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The Shock-Wave Structure in a Gas–Particle Mixture with Chaotic Pressure

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Abstract—We consider the propagation of a shock wave in a mixture of a gas and fine solid particles with allowance for the difference in their velocities and the availability of the proper pressure of the phase of particles; here, equations of the Anderson type and others are used. We propose an approximate mathematical model of the flow; in this model, the dependence of the pressure of the first (gaseous) phase from the particles' volume-concentration can be ignored, but the terms that present the phase volume-concentration multiplied by the pressure gradient of the gas are taken into account. It turns out that with this representation of the equation of state, the mathematical model has the hyperbolic type. For this system of equations of mechanics of heterogeneous media, we carry out the classification of the types of shock waves implemented in the considered mixture. The presented statements about the types are illustrated by numerical computations in stationary and nonstationary formulations; for this purpose, the numerical method of the TVD type is developed.

Keywords: mixture of gas and solid particles, particle-phase pressure, shock-wave structure, frozen and dispersion shock waves, numerical methods

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1. INTRODUCTION

The problem of the physicomathematical description of wave processes in mixtures of gases and fine particles (droplets) is relevant in the description of many industrial technological processes. In the context of rational mechanics based on the conservation laws of mechanics of heterogeneous media (MHM), it is possible to study various problems related to drying capillary and porous media, phenomena of the heterogeneous detonation, the ignition and the combustion of clouds of micro- and nano-sized particles or condensed explosives, the transportation of granular media, etc. In order to describe these processes, the following models are used in the literature and by us: (1) unit particles, (2) interacting continua with and without allowance made for the volume concentration of incompressible particles, and (3) compressible gases and particles in a continual approximation with allowance for the difference in the velocities, the temperatures, and the pressures of the phases (see [1–7], where a fairly complete bibliography on these issues is presented). Certain models listed in this hierarchy are described by equations of the composite type. Here, e.g., the Baer–Nunziato equations for describing the motion in MHM are hyperbolic. They determine two sound velocities that correspond to the propagation of perturbations in the considered compressible phases.

Then, we focus our attention on the well-known Anderson model of a heterogeneous mixture with incompressible particles and with allowance for the chaotic pressure of the particles. Using this model, we have studied the problem of the structure of a stationary combination discontinuity [8] and in the recent works, we consider the problem of the structure of a shock wave (SW) [9, 10]; we also mention the work [11], which explores the structure of an SW in the mixture of a gas and solid micro- and nanoparticles.

In the first approximation, for the last problem in [9, 10], we investigate the mathematical model of an SW; in this model (in the conservation equation of the particles' momentum), we neglect the term associated with the inclusion of the volume concentration of theparticles. This makes it possible to obtain a "hyperbolic approximation" of this model and (based on the formal similarity with the models of two compressible gases) classify the types of SWs analogously to [6].

Note that in practice, the problem of the interaction of the layers of a porous material, e.g., the cellular–porous structure and layers of particles of the poured density with strong discontinuities, is of significant interest (from the point of view of reducing the impact of dynamic phenomena on the environment). Therefore, the focus of this work, which is devoted to finding the possible configurations of SWs in the context of a more advanced model (taking into account the gas pressure gradient in the conservation equation of the particles' momentum), represents a certain practical interest. In fact, in order to estimate the impact of an SW on rigid walls in a two-phase medium, researchers need information on the types of strong discontinuities that can be implemented in a heterogeneous environment and the conditions under which they exist.

2. PHYSICOMATHEMATICAL FORMULATION OF THE PROBLEM, AND BASIC EQUATIONS

2.1. Basic Equations

Consider a mixture of a gas and solid particles that fill a one-dimensional channel. A mathematical model that describes the motion of the mixture represents a model of an interpenetrating flow of two interacting continua, during which the parameters of each of them (such as velocity, density, and pressure) are averaged for volume. The first continuum is a carrier gas characterized by its own velocity, pressure, and volume concentration.

The second continuum (the phase of the particles) also has its own pressure (as a result of the exchange of the momentum among particles due to their chaotic motion in the gas), velocity, and volume concentration that are different from the parameters of the gas. The isothermal motion of the considered two-phase medium (with allowance for the proper pressure of the phase of particles) is described by the equations of conservation of mass and momentum that are presented for each phase, and is supplemented by the equations of state (the Anderson model):

$$\begin{gathered}
\frac{\partial \rho_i}{\partial t} + \frac{\partial(\rho_i u_i)}{\partial x} = 0, \\
\frac{\partial(\rho_1 u_1)}{\partial t} + \frac{\partial(\rho_1 u_1^2 + m_1 p_1)}{\partial x} = p_1 \frac{\partial m_1}{\partial x} + f_1, \\
\frac{\partial(\rho_2 u_2)}{\partial t} + \frac{\partial(\rho_2 u_2^2 + m_2 p_1)}{\partial x} + \frac{\partial p_2}{\partial x} = p_1 \frac{\partial m_2}{\partial x} + f_2, \\
p_1 = p_1(\rho_1, \rho_2), \quad p_2 = p_2(\rho_2).
\end{gathered} \tag{1}$$

In [10], we describe certain forms of equations of state of a discrete phase. Here we present the equations of state for both phases as follows:

$$p_1 = \frac{a_1^2 \rho_1}{1 - \rho_2/r}, \quad p_2 = a_2^2 \rho_2, \quad \rho_{22} \equiv r = \text{const}, \tag{2}$$

where $\rho_i = \rho_{ii} m_i$, ρ_{ii}, m_i, u_i, p_i, and a_i are the average and real densities, the volume concentration, the velocity, the pressure, and the sound velocity of the ith phase ($i = 1, 2$). Index 1 denotes the parameters of a gas; and index 2, the phases of particles; f_1 is the force acting on the particles from the gas and $f_2 = -f_1$ is the force acting on the gas from the particles. We close system (1) and (2) by the main equality of MHM

$$m_1 + m_2 = 1. \tag{3}$$

2.2. Determination of the Interaction Force of the Phases

For the force interaction of the phases, we have the relation

$$f_1 = -\frac{3}{8} \frac{m_2 \rho_{11}}{r} C_D (u_1 - u_2)|u_1 - u_2|, \tag{4}$$

where C_D is the resistance coefficient of a spherical particle. Under the Stokes flow conditions, the resistance coefficient is determined as $C_D = 24/\text{Re}$, where $\text{Re} = (2r|u_1 - u_2|\rho_{11})/\mu$ is the relative Reynolds number, μ is the gas viscosity, and r is the radius of the particle. Expression (4) under the Stokes flow conditions has the form $f_1 = (u_1 - u_2)\rho_2/\tau_{st}$, where $\tau_{st} = (2/9)(\rho_{22}r^2/\mu)$ is the Stokes relaxation time of velocities. The deviation from the Stokes law of resistance can be taken into account, e.g., as follows:

$$C_D = \frac{24}{\mathrm{Re}}\frac{\left(1+0.15\,\mathrm{Re}^{0.687}\right)\left(1+\exp\{-0.427/\mathrm{M}^{4.63}-3.0/\mathrm{Re}^{0.88}\}\right)}{\left(1+\frac{\mathrm{M}}{\mathrm{Re}}\left(3.82+1.28\exp\left\{-1.25\frac{\mathrm{Re}}{\mathrm{M}}\right\}\right)\right)}. \quad (5)$$

2.3. On the Type of the Considered System of Equations of MHM

After determining all the empirical functions that appear in this mathematical model, its type should be specified. This question is quite widely covered in the literature; a fairly detailed list of works devoted to this subject is presented in [10]. We study the type of a system for several special cases of the equation of state. Consider a one-dimensional stationary flow of the mixture. The type of system depends on the number of real and imaginary eigenvalues of the corresponding matrix of coefficients of Eqs. (1)

$$\begin{vmatrix} u_1-\lambda & \rho_1 & 0 & 0 \\ \frac{m_1}{\rho_1}p_{1,\rho_1} & u_1-\lambda & \frac{m_1}{\rho_1}p_{1,\rho_2} & 0 \\ 0 & 0 & u_2-\lambda & \rho_2 \\ \frac{m_2}{\rho_2}p_{1,\rho_1} & 0 & \frac{m_2}{\rho_2}p_{1,\rho_2}+\frac{1}{\rho_2}p_{2,\rho_2} & u_2-\lambda \end{vmatrix} = 0.$$

Hence, we obtain the polynomial of the fourth degree

$$\Delta = (u_1-\lambda)^2(u_2-\lambda)^2 - (u_1-\lambda)^2(m_2 p_{1,\rho_2} + p_{2,\rho_2}) - (u_2-\lambda)^2 m_1 p_{1,\rho_1} + m_1 p_{1,\rho_1} p_{2,\rho_2} = 0.$$

(1) Consider this model in the case where the proper pressure of the phase of particles can be neglected. The equation for determining the characteristics of this limiting case of system (1) and (2) has the form

$$(u_1-\lambda)^2(u_2-\lambda)^2 - m_2 p_{\rho_2}(u_1-\lambda)^2 - m_1 p_{\rho_1}(u_2-\lambda)^2 = 0. \quad (6)$$

Here,

$$m_1 p_{,\rho_1} = a_1^2,\quad m_2 p_{,\rho_1} = \frac{m_2}{m_1}a_1^2,\quad m_2 p_{,\rho_2} = a_1^2\frac{\rho_{11}}{r}\frac{m_2}{m_1},$$

$$m_1 p_{,\rho_2} = a^2\frac{\rho_{11}}{r},\quad \theta = a^2\frac{\rho_{11}}{r}\frac{m_2}{m_1}.$$

Then polynomial (6) is presented as follows:

$$(u_1-\lambda)^2(u_2-\lambda)^2 - \theta a_1^2(u_1-\lambda)^2 - a_1^2(u_2-\lambda)^2 = 0.$$

The investigations of the roots of this equation are repeatedly described in the literature by different researchers (note once more that the corresponding bibliography is presented in [9, 10]). This made it possible to obtain conditions that specify the domains of the hyperbolicity, the ellipticity, and the composite type of this system of equations in the finite volume-concentration of the particles.

(2) The second simplified form of system (1) of equations is also considered in [9, 10]. This form is obtained if we formally ensure that the volume concentration of the particles tends to zero and the concentration of the gas tends to one. In this case, we have the hyperbolic type of equations, during which the system is close to the equations of two compressible gases with the Mach lines corresponding to the sound velocities in two compressible phases.

(3) At last, the third variant of the mathematical model is obtained if we assume that the equation of state of the first phase does not depend on the density of the second phase. Then the type of the considered mathematical model is defined by the following equation:

$$(u_1-\lambda)^2(u_2-\lambda)^2 - (u_1-\lambda)^2 p_{2,\rho_2} - (u_2-\lambda)^2 m_1 p_{1,\rho_1} + m_1 p_{1,\rho_1} p_{2,\rho_2} = 0.$$

Here, $m_1 p_{1,\rho_1} = m_1 a_1^2$ and $a_2^2 = p_{2,\rho_2}$ are the sound velocities in the first and second phases of the mixture. In this case, we obtain four valid characteristics

$$(u_1-\lambda)\|_{1,2}=\pm p_{2,\rho_2}=\pm a_2,\quad (u_2-\lambda)\|=\pm\sqrt{m_1}p_{1,\rho_1}=\pm\sqrt{m_1}a_1.$$

Hence, the system of equations is hyperbolic, as in the previous point.

2.4. Problem of the Structure of a Traveling Wave: Normal Form of the System

Using the last mathematical model of a mixture with two pressures, consider the problem of the structure of a traveling SW in the case where the covolume in the equation of state of the gaseous phase can be neglected. To put it differently, we assume that this equation of state depends only on the average density of the gaseous phase. Then Eqs. (1) in the coordinate system associated with the front of an SW are presented as follows:

$$\begin{gathered}\rho_i u_i = C_i,\\ C_1\dot{u}_1+m_1\dot{p}_1=f_1,\\ C_2\dot{u}_2+m_2\dot{p}_1+\dot{p}_2=-f_1.\end{gathered}\tag{7}$$

A dot over the variables of interest denotes differentiation with respect to a self-simulated variable. Solving system (7) with respect to derivatives, we can reduce it to the normal form

$$\begin{gathered}\frac{du_1}{dx}=\frac{u_1 f_1}{\rho_1(u_1^2-m_1a_1^2)},\\ \frac{du_2}{dx}=-\frac{u_2 f_1(u_1^2-a_1^2)}{\rho_2(u_2^2-a_2^2)(u_1^2-m_1a_1^2)}.\end{gathered}\tag{8}$$

We can notice that the considered system has the following singular points.

The *first type of such points* appears if the velocities of the phases are equal. In this case, the interaction force is zero. Here, we can otherwise have the state of rest (stable or instable, depending upon the type of assigned singular point).

The *second type of singular points* appears if the velocity of the first phase approaches the speed of sound in the first phase. However, such a point within a flow of a mixture can be stable (unstable), because here the gradient of the velocity of the first phase is not necessarily zero.

In addition, at points where the velocity of the first phase is $\sqrt{m_1}a_1$ and the velocity of the second phase equals the speed of sound in the phase of the particles, the solution turns over. In other words, the acceleration of the phases becomes infinite and a flow with sharpening takes place. All this determines the relatively complex behavior of the solution of boundary value problems for such a simple system of equations such as (8).

2.5. Formulation of the Boundary Value Problem

The system of equations (8) must satisfy the following boundary stationary conditions for the solution vector $\Phi(u_1,u_2,\rho_1,\rho_2)$:

$$\Phi\to\Phi_0,\quad \dot{\Phi}\to 0\ (x\to-\infty),\quad \Phi\to\Phi_k,\quad \dot{\Phi}\to 0\ (x\to+\infty).\tag{9}$$

This is consistent with the one before the SW front and far beyond it the mixture is in equilibrium when the velocities of the components are equal. Thus, the problem of the structure of an SW in a gas–particle mixture with allowance for the proper pressure of the particles is reduced to solving the boundary value problem (8) and (9) over an infinite interval. Our task is to determine the possible structures of this transition. We preliminarily determine certain characteristic values of the flow parameters of the mixture.

2.6. Equilibrium and Frozen Sound Velocities in the Mixture

When the velocities of the phases are equal, the sound velocity for an *equilibrium* mixture is determined as follows: $dP/d\rho=a_1^2\xi_1+a_2^2\xi_2=C_e^2$, where $P=p_1+p_2=C_e^2\rho$ is the pressure for the entire mixture, $\rho=\rho_1+\rho_2$ is the average density of the mixture, and $\xi_i=C_i/(C_1+C_2)=\mathrm{const}$ is the relative mass concentration of the *i*th phase in an equilibrium flow of the mixture. The *frozen* sound velocities in the first and second phases are $\sqrt{m_1}a_1$ and a_2, respectively.

2.7. Frozen and Equilibrium States of the Mixture

Consider certain inherent types of phase flows that arise when changing the relaxation times of the velocities and the concentrations of components of a two-phase mixture. For this purpose, we preliminarily obtain the conservation laws for strong discontinuities.

Conditions on a frozen SW. First of all, we obtain the conditions of conservation on a frozen SW for $u_1 \neq u_2$. The equations of conservation of mass and momentum for the phases give the following integrals and one differential equation presented in the nonconservative form:

$$\begin{gathered} \rho_i u_i = C_i = \rho_{i0} u_{i0}, \quad i = 1,2, \\ C_1 u_1 + C_2 u_2 + p_1 + p_2 = C_3 = (C_1 + C_2) u_0 + p_{10} + p_{20}, \\ C_2 \dot{u}_2 + m_2 \dot{p}_1 + \dot{p}_2 = f_2. \end{gathered} \tag{10}$$

The first two equations are the continuity equations of the phases, the third expression represents the equation of momentum conservation of the entire mixture, and the last expression is the equation of momentum conservation of phase particles. The last nonconservative equation admits, however, the integral that describes the condition of momentum conservation of the second phase. In fact, we divide this equation by the average density of the second phase. We take into account the fact that the true density of the particles is constant and integrate the obtained equation with respect to the self-similar variable in a small neighborhood of the strong discontinuity (−ε, +ε). Since the resistance force has no singularities over this interval, we obtain the equation

$$\frac{u_2^2}{2} + \frac{p_1}{r} + \int_{\rho_{20}}^{\rho_2} \frac{dp_2}{\rho_2} = C_{22} = \frac{u_0^2}{2} + \frac{p_{10}}{r}.$$

We use the expression for the density of the first phase in terms of its velocity to perform further rearrangements. As a result, the last equation (the equation of momentum conservation for particles) takes the form

$$u_2^2/2 + a_1^2 C_1/(r u_1) + a_2^2 \ln(u_0/u_2) = C_{22} = u_0^2/2 + p_{10}/r. \tag{11}$$

The equation of momentum conservation of the entire mixture can be presented as the function

$$\begin{gathered} \Phi(u_1, u_2) = C_1 u_1 + C_2 u_2 + a_1^2 C_1/u_1 + a_2^2 C_2/u_2 - C_3 = 0, \\ C_3 = (C_1 + C_2) u_0 + p_{10} + p_{20}. \end{gathered}$$

Its form can be simplified to

$$\Phi(u_1, u_2) = \frac{C_1}{u_1}(u_1 - u_0)(u_1 - \tilde{u}_1) + \frac{C_2}{u_2}(u_2 - u_0)(u_2 - \tilde{u}_2). \tag{12}$$

Equation (12) was obtained earlier in the work of A.V. Fedorov [6] for a model of the Baer–Nunziato type; this model describes a flow of a mixture of a gas and solid particles with different velocities and pressures. The solid-phase particles are assumed to be compressible. Besides in [6], the transfer equation of the solid phase is used for the closure. This predetermines the fact that the concentrations of the phases on the wave front are frozen. As a result, Eq. (12) is transformed into the equation

$$\Phi(u_1, u_2) = \frac{\xi_1}{u_1}(u_1 - u_0)(u_1 - \tilde{u}_1) + \frac{\xi_2}{u_2}(u_2 - u_0)(u_2 - \tilde{u}_2) = 0. \tag{13}$$

The analogy between Eqs. (12) and (13), which describe the motion in the phase plane (u_1, u_2), makes it possible to reveal the properties of function (12). In particular it reveals the following information:

(1) This function is closed.

(2) The domain of its definition we are interested in is found analytically.

(3) There exist four points $u_1 = \sqrt{m_1} a_1$ and $u_2 = a_2$, at which the derivative du_1/du_2 becomes zero or infinity. All this can be seen in [6]. In fact, if we introduce the function $P_2 = P_2(u_2) = \dfrac{C_2}{C_1 u_2}(u_2 - u_0)(u_2 - \tilde{u}_2)$, the last equation can be presented as the second-degree polynomial

$$u_1^2 - (u_0 + \tilde{u}_1 + P(u_2)) u_1 + u_0 \tilde{u}_1 = 0, \tag{14}$$

Table 1. Types of SWs and their parameters according to particles' volume-concentration

Type of SW	m_2	ξ_{10}	u_0, m/s	a_e, m/s	u_k, m/s
1. Dispersion and frozen	5×10^{-5}	0.9	430	396	365
2. Dispersion for both phases	3×10^{-4}	0.6	430	415	401
3. Frozen two-front	4×10^{-4}	0.53	460	420	383
4. Frozen and sound	5×10^{-4}	0.46	460	424	390
5. Frozen one-front	7×10^{-4}	0.4	460	427	396
	0.2	0.002	460	449.9	440

which has two branches of solutions under certain conditions relative to the velocity of the second phase. These conditions depend on the sign-definiteness of the discriminant, analogously to [6]. As a result, it can be argued that function (12) is a closed curve in the plane (u_1, u_2). The other properties of the function $\Phi(u_1, u_2)$ are proved analogously.

The velocities of the phases behind a frozen SW are found from the solution of the system of Eqs. (11) and (12). In fact, by choosing some interval of the change in the propagation velocity u_0, we can determine these velocities. Further their values are found numerically.

Conditions on an equilibrium SW. In such a flow of the mixture, the condition $u_1 = u_2$ is fulfilled. It is implemented if the time of the equalization of the velocities is much smaller than the characteristic times of the wave-processes' propagation in the phases of the mixture. We find the equilibrium values of the velocities in the mixture behind the front of the SW. We put $u_1 = u_2 = u$ and obtain the quadratic equation $u^2 - u(u_0 + C_e^2/u_0) + C_e^2 = 0$. It is clear that there are only two equilibrium states, u_0 and u_k, during which the velocity of the phases in the final equilibrium state is $u_k = C_e^2/u_0$. The Zemplen theorem immediately follows from this representation of the equilibrium flow. Relying on the points above, we determine the parameters of an SW by increasing the volume concentration of the particles for the following parameters of the mixture: $a_1 = 390$ m/s, $a_2 = 450$ m/s, $\rho_{11} = 1.2$ kg/m, and $\rho_{22} = 2700$ kg/m (see Table 1).

3. DISCUSSION OF THE COMPUTATIONAL RESULTS IN THE STATIONARY APPROACH

3.1. Flow in a Dispersed and Frozen SW ($a_1 < u_0 < a_2$, $m_{20} = 5\times10^{-5}$)

A continuous transition in the structure of an SW is performed in both phases. Then when moving in the phase plane, the destruction of the continuous flow in the first phase takes place. Therefore, it is necessary to introduce an SW in the tail of the SW. In this case, the velocity behind the frozen SW in the tail must be identical with the equilibrium velocity of the phases at the end of the SW. This is achieved by determining the free parameter ξ_k (we find the parameter $u_1(\xi_k)$ from the numerical solution of the Cauchy problem with the corresponding initial conditions) so that $u_k = a_1^2/u_1(\xi_k)$. After reaching the final value of ξ_k, the computation stops. For the first phase, the velocity behind the front of the head discontinuity is equal to the initial velocity. Then when moving on the curve $\Phi(u_1, u_2) = 0$, we reach the mixture's equilibrium point, at which the velocity of the first phase is identical with the velocity of the second phase and is equal to the value $u_k > \sqrt{m_1}a_1$. The corresponding illustrations are shown in Fig. 1. At the left of this figure, the dependences of the velocities from the self-similar variable are presented. It is seen that in front of the frozen SW, the compression wave is propagated in both phases, and the gas and the particles in the relative coordinate system in it slow down. After that, a gradient catastrophe arises in the gas phase, during which the compression wave is overturned and a tail SW is created. This structure is stationarily propagated in the physical region of the flow. At the right we see the flow pattern (in the phase plane (u_1, u_2)), which justifies the numerical results. Here, the slant line $0, k$, the closed curve, and the straight lines present the Rayleigh–Michelson line, the function $\Phi(u_1, u_2) = 0$, and the characteristic parameters of the mixture (the sound velocities of the phases), respectively. An analogous notation is used in the subsequent figures; we will not comment on it further.

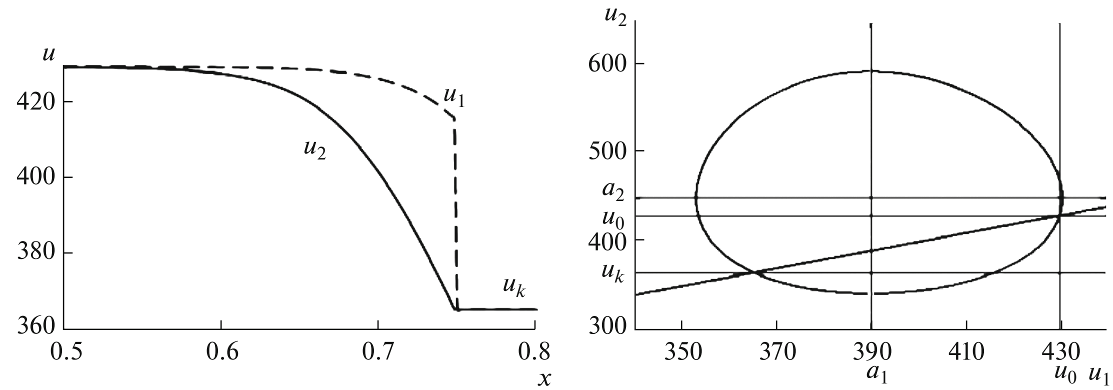

Fig. 1. Dispersion and frozen SW (dispersion SW in both phases and trailing SW in first phase; $u_0 = 430$ m/s and $m_{20} = 5 \times 10^{-5}$).

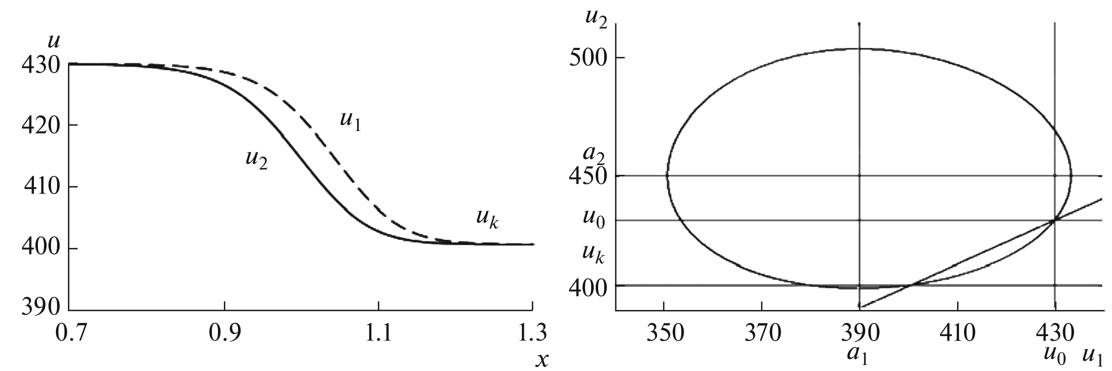

Fig. 2. Dispersion SW in both phases ($u_0 = 430$ m/s and $m_{20} = 3 \times 10^{-4}$).

3.2. Flow in an Entirely Dispersed SW ($a_1 < u_k < a_e < u_0 < a_2$, $m_{20} = 3 \times 10^{-4}$)

The flow is supersonic for the first phase but subsonic for the second phase. Figure 2 shows that by increasing the volume concentration of particles by one order of magnitude, the tail SW disappears, during which the structure of the flow degenerates into an entirely dispersed structure. Although, in the previous case, the velocity of the second phase was adjacent to the zone of a constant flow (the equilibrium state) through a weak discontinuity, now such a junction is realized continuously when the self-similar variable tends to infinity. Naturally the junction is realized exponentially, as is typical for the structure of an SW in classical gas dynamics. The velocity of the first phase here also changes continuously.

3.3. Flow in a Frozen Two-Front SW ($a_1 < a_2 < u_0$, $m_{20} = 4 \times 10^{-4}$), Fig. 3

The initial state in the medium is supersonic for both phases. As a result, in the head of the wave, an SW in the second phase is implemented and then (in passing through the zone of the velocity relaxation), the phases come into equilibrium by means of the tail SW in the gaseous phase.

3.4. Flow in an SW with a Sound Flow for the Tail in the First Phase ($a_1 < a_2 < u_0$, $u_k = a_1$, $m_{20} = 5 \times 10^{-4}$), Fig. 4

In this flow, the final velocity of the phases in the equilibrium state is equal to the sound velocity in the gas. With the phase velocities of phases changing continuously and once the sound velocity is achieved, the flow in question resembles the Chapman–Jouguet flow. This is only merely formal closeness, because such a motion is not self-sustained. This state is not stable in relation to second-phase rarefaction waves whose leading front moves with the speed of sound a_2. The state in question is destroyed by such a rarefac-

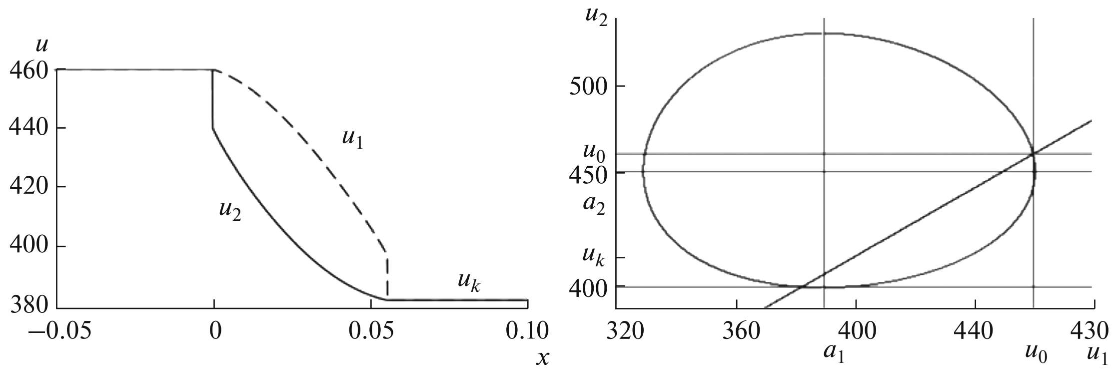

Fig. 3. Frozen two-front SW ($u_0 = 460$ m/s and $m_{20} = 4 \times 10^{-4}$).

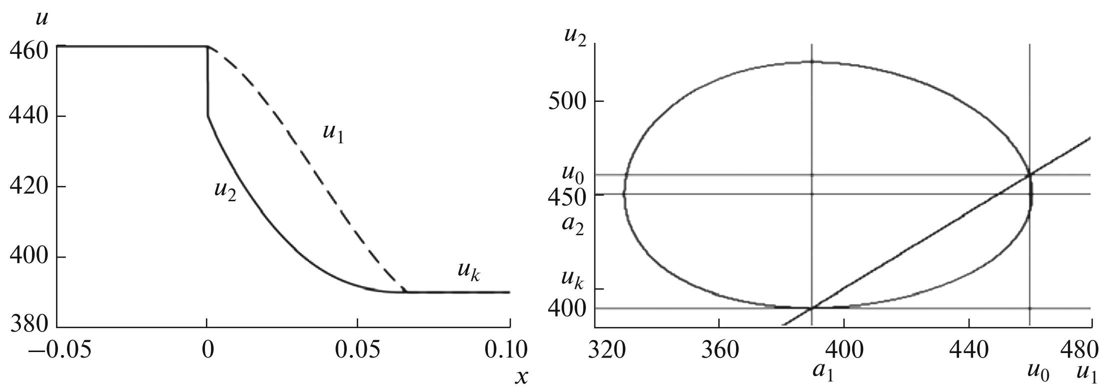

Fig. 4. Frozen and sound SW ($u_0 = 460$ m/s and $m_{20} = 5 \times 10^{-4}$).

tion wave. Just like all the previous flows, this flow is stable if supported by a piston that moves with the corresponding velocity (i.e., in this case, with the sound velocity in the second phase).

3.5. On the Influence of the Term with a Gradient of a Continuous Phase in an SW

The considered flow is implemented for the initial flow velocity greater than the speed of sound in the phases, and a further increase of the concentration of the particles, e.g., for $a_1 < a_2 < u_0 = 460$ m/s and $m_{20} = 7 \times 10^{-4}$. In this case, the final velocity of the mixture becomes greater than $\sqrt{m_1} a_1$. Hence, there is no need to cross the point at which the solution turns over. Here, when $a_1 < a_2 < u_0$ and $m_{20} = 0.2$, a partially frozen SW arises. The latter concentration is very high: of the same order of magnitude as the bulk concentration. Figure 5 presents the velocity distributions of the phases implemented here in the cases when the terms of the form $m_i \nabla p_1$ in the conservation equations of the momenta of the phases are taken into account (curves *1*) and when they are not taken into account (curves *2*). It is seen that with the increasing particle concentration, the difference of the gas velocities in the mixture for different models becomes significant. Obviously, this is caused by the increasing effect of the term $m_i \nabla p_1$ in the momentum equations. In Fig. 5, the SW in the particles is retained without its configuration being changed. The gas is filtered through the set of particles; here, the relaxation zone of the velocities for the gas is changed about two times taking the term $m_i \nabla p_1$ into account. Figure 5 clarifies the effect of this term on the velocity distribution. It is seen that the velocity of the particles in the leading SW varies very little. The width of the SW is almost unchanged with the increasing volume concentration. The filtering gas slows in particles in the relaxation zone at a greater distance.

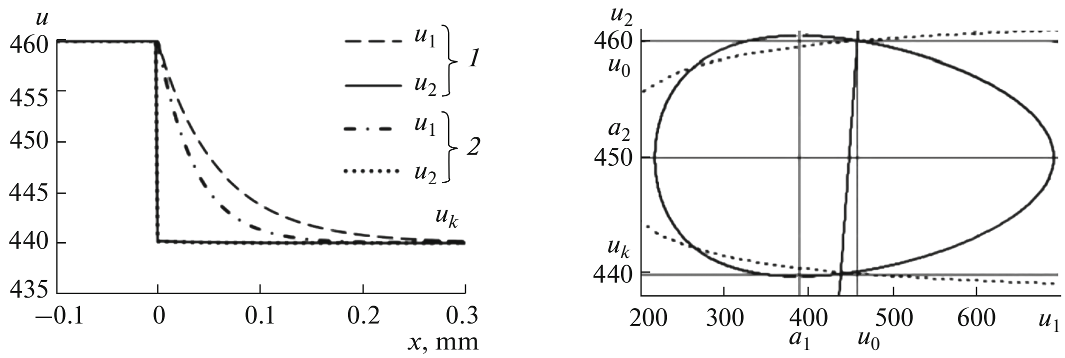

Fig. 5. Frozen one-front SW for $m_2 = 0.2$ ($u_0 > a_i$; head SW in second phase and continuous flow in first phase). Dashed lines represent two branches of curve that specifies condition behind frozen SW.

Remark. The questions relative to the existence of solutions of the considered types can be analyzed in a standard way using the qualitative study of the stationary points of the dynamical system (8) of ordinary differential equations for the velocities of the phases, or these equations can be reduced to one equation. In fact, from the integral of the momentum conservation of the mixture, we can find, e.g., the velocity of the first phase as a function of the velocity of the second phase. Based on this, it is easy to obtain the equation

$$\frac{du_1}{dx} = \frac{u_1 f_1}{\rho_1(u_1^2 - m_1 a_1^2)} = g(u_1, u_2(u_1)).$$

We determine the eigenvalue $\lambda_1 = dg/du_1$ at the equilibrium points for $u_1 = u_2 = u_{0,k}$ by differentiating the right side of this equation. For this purpose, from the momentum-conservation equation, we find the derivatives

$$\frac{du_2}{du_1} = -\frac{\xi_1}{\xi_2}\frac{(u_1^2 - a_1^2)}{(u_2^2 - a_2^2)}, \quad \frac{\partial g}{\partial u_1} = -\frac{\rho_2 u_1}{\rho_1 \tau D_1}, \quad \frac{\partial g}{\partial u_2} = -\frac{\rho_2 u_2}{\rho_1 \tau D_1}, \quad D_1 = u_1^2 - m_1 a_1^2$$

and then the eigenvalue

$$\lambda = -\frac{u}{\tau\xi_1}\frac{u^2 - C_e^2}{(u^2 - a_1^2)(u^2 - a_2^2)}. \tag{11}$$

It is noted above that here the velocity takes values equal to the initial or final equilibrium values of the mixture. Note that the sign of the eigenvalue depends on the correlation of the equilibrium velocity of the mixture at the initial or final state, with the sound velocities in both phases and the equilibrium sound velocity. This also depends on the dependence of the final equilibrium velocity of the mixture on its initial parameters ξ_1 and u_0. Taking into account that the final velocity of the mixture $u_k = C_e^2/u_0 = (\xi_1 a_1^2 + (1 - \xi_1)a_2^2)/u_0$, we impose the corresponding conditions on u_k and obtain various inequalities that give estimates (of the velocities behind the SW) that are analogous to the estimates from [6–10].

4. DISCUSSION OF THE COMPUTATIONAL RESULTS IN A STATIONARY FLOW

In order to study the stability of the obtained solutions in the mathematical model (1), the mathematical technology for solving problems of MHM with two pressures and velocities based on the total variation diminishing (TVD) approach was designed and implemented.

4.1. Numerical Method

Note that the boundary value problem for the system of ordinary differential equations of the mathematical model was solved by the RADAU5 solver of stiff systems of equations, which uses the implicit fifth-order Runge–Kutta method with step selection. Previously we had turned our attention to the issue of constructing difference schemes for the equations of MHM without considering the terms $m_i\nabla p_1$ in [12, 13] both for the model with one pressure and for the model with two pressures [10]. Besides, for the mixture of gases with widely different molecular weights, we propose in [14] the computational scheme in the two-velocity two-temperature approximation (with allowance for the differences of the pressures of the components) and solve several problems of the interaction of an SW with contact discontinuities [15]. In the present paper, we take these terms (i.e., $m_i\nabla p_1$) into account.

As an approximation in time for the system of Eqs. (1), we use a fifth-order scheme of the Runge–Kutta type [16]. The scheme of the mth order of approximation can be presented as follows. Assume that y is one of the unknown functions, ρ_i or $(\rho u)_i$, and $Q_y(t)$ are the corresponding components of the vector $\vec{Q}$ in Eqs. (1). Using these designations, we can present Eqs. (1) as $dy/dt + Q_y(t) = 0$. Then the m-stage scheme has the form

$$\begin{aligned} y^{(0)} &= y^{(n)}, \\ y^{(1)} &= y^{(0)} - \gamma_m \tau Q_y^{(0)}, \\ y^{(2)} &= y^{(0)} - \gamma_{m-1} \tau Q_y^{(1)}, \\ y^{(m)} &= y^{(0)} - \gamma_1 \tau Q_y^{(m-1)}, \\ y^{(n+1)} &= y^{(m)}. \end{aligned} \tag{12}$$

The values of the parameters $\gamma_1, \gamma_2, \ldots, \gamma_m$ are chosen from the conditions of approximation and maximum stability. Because of the significant difficulties that emerged in analyzing these conditions for a system of nonlinear differential equations, the values of the parameters $\gamma_1, \gamma_2, \ldots, \gamma_m$ are computed based on the analysis of the linear transport equation. For a five-stage scheme, these parameters have the following values: $\gamma_1 = 1$, $\gamma_2 = 1/2$, $\gamma_3 = 3/8$, $\gamma_4 = 1/6$, and $\gamma_5 = 1/4$. It is found that for this problem, the use of such a scheme allows us to significantly extend the interval of stability and computing with large values of the Courant number. Here, the increase of the interval of stability is nonlinear in character. The use of a scheme of the fifth order of accuracy made it possible to increase the Courant number 40 times as compared with the scheme of the first order.

In order to construct a spatial approximation of system (1) by using the TVD approach [17], it is necessary to split the flow vector $\vec{Q}_i$ for each component. There are many ways of doing this. We use the following designations: $\tilde{p}_1 = m_1 p_1$ and $\tilde{p}_2 = m_2 p_1 + p_2$. In the subsequent discussion, index i for flows is omitted. Here, in order to obtain a stable upwind approximation of the right sides of the differential schemes, we divide the flow vector $\vec{Q}$ into positive and negative components: $\vec{Q} = \vec{Q}^+ + \vec{Q}^-$. For this purpose, we use the method of splitting a flow vector by physical processes [18]. In accordance with this method, we divide the flow vector $\vec{Q}$ into the components $\vec{Q}^+$ and $\vec{Q}^-$, depending on the sign of the velocity, in such a way that the pressure is approximated for the flow; and all other variables, against the flow:

$$\begin{aligned} Q_\rho^+ &= \begin{cases} \rho u, & u > 0, \\ 0, & u \le 0, \end{cases} & Q_\rho^- &= \begin{cases} 0, & u > 0, \\ \rho u, & u \le 0, \end{cases} \\ Q_u^+ &= \begin{cases} \rho u^2, & u > 0, \\ \tilde{p}, & u \le 0, \end{cases} & Q_u^- &= \begin{cases} \tilde{p}, & u > 0, \\ \rho u^2, & u \le 0. \end{cases} \end{aligned} \tag{13}$$

A higher order approximation is obtained if we use the formulas

$$\frac{\partial \vec{Q}}{\partial x} \approx \frac{\left[\vec{Q}^+_{j+1/2} - \vec{Q}^+_{j-1/2} + \vec{Q}^-_{j+1/2} - \vec{Q}^-_{j-1/2}\right]}{\Delta x}, \tag{14}$$

where

$$\begin{aligned} \vec{Q}^-_{j+1/2} &= \vec{Q}^-_{j+1} - \frac{1}{4}\left[(1-\kappa)\Delta^+(\vec{Q}^-_{j+1}) + (1+\kappa)\Delta^-(\vec{Q}^-_{j+1})\right], \\ \vec{Q}^+_{j+1/2} &= \vec{Q}^+_{j+1} + \frac{1}{4}\left[(1-\kappa)\Delta^-(\vec{Q}^+_{j}) + (1+\kappa)\Delta^+(\vec{Q}^+_{j})\right], \end{aligned} \tag{15}$$

$$\Delta^+(\vec{Q}_j) = (\vec{Q}_{j+1} - \vec{Q}_j), \quad \Delta^-(\vec{Q}_j) = (\vec{Q}_j - \vec{Q}_{j-1}).$$

The expressions for $\vec{Q}^-_{j-1/2}$ and $\vec{Q}^+_{j-1/2}$ are obtained by shifting the index by one unit.

Formulas (14) and (15) approximate the spatial derivatives with the third ($\kappa = 1/3$) or second ($\kappa = -1$, 0, 1) order. For $\kappa = -1$, $\kappa = 0$, and $\kappa = 1/3$, 1, the approximation is reduced to completely one-sided differences, central differences, and differences shifted against the flow, respectively.

To retain the monotonicity of the solution in the domains of large gradients, the order of the approximation is reduced by the application of the minmod limiter to the operators Δ^+ and Δ^- [17]:

$$\delta^+ = \begin{cases} 0, & \operatorname{sign}\Delta^+ \operatorname{sign}\Delta^- \le 0, \\ \min\left(\left|\Delta^+\right|, \Theta\left|\Delta^-\right|\right), & \operatorname{sign}\Delta^+ \operatorname{sign}\Delta^- \ge 0. \end{cases} \tag{16}$$

$$\delta^- = \begin{cases} 0, & \operatorname{sign}\Delta^+ \operatorname{sign}\Delta^- \le 0, \\ \min\left(\left|\Delta^-\right|, \Theta\left|\Delta^+\right|\right), & \operatorname{sign}\Delta^+ \operatorname{sign}\Delta^- \ge 0, \end{cases}$$

where the parameter Θ varies over the following range:

$$1 \le \Theta \le \frac{3-\kappa}{1-\kappa}. \tag{17}$$

With allowance for introducing a limiter, formulas (15) take the form

$$\left.\begin{aligned} Q^-_{j+1/2} &= Q^-_{j+1} - \frac{\sigma}{4}\left[(1-\kappa)\delta^+ + (1+\kappa)\delta^-\right](Q^-_{j+1}) \\ Q^+_{j+1/2} &= Q^+_j + \frac{\sigma}{4}\left[(1-\kappa)\delta^- + (1+\kappa)\delta^+\right](Q^+_j) \end{aligned}\right\}. \tag{18}$$

Using the proposed splitting by formulas (14)–(18), we can construct approximations for the derivatives of the flow components of Eqs. (1), whose numerical solutions we employ in order to analyze the resolving abilities of several schemes.

For approximating the nondivergent terms of Eq. (1), we use a scheme of the second-order accuracy with the central differences

$$\frac{\partial m_i}{\partial x} \approx \frac{m_{i,j+1} - m_{i,j-1}}{2\Delta x},$$

where i specifies the number of a phase.

4.2. Reproducing the Propagation of SWs of Different Types by the Numerical Method

Consider the problem of the motion of a certain SW structure that propagates along a one-dimensional channel. The problem consists of implementing the obtained stationary solution in the nonstationary formulation of model (1), i.e., by the relaxation method. Thus, we consider the Cauchy problem for the system of nonstationary Eqs. (1) in the case where, as the initial data, we take the numerical solution of the boundary value problem formulated above for a system of ordinary differential equations describing a traveling wave. As is known, a numerical solution contains varied infinitesimal perturbations. Therefore, the computation of such a problem can also be interpreted as solving the question of the stability of a traveling wave with respect to these perturbations. However, first, we dwell on the question of the convergence of the numerical solution.

4.3. Convergence in a Grid

Figure 6 illustrates the convergence of the numerical solution of the problem of propagating a *two-front SW* on a sequence of nested grids. It is seen that with a decreasing step of the computational grid, approximate solutions become indistinguishable from each other. Here, the Courant number is 1.4.

4.4. Illustrations to Dynamics of SWs of Different Types

We dwell on one example of the dynamics of a *one-front SW* at high concentrations of the second phase in the mixture ($m_2 = 0.2$). As described above, there is a strong discontinuity in the head of the wave in the

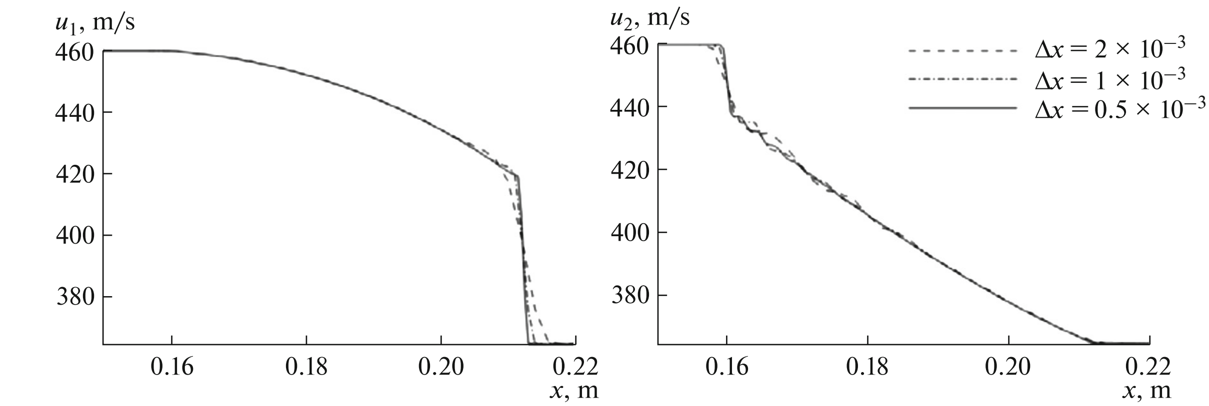

Fig. 6. Convergence in grid with computation of unsteady problem of propagating two-front SW.

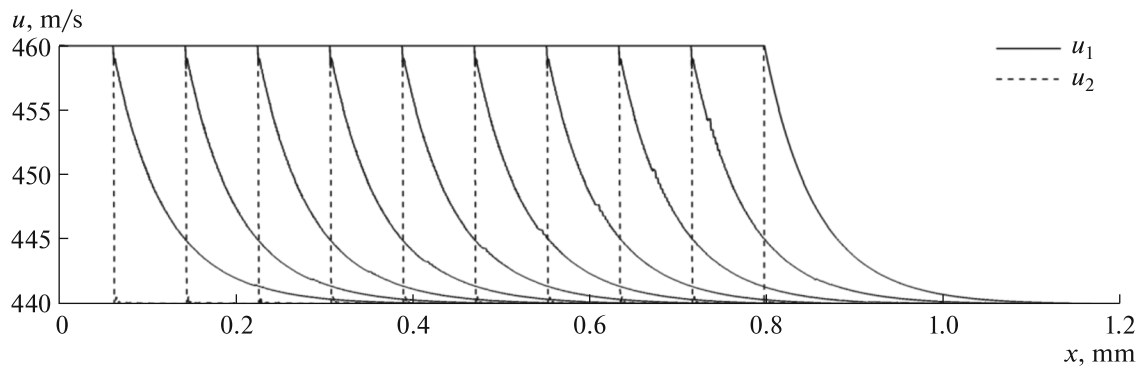

Fig. 7. Stability of propagation of frozen one-front SW for $m_2 = 0.2$ and $u_0 = 460$ m/s.

second phase. In the first phase, this discontinuity is followed by the relaxation zone of the velocities. The adjacency to the initial state occurs continuously through the weak discontinuity. All these features are satisfactorily transferred by the proposed numerical scheme; this can be seen in Fig. 7, where for different points in time, the distributions of parameters in the flow field of the mixture are presented. Figures 8 and 9 show that *two-front* and *dispersed and frozen SWs* are also steadily propagated in the mixture. Here, of course, the movement of all types of SWs is supported by a piston.

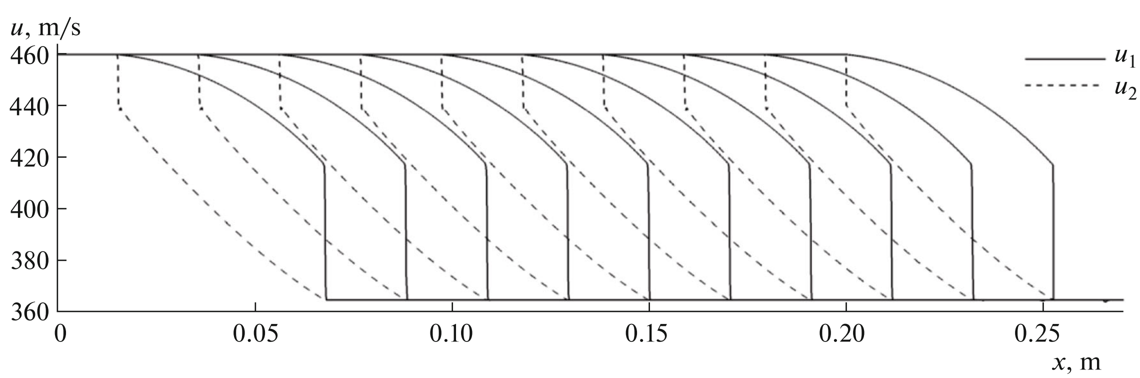

Fig. 8. Stability of propagation of frozen two-front SW for $m_2 = 2 \times 10^{-4}$ and $u_0 = 460$ m/s.

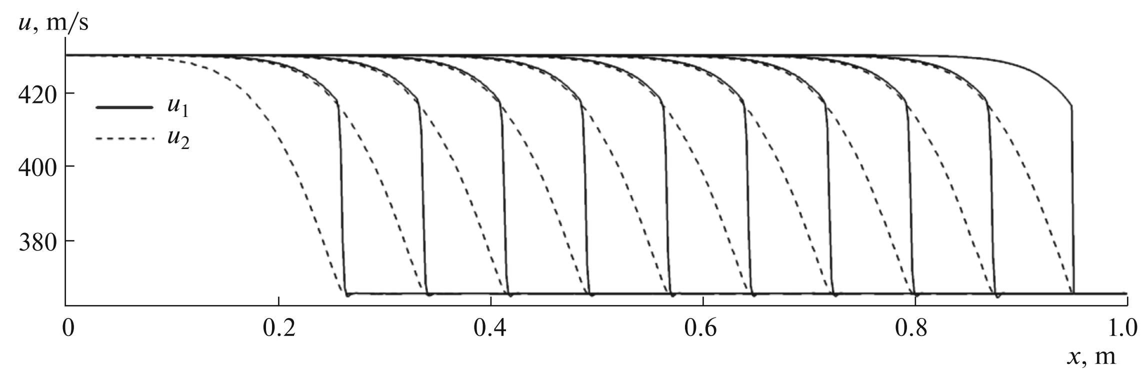

Fig. 9. Stability of propagation of dispersion and frozen SW for $m_2 = 5 \times 10^{-5}$ and $u_0 = 430$ m/s.

5. CONCLUSIONS

In a mathematical model of the Anderson type for describing a flow of a mixture of a gas and solid particles with respect to their own pressure (the terms $m_i \partial p_1/\partial x$), in the present work, we develop the theory of a stationary strong discontinuity and numerically implement this theory in an unsteady formulation. This allows us to describe possible types of SWs.

To perform the computations, we elaborate the version of a TVD scheme of a high-order accuracy for solving a system of nonstationary equations of MHM with different velocities and pressures (the Anderson model, etc.). The convergence of the solution on a sequence of thickening grids is checked. The numerical computations that illustrate these structures show the stability of their propagation with the flows supported by a piston.

ADDITION: THE HEURISTIC DEVELOPMENT OF THE CONDITIONS ON A SHOCK WAVE IN THE DISCRETE PHASE OF A GAS–PARTICLES MIXTURE

Let us rewrite the motion equation of particles in a dense gas–particles mixture as follows:

$$C_2 u_2' + m_2 p_1' + \frac{m_2 p_2'}{m_2} = f_2. \tag{a}$$

We introduce the function $P_2 = \int \frac{dp_2}{m_2}$; then, (a) can be presented as

$$r m_2 \left(u_2^2/2 + p_1/r + P_2/r \right)' = f_2. \tag{b}$$

We integrate (b) by parts on the discontinuity located over the interval $(-\varepsilon, \varepsilon)$ and, in accordance with [19], find the limit of this expression for ε tending to zero. We have

$$[m_2 I_2] - [m_2] I_2(0) = \lim_{\varepsilon \to 0} f_2 = 0. \tag{c}$$

The value $I_2(0) = \lim_{\varepsilon\to 0} \int_{-\varepsilon}^{+\varepsilon} I_2 [m_2] \delta(\xi) d\xi$ corresponds to the momentum inflow on a strong discontinuity and can be determined. For this purpose, we use the relation

$$[m_2 I_2] = m_2 [I_2] + I_{20} [m_2].$$

We substitute it into (c), require that $I_2(0) = I_{20}$, and obtain $[I_2] = 0$ on condition that the volume concentration of particles never becomes zero.

Note that such an approach for determining $I_2(0)$ at the physical level of strictness is used in the works of V.G. Dulov, I.K. Yaushev, and A.N. Kraiko, along with P.G. LeFloch, Mai Duc Thanh, et al. (see publications of these researchers, where the detailed bibliography is presented).

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