. (8), using the proposed iterative procedure, on the segment $x \in [0,1]$ with the boundary conditions $y(0) = 0$, $y(1) = 1$ and with the zero source $f(x) = 0$. For the parameters τ_k, $k = 1,\ldots,s$, let us use the Chebyshev set. Note that the number of iterations in our argument should be a power of two:

$$s = 2^n, \quad n = 3,4,5,\ldots. \tag{10}$$

For the stability of the algorithm, we must order the Chebyshev set of parameters for the Richardson method from 1 to s [5, 6, 9]. Ordering an array of integers index(k)=1,2,…,s is made by the following sorting procedure:

```
subroutine RSort(index, s)
integer, intent(in):: s
integer, intent(inout):: index(s)
integer, allocatable:: prev(:)
integer log2_s, pow, k
log2_s = INT(DLOG(DBLE(s))/DLOG(2.d0))
allocate(prev(s))
prev(1) = 1
do pow = 1, log2_s
do k = 1, 2**(pow – 1)
index(2*k –1) = prev(k)
index(2*k) = 2**(pow + 1) – index(2 * k – 1)
enddo
prev = index
enddo
deallocate(prev)
end subroutine RSort
```

Then we compute $x_{k,s}$, where instead of formula (7), we use

$$x_{k,s} = -\cos\left(\pi\frac{\text{index(k)}}{2s}\right).$$

For example, let us provide the set of ordered $x_{k,s}$ for $s = 8$.

$x_{1,8} = -0.9807852$	$x_{2,8} = 0.9807852$	$x_{3,8} = -0.1950903$	$x_{4,8} = 0.1950903$
$x_{5,8} = -0.8314696$	$x_{6,8} = 0.8314696$	$x_{7,8} = -0.5555702$	$x_{8,8} = 0.5555702$

These values are grouped in fours of the form

$$x_{1,s} = -\cos\mu, \quad x_{2,s} = \cos\mu, \quad x_{3,s} = -\sin\mu, \quad x_{4,s} = \sin\mu.$$

The value

$$\Delta_k = \|\mathbf{y}^k - \mathbf{y}^{k-1}\|_C \equiv \max_i |y_i^k - y_i^{k-1}|, \tag{11}$$

as a rule, decreases for each subsequent iteration inside the four, but it increases when passing from one group-of-four to another:

$$\Delta_4 < \Delta_3 < \Delta_2 < \Delta_1, \quad \Delta_5 > \Delta_4. \tag{12}$$

Table 1

k	Δ_k	k	Δ_k	k	Δ_k	k	Δ_k
1	95.8900	5	**24.6206**	9	**25.5387**	13	5.7477
2	47.8101	6	10.7291	10	12.4832	14	**2.6924**
3	39.4199	7	8.0491	11	12.2304	15	**2.8143**
4	11.8851	8	3.1236	12	3.0376	16	0.9011

The error for the first iteration at the beginning of each group-of-four, as a rule, is smaller than the error for the first iteration at the beginning of the previous group-of-four, which yields the convergence of the algorithm

$$\Delta_{13} < \Delta_9 < \Delta_5 < \Delta_1. \tag{13}$$

The behavior of the iterative error in solving Eq. (8) for $s = 16$ iterations is represented in Table 1. Note that in this example, we have $\Delta_9 > \Delta_5$ and $\Delta_{15} > \Delta_{14}$ (they are in bold), which contradicts (13) and (12), respectively. However, this does not influence the general convergence.

After the choice of the number of iterations s and the corresponding set $x_{k,s}$, $k = 1,\ldots,s$, the remaining necessary parameters are determined by formulas (7). The bounds for the spectrum of the Laplacian on a uniform grid are known [5]:

$$\gamma_1 = \frac{4}{h^2}\sin^2\frac{\pi h}{2l} > \frac{8}{l^2}, \quad \gamma_2 = \frac{4}{h^2}\cos^2\frac{\pi h}{2l} < \frac{4}{h^2}, \tag{14}$$

where $l = b - a$.

4. COMPETING PROCESSES ALGORITHM

The Gershgorin theorem on circles does not give a correct estimate for the lower bound for the spectrum of a strictly positive operator of a problem. At the same time, the Richardson iterative method requires the estimate $\gamma_1 > 0$. Hence, in this paper, taking into account the proven convergence of iterations in the Richardson method with an overestimated spectral boundary $\gamma_1 > 0$ (see Theorem 1), we propose a tuning algorithm at the a priori unknown positive γ_1 based on competing iterative processes with different γ_1 operating simultaneously:

$$\gamma_1^* = q\gamma_2, \quad \gamma_1^{**} = p\gamma_2,$$

where the upper bound γ_2 is determined from the Gershgorin Circle theorem and is given, for instance, by estimate (14); $q < 1$ and $p < 1$ are certain factors. We propose to choose $q = 1/10$ and $p = q/5$. For the case of the Laplace equation considered on segment [0,1] on a grid with $N = 100$ cells, this choice of parameters gives $\gamma_1^* = 4000$, which is much greater than the analytic estimate $\gamma_1 = 8$. This process will converge very rapidly at the initial iterations; however, the convergence speed in the considered norm will rapidly decrease due to the cutoff of the large lower part of the spectrum from the Richardson iterative optimization process. At some moment the competing process with $\gamma_1^{**} = 800$, which initially converged more slowly due to the optimization in the more representative scope of the whole width of the spectrum, will reach the process with γ_1^*, which (as proved at the end of Section 2) although it converges, does not do so optimally in the sense of the Richardson method in the lower part of the spectrum.

In Fig. 1a we represent relative errors (11) in the two competing processes γ_1^* and γ_1^{**} with the parameters $q = 1/10$ and $p = q/5$ for the number of iterations $s = 64$. We see that starting with the iteration $k = 27$, the process with the lower value of the lower boundary of the spectrum γ_1^{**} becomes more rapid.

However, for analysis it is convenient to compare the monotonic graphs of convergence presented in Fig. 1b. Since the error values are grouped in fours, in order to construct monotonic graphs, we put the initial points Δ_4^* and Δ_4^{**} for both processes γ_1^* and γ_1^{**}, and then we choose only the points at the end of each group-of-four, for which the conditions

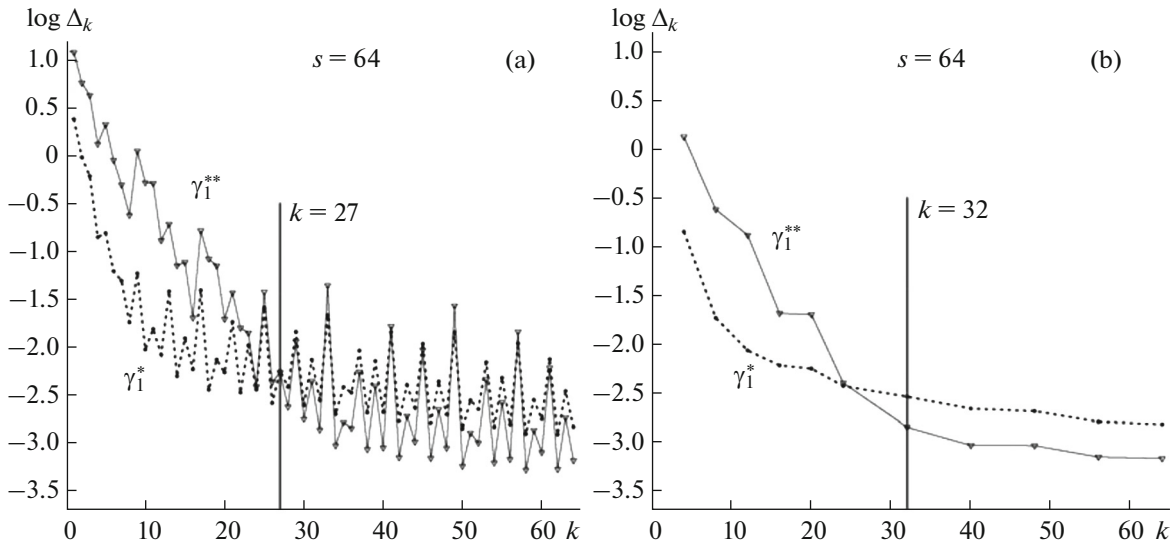

Fig. 1. Convergence of competing processes γ_1^* and γ_1^{**} with parameters $q = 1/10$ and $p = q/5$ for $s = 64$ iterations. (a): We show relative errors at each iteration; (b) we show monotonic graphs constructed from selected points.

$$\Delta_k^* < \Delta_{k-4}^*, \quad \Delta_k^{**} < \Delta_{k-4}^{**} \tag{15}$$

hold for both processes. If at the end of a group-of-four, (15) does not hold, then such a point is missed, and at the end of the next group-of-four, we check $\Delta_k^* < \Delta_{k-8}^*$ and $\Delta_k^{**} < \Delta_{k-8}^{**}$, etc. Comparing monotonic graphs (Fig. 1b) yields the critical iteration number $k = 32$, at which the process γ_1^{**} overtakes γ_1^*. If we assume that at the beginning the spectral boundary γ_1^* is more appropriate for the unknown actual boundary $\gamma_1 > 0$ (in the sense of the error estimated over all the spectral components), then further this value γ_1^* is thrown away as an unnecessary one. Its role starts to be played by the value γ_1^{**}. The algorithm of changing processes is realized in this way. However, if the positive value γ_1^* is already smaller than the true unknown value $\gamma_1 > 0$, then the Richardson γ_1^*-process of the error's spectral cutoff is more advantageous than the process with γ_1^{**} smaller than γ_1^*. In order to bar the competing processes to change in the last situation, it is necessary that, at the critical number of iterations described above (if it arises) the discontinuity of the monotonic graphs after their intersection would be on some strictly positive threshold quantity. Let us also allow the competing iterative processes to change only after the end of the s-cycle of Richardson iterations.

Taking the points made above into account, the algorithm for tuning the competing processes on an unknown positive lower bound of the operator spectrum consists of checking the following conditions:

(1) If at all the points of the s-cycle, over which we construct monotonic graphs (Fig. 1b) in the spectral-integral norm, the condition $\Delta_k^{**} > \Delta_k^*$ holds, i.e., the graphs do not intersect, then the processes restart with the same values γ_1^* and γ_1^{**}, but with twice as many s iterations (see (10), where we put $n \to n+1$). Each time when restarting the processes after an s-cycle, as the initial approximation, we take the solution obtained at the previous step in the maximally exact process γ_1^* or γ_1^{**}.

(2) If at some point k of an s-cycle the graphs intersect, i.e.,

$$\Delta_k^{**} < \Delta_k^* \tag{16}$$

(see the point $k = 32$ in Fig. 1b), then from this moment, we start checking the proximity of the errors given by both processes with respect to the relative criterion

$$\Delta_k^* - \Delta_k^{**} > \varepsilon_\Delta \Delta_k^*, \tag{17}$$

where $0 \le \varepsilon_\Delta < 1$. Criterion (17) means that the graphs differed significantly in the value determined by the parameter ε_Δ.

(3) If (17) does not hold for all points of the graph (Fig. 1b) corresponding to the s-cycle, then the processes restart with $s \to 2s$ iterations. This time the value of γ_1^* is preserved, and γ_1^{**} declines by a factor of four ($\gamma_1^{**} \to \gamma_1^{**}/4$). The choice of these coefficients is explained in the Appendix.

(4) If (17) holds at some point of the s-cycle, then from this moment, we stop the process γ_1^*, and we continue only the process γ_1^{**} until the end of the s-cycle. After this both processes are restarted with the $s \to 2s$ iterations. This time the value γ_1^* is made equal to the value γ_1^{**} from the previous step, and γ_1^{**}, as in p. 3, decreases by a factor of four ($\gamma_1^{**} \to \gamma_1^{**}/4$).

If in (17) we put $\varepsilon_\Delta = 0$, then p. 3 of the algorithm will be ignored. While the algorithm is running, at each iteration, we check the absolute (applied) convergence ε-criterion of the problem. If in some process, the inequality $\Delta_k^* < \varepsilon$ (or $\Delta_k^{**} < \varepsilon$) holds, where ε is the given precision of the solution, then the problem is considered solved.

5. A COMPUTATION EXAMPLE BY THE ALGORITHM OF COMPETING PROCESSES

Below we present the result of the work done by the algorithm. It consists of six iterative steps. Three of them are illustrated in Fig. 2, where we show monotonic graphs of the errors in the spectral-integral norm. The computation parameters are: $\varepsilon = 1\times10^{-12}$, $\varepsilon_\Delta = 0.3$, and the initial number of iterations $s = 16$. The initial values of the parameters determining the assumed lower boundary of the matrix spectrum are $q = 1/5$ and $p = q/5$.

(1) The number of iterations is $s = 16$, Fig. 2a. For the initial values of q and p, we have computed $\gamma_1^* = 8000$ and $\gamma_1^{**} = 1600$. The graphs intersected (condition (16) holds) but the relative criterion (17) did not hold. For the next step, the maximal number of iterations increased to $s = 32$, and the quantity γ_1^{**} decreased by a factor of four according to p. 3 of the algorithm.

(2) The number of iterations is $s = 32$, Fig. 2b. We use $q = 0.2$ and $p = 0.01$, corresponding to $\gamma_1^* = 8000$ and $\gamma_1^{**} = 400$. The graphs intersected at the iteration $k = 8$. Criterion (17) holds for $k = 16$: $\Delta_{16}^* - \Delta_{16}^{**} = 1.183601318917066\times10^{-3}$, $\varepsilon_\Delta\Delta_{16}^* = 7.642181884604249\times10^{-4}$. We pass to p. 4 of the algorithm. The process γ_1^* is stopped. Since the solution with the given precision is not found, the maximal number of iterations for the next step is increased to $s = 64$.

(3) The number of iterations is $s = 64$. According to p. 4 of the algorithm, we use the values $q = 0.01$ and $p = 2.5\times10^{-3}$, and the corresponding numbers $\gamma_1^* = 400$ and $\gamma_1^{**} = 100$. Condition (16) holds for the very first point for which we check it, i.e., at the iteration $k = 4$. In this case the graph of γ_1^* is everywhere higher than γ_1^{**}. Criterion (17) holds for $k = 24$: $\Delta_{24}^* - \Delta_{24}^{**} = 1.510476177942477\times10^{-4}$, $\varepsilon_\Delta\Delta_{24}^* = 1.331132953752201\times10^{-4}$. We pass to p. 4 of the algorithm. The process γ_1^* is stopped. Since the solution with the given precision is not found, the maximal number of iterations for the next step is increased to $s = 128$.

(4) The number of iterations $s = 128$. According to p. 4 of the algorithm, we use the values $q = 2.5\times10^{-3}$ and $p = 6.25\times10^{-4}$, and the corresponding numbers $\gamma_1^* = 100$ and $\gamma_1^{**} = 25$. Condition (16) holds at the iteration $k = 4$. Criterion (17) holds for $k = 40$: $\Delta_{40}^* - \Delta_{40}^{**} = 3.696396509766942\times10^{-5}$, $\varepsilon_\Delta\Delta_{40}^* = 3.182103036942774\times10^{-5}$. We pass to p. 4 of the algorithm. The process γ_1^* is stopped. Since the solution with the given precision is not found, the maximal number of iterations for the next step is increased to $s = 256$.

(5) The number of iterations $s = 256$. According to p. 4 of the algorithm, we use the values $q = 6.25\times10^{-4}$ and $p = 1.5625\times10^{-4}$, and the corresponding numbers $\gamma_1^* = 25$ and $\gamma_1^{**} = 6.25$. Condi-

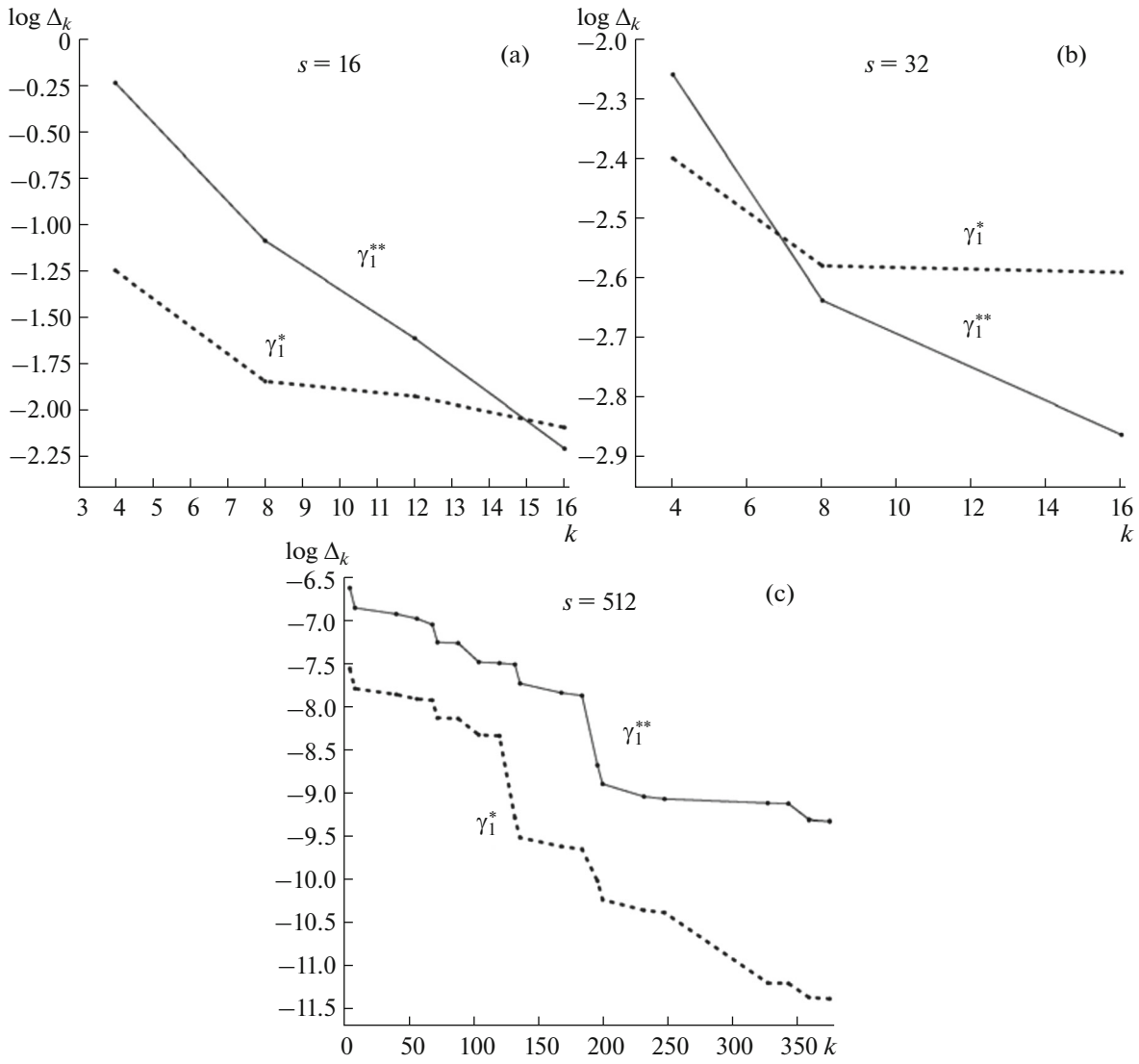

Fig. 2. Example of computing by competing processes algorithm up to moment when solution with given precision is obtained. We show monotonic graphs of errors in spectral-integral norm in three out of six iteration steps.

tion (16) holds at the iteration $k = 4$. Criterion (17) holds for $k = 72$: $\Delta_{72}^{*} - \Delta_{72}^{**} = 3.546534794174327 \times 10^{-6}$, $\varepsilon_{\Delta}\Delta_{72}^{*} = 2.953019525409051 \times 10^{-6}$. We pass to p. 4 of the algorithm. The process γ_1^{*} is stopped. Since the solution with the given precision is not found, the maximal number of iterations for the next step is increased to $s = 512$.

(**6**) The number of iterations $s = 512$, Fig. 2c. According to p. 4 of the algorithm, we use the values $q = 1.5625 \times 10^{-4}$ and $p = 3.90625 \times 10^{-5}$, and the corresponding numbers $\gamma_1^{*} = 6.25$ and $\gamma_1^{**} = 1.5625$. The solution is found for $k = 390$ in the process γ_1^{*}: $\Delta_{390}^{*} = 7.692180226115397 \times 10^{-13}$.

The total number of iterations made in process γ_1^{*} is 558; and in process γ_1^{**}, 886. Let us emphasize that from the moment of the intersection of graphs at the step $s = 32$, Fig. 2b, at the subsequent steps, after γ_1^{*} and γ_1^{**} approach the actual lower spectral boundary (see (14)), the graph γ_1^{*} continues to be higher than the graph γ_1^{**}. This continues until the quantity γ_1^{**} is closer to the actual lower spectral value. At the step $s = 512$ (Fig. 2c), the quantity γ_1^{*} gets closer, and the graphs exchange places.

When γ_1^* and γ_1^{**} approach the actual lower spectral value, the iterations start to converge very rapidly: at the step with $s = 512$, the norm of the error is $\Delta_k^* \sim 10^{-13}$.

Note that when computing with a large initial number of iterations, the iterative process can be less effective due to the the value of the spectrum's lower boundary being extremely overestimated.

6. CONCLUSIONS

It has been shown that the transition operator of the Richardson method also acts as a compressing operator in the spectrum's domain from 0 to $\min \lambda_n$. Therefore, for an overestimation of the lower boundary γ_1^* of the spectrum, the Richardson method's iterative process converges. The latter substantiates the possibility of developing a class of converging iterative algorithms and tuning the a priori unknown lower spectral bound of the problem.

APPENDIX

For the classical Richardson method (see (2)–(7)) with $\gamma_1 > 0$, we have

$$\|x^s\| \le q_s \|x^0\|, \quad q_s = \frac{2\rho_1^s}{1+\rho_1^{2s}},$$

so that

$$s = \ln\left(2/[q_s(1+\rho_1^{2s})]\right)/\ln\left(1/\rho_1\right).$$

And as $\xi = \gamma_1/\gamma_2 \to 0$, we see that $s \to \ln(1/q_s)/\left(2\sqrt{\gamma_1/\gamma_2}\right)$. Hence, in order to have the agreed error decreasing by a factor of q_s, it is necessary, after doubling the number of iterations of the s-cycle, to decrease the estimate for γ_1 by a factor of four. Hence in p. 3 and p. 4 of the algorithm of competing processes, we propose to decrease the value γ_1 by a factor of four, while doubling the corresponding number of iterations.

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