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Coherent Hydrodynamic Structures and Vortex Dynamics

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Abstract—Possible approaches to modeling two-dimensional coherent hydrodynamic structures based on the statistical mechanics of local vortices are considered. The exact definitions of coherent structures are given and the mechanisms of their formation are shown. The bases of the kinetic theory of Onsager vortices are given and the possibility of applying the classical molecular-kinetic theory for the explanation of the origin of vortex meso-structures in the shear flows is considered.

Keywords: coherent hydrodynamic structures, Onsager vortices, Vlasov equation, Poisson systems, variation problem, Joyce-Montgomery equation, solution branching

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1. INTRODUCTION. CURRENT APPROACHES TO THE SIMULATION OF COHERENT HYDRODYNAMIC STRUCTURES

The class of hydrodynamic problems associated with various aspects of the evolution of vortex movements of liquid and gas is exceptionally important both in terms of practical applications and the study of the theory of evolution of multiobject statistical systems. The subclass of problems significantly associated with the presence of some ordered ensembles of elementary vortices of various types (including processes of their confluence into agglomerations and disintegration), known as (composite) *coherent macro and meso structures* is of special importance. This subclass can be viewed from different viewpoints and by the use of considerably different approaches in the description procedures. For the most part, this is due to the possibility of analyzing and simulating the processes going on in gas-liquid coherent systems of (1) direct classical equations of hydrodynamics reduced to a form suitable for research in this particular case; (2) the latter's complexes (transformed in a certain way in order to smooth and/or to average) and equations describing the chaotic movement (which, generally speaking, may possess a quality of separability by some parameters). It should be noted that the degree of validity of the existing concepts underlying the physical substantiation of this approach is a problem for discussion; leaving aside the *pro* and *contra* of this veracity, according to the latter approach, it is possible to present the moving vortical medium (in particular, in phenomenological simulation by a pseudo-structured shift of the turbulence) as a thermo-hydrodynamic complex, consisting of two or three interpenetrating continua (subsystems), which fill up one and the same volume of the configuration space continually. This procedure, in turn, can be divided into two classes of subprocedures, one of which can be descriptionally called a decomposition-hydrodynamic subprocedure (DHSP) and the other a hybrid subprocedure (HSP). The joint DHSP class may be thought of as including the approaches based on binary and ternary decompositions of turbulent flows, whose bases may have been formulated for the first time as early as in the classical works by O. Reylnolds and L. Prandtle; today's language of decomposition studies was formulated in the 1970s [1–4] (later the mathematical apparatus was the topic for a great number of publications, among which [5–9] should be noted specially). Among the similarly large number of works associated with the class of hydrid subprocedures, using both the Navier-Stokes equations (NSEs) and the stochastic-termodynamic approach (see, for example, [10–13] and its review in the literature), we note [14–17], considering the subsystems of the averaged movement of liquid (arising as the result of the theoretical probability averaging of instantaneous hydrodynamic variables) and composite turbulent chaos (associated with the stochastic low-scale pulsation motion of vortical liquid and with a coherent component—frequency clusters, whose image in the space of the state of equivalent dynamic systems are the limit cycles); as a result, equations of the Focker Planck-Kholmogorov type are obtained for the functions of probability distributions of stochastic characteristics of vortical formations in the space of internal coordinates (these equations describe Markov diffusion processes, , taking into account transitions among stationary states of the system as a result of the cycle of consecutive losses of consistence—in a sense, it is possible here to speak of analogies with the evolution of the quasi-stationary states of the statistical vortical systems).

As to the approach based on the use of directly classical hydrodynamic equations, we note that with respect to the general calculation procedures, it is at present relatively limited, because the analysis of the behavior of multivortical systems in postlaminary modes, without auxiliary simplifying assumptions on the structure of these equations themselves is extremely difficult: for example, the division in a certain way of the corresponding spatial and time scales. These simplifications are a transition to the above-mentioned decomposition level in the description of hydrodynamic processes. However, the various operations of averaging, closing, and accentuation of high frequency pulsations (generally speaking, ambiguously introduced) have a fairly significant effect on the end result. It is possible to say that these operations introduce a system of equations of coherent-vortex motion, independent of the original hydrodynamic laminar equations of coherent-vortex motion [18]. As an example, we can refer to differences in the approaches between the known and widely used *DNS* and *RANS* calculation procedures [19–21]: direct inclusion into consideration of all the levels of motions of a medium (leading to a considerable increase of time for computer simulation), or the preliminary closing of the equation hierarchy for the highest moment in a rather arbitrary way. We can also mention algorithms (specialized by scales) of the *LES* type (see, for example, [22–24]) and many of their varities, in particular, the vortonic approaches $VEM/VBM/VFM$ [25, 26]. In this work, we limit ourselves only to mentioning procedures of this kind, since for their detailed review a special, rather long, paper is needed.

However, it is necessary to note that, within the approach of classical hydrodynamic equations, procedures with algorithms based on the Euler system equations with certain modifications, caused by the introduction of certain semiphenomenological assumptions (see, for example, O.M. Belotserkovskii [27–29]), are being successfully developed. They are characterized by the possibility of obtaining in the hydrodynamic modeling of large-scale coherent structures very accurate (in terms of coincidence with really observable manifestations of natural processes) results.

At the same time, the expansion of the class of practical tasks, which is unavoidable as a result of the progress in basic research and technologies, evidently leads to considerable problems in the available approaches to the hydrodynamic simulation, such as the impossibility of mass practical calculations because of the difficulty of the algorithm, insolvability of high frequency pulsations, reasoning for the choice in the way of closing the chain of moment equations, etc. Thus, it seems necessary to turn to the analysis of a set of problems associated with the vortical statistical mechanics, including disequilibirum (kinetics of multivortical systems).

2. SIMULATION OF VORTICAL HYDRODYNAMIC FLOWS BY USING METHODS OF STATISTICAL MECHANICS

The conceptual aspects of the mechanics of liquid flows of the vortex kind have been the subject for intensive research already for a rather long time (at least since the middle of the 19th century, starting with the pioneering works of H. Helmholtz and G. Kirchhoff), and accordingly the results obtained during the past one-and-a-half century can be used as the basis for the construction of evolution dynamics of large scale (coherent) structures. The use of methods of the theory of vortices with a maximally simple structure, i.e., Onsager and Rankin vortices, is a natural mathematical formalism here; however, for the cases where a medium's motions may considerably differ from the global states of equilibrium in the system, i.e., in processes associated with transport in the presence of quite a significant degree of anisotropy/inhomogeneity in the system, it is necessary to take into consideration the possibility of the emergence in the considered system of compound evolving subsystems, whose description requires a principal modification of the trivial vortical approach. A procedure based on the mean field theory (G. Joyce and D. Montgomery [30], P. Chen and M.C. Cross [31], etc.) is the simplest and at the same time efficient method here; it can be introduced based on the variational formalism (in particular, under the conditional extremality of the system's entropy) mentioned earlier. The efficiency of this approach in the investigation of about-equilibrium statistical systems is quite evident, which is confirmed by a considerable number of publications by different authors that are devoted to this topic (see, for example, [32–35]); we can state quite definitely that at present, in the construction of the calculation algorithms of the vortical models of flows of different genesis, it seems that the most progressive procedure is the procedure that takes into account just the presence of the self-agreed interaction of the elementary substructures. The method of the mean field can be taken as the formal basis in the construction of the consistent kinetic approach of the analysis of interactions in multivortical ($N \gg 1$) systems having a set of quasi-stationary meta-stable states, possibly having a tendency to relax to the asymptotic collisional equilibrium. Nevertheless, the problem of investigating the appearance and formation of the structure of ordered complexes of vortex-like objects in the process of their interaction in hydrodynamic (or, in the general case, pseudo-hydrodynamic) systems (specifically, those described by the Euler-Helmholtz, Bragg-Hawthorne, Grocco-Vazsonyi, etc., equations) having a set of local states of relative equilibrium with the presence of restrictions—in particular, as the stationarity of the local values of energy, circulation, etc., under the conditions of an extreme value of the spatial distribution of enstrophy (generalized enstrophy, entropy, etc.) of the evolving medium—is at the present stage very far from completion. Approaches to the statement and analysis of this problem (even in the simplest version) are at present at first sight very diverse, although actually they are based on the apparatus of the generalized statistical theory of L. Onsager [36] and its various extensions—formalism of quasi-equilibrium statistical mechanics with limitations in the treatment of the *RSM* (R. Robert-Sommeria-Miller) [37, 38], CVP/EHT (Ellis-Haven-Turkington) [39] methods, or the use of the latter's closely related approach based on the variation of the Casimir-Chavanis functionals [40], as well as the method of variation of the stream-oriented $SFVP$-functional [41] (in a sense similar to the Van der Waals-Cahn-Hillard model [42], describing the coexistence of various phases of matter in ordinary thermodynamics), as also other such approaches (NCD, $MaxS$, $Min\Gamma_2$, etc.), with no more than insignificant differences from the above mentioned ones. It should be noted that actually beyond these procedures, the problem of the direct description of the primary genesis and nonequilibrium development of physical macroobjects of the type considered in this work remains untouched; i.e., the description of (composite) coherent systems, including in themselves many substructures, each corresponding to the local state of the relative stylistic equilibrium of the given system (for example, Onsager vortices, or the carrier's vorticity of a finite size in the fluid flow, described by Euler's equation or by the NSE). In other words, the microscopic dynamics of the aggregation of the substructures, by which the vortices of various types are understood (or of their aggregation ordered in a certain way), was not investigated sufficiently. In order to achieve agreement among the phenomenological asymptotes of specific applied problems, many authors make additional assumptions on the structure of the vertical substructures of the total system and on the introduction of effective (adjustable) parameters into the calculation equations (suitable examples can be found in [43–45]). Although this is logical, it reduces the vortical dynamics to a form of one of the possible calculation procedures of hydrodynamics. Maintenance of the optimal balance between the universality of the approach and suitability for obtaining reasonable results during numerical simulation (using the algorithms based on this approach) is in this case a very difficult problem. This is because of a number of factors, among which it is now necessary to find the stochastics of the properties of vortex systems at their very small number ($N \geq 4$ on a plane), the difficulty of the discernability of the instability of the simulated flow because of the instability of the system of discrete vortices, and the general doubtfulness of the catholicity in the application of approximations of the a priori type factorization of multivortex functions of distribution or of account for correlations among vortices by means of some optimization assumptions (often having a very limited range of applicability in changing the set of the external and internal parameters of the problem).

In particular, we should mention models of the distribution of vorticity in the vortex core, used in practical calculations of the distribution model of vorticity in the vortex core: those of the fractional-power Gussian type, with an empirical cyclon profile, Q-type Leibovich, Rosenhead regularization type, etc. Each of these models can describe sufficiently reliably only some particular types of problems (relating to laminary flows, laminary-turbulent transition in shear flows, detachable turbulent mode of the afterflow, etc.); however, the definition of the elementary vortex geometry by the kind of problem would be difficult without information on the details of the real physics of the process in each case (adaptation of the specific type of vortex structures, optimal for calculating a specific type of vortex structures, is often made a posteriori by iterative selection. In particular, it is necessary to identify the specific class of works for researchers who are members of different research teams, depending on the opportunity and various ways of the regulation of calculations, based on the pointlike vortices by ascribing to them the carriers with a nonzero measure, formal desingularization of the Hamilton-Routh system, etc. The possibility of forming a scale hierarchy of the clusters of the hydrodynamic mesolevel and macrostructures of desingularized pseudo-point vortices seems to have been ignored up to now. The improvement of the formalism of the vortical kinetics in the direction of the use of nonlocal vortical carriers (the development of special numerical algorithms) without adequate inclusion of the effect of the intervortical correlations and/or introduction of strictly justified assumptions on the effect of the dynamic properties of the vortical system on its local topological properties is obviously not very promising; as a means of overcoming this barrierin the development of the above-mentioned formalism, it is possible to propose a geometrodynamic approach, using methods of the Lagrange and Hamilton physical geometries. It is clear that the situation with vortical non-equilibrium transport is in many ways similar to that in the ordinary multiparticle theory (taking into account the logarithmic weakness of the interaction on remote distances and the possibility to change the internal structure of the substructures when they are found to be close to each other—at a kind of inelastic scattering using the analogy with standard processes of molecular atomic theory). In the system of vortices, as in the case of the ordinary multiparticle system, there are two modes of relaxation to equilibrium: collisional and noncollisional (collective interaction through a self-agreed field). However, there are also principal differences, of which the main one is the fact that the idea of equilibrium requires in our case a somewhat different meaning than the Boltzmann distribution in the classical case: the quasi-stationary state characterized by the presence of a local maximum entropy that in the vortical system is not the only one (and depends in the general case on the initial conditions), and it is possible to admit the case of the existence of consecutive transitions between such states, including those with a hierarchical complication of the internal structure of flow: it should be noted that a similar (in a sense) situation arises also in the astrophysical systems with the Lynden-Bell statistics [46], whose evolution is described by the Vlasov-Poisson and Landau equations. The collisional mode in a vortical system can be described using the analog of the Bogol'ubov-Born-Green- Kirkwood (BBGKI) chain for reduced functions of the vortices' distribution in the expanded phase space (extended on the bases of the cotangential over the configuration spaces of the external and internal coordinates, where the latter describe the change in the shape and level of anisotropy of a particular vortex), deduced from the corresponding Lioville equation for the monogenic vortical function N.

The situation with the analysis of the properties of kinetic equation chains for vortical systems has been up to now not quite satisfactory, since practically all the obtained results apply only to the point vortices (see, for example, [47, 48]). It is because of the extreme labor intensity of the preliminary analytical constructions and estimations, as well as the coexisting narrow practical orientation of the research in this direction: the type of vortices used in the course of the numerical simulation, as has been already mentioned, is determined by the physical content of the considered problem, so that actually up to now it has been considered more reasonable to introduce empirical corrections than trying to reveal through the BBGKI chain the specific form of the correlations between the vertical substructures of various forms and structure. A pragmatic approach of this kind evidently leads to the fact that the overall picture does not appear complete here, including in terms of the principal need for the rigorous substantiation of the simulation algorithms; the Onsager vortices are a mathematical abstraction of a high level; thus, the use of the corresponding formalism of the calculation has limits of its own, and in a number of cases they are very important, down to the actual discrepancy with the physical picture of the hydrodynamic processes. For simulation of the fine characteristics of the flows, in particular, those associated with the pulsation character of the flow, with the development of the instability, and with the investigation of the detachable zones, the calculation methods by the use of point vortices are clearly not optimal, since for a description of these effects, one has to considerably increase the number of vortices, which, among other things, leads to the stochastization of their trajectories because of the unlimited increase of the speeds of the point carriers at their mutual approximation (which generates the need for the use of the apparatus of the corresponding fluctuation-dependent equations, for example, those of the Boltzmann/Enskog-Langevin type [49, 50], which in fact turns us back, at a higher level of the complexity of the approach, to the earlier mentioned calculation hybrid subprocedure, HSP).

3. PROTERTIES OF COHERENT STRUCTURES OF HYDRODYNAMIC TYPE

In terms of the physical content of the processes, the coherence is associated first of all with the preserved order of the evolving medium in combination with the corresponding scale of the change in the spatial characteristics of the system. Actually, this event, seen as a special kind of organized movement of liquid, was for the first time described in [51–53]. In today's hydrodynamics, the coherent structures are defined, following [54], as *connected, large-scale, turbulent, liquid masses with the vorticity correlated by phase over the whole area of the space occupied by them*. Similar definitions are given also in [55] and [56], whose authors think that the most characteristic properties of coherent structures are their isolation from the outside flow in the sense that inside these structures the vorticity and macro characteristics are distributed in a quasi-deterministic way, i.e., on average, orderly and, moreover, these structures have a sufficiently long time of existence without a principal change of their species.

The presence of the core descriptive elements in the definition of the considered physical phenomenon brings into it a considerable level of uncertainty. Also, we have to take into consideration the probability that in the taken definition limits of coherence, the medium's motions are described not as an undifferentiated set of structures with homotopically changing sets of parameters but as a whole range of basically different objects, formally collected by a number of attributes, related to the features of quasi-determinism in the given processes (the processes of the occurence of orderliness in the structure of the hydrodynamic flow on the marked scale).

Thus, the use of the mathematical formalism of Hamilton systems of hydrodynamics for descriptive purposes is in this case not only reasonable but possibly quite necessary—first of all, in order to have a full understanding of the flow coherence in a hydrodynamically natural conceptual context, excluding arbitrariness in interpretations (typical of descriptive definitions) and adequately separating phenomena that are externally similar but internally substantially different, which can be put into the class of coherent ones. In other words, in the investigation of this circle of problems, to move from the general physical level to a specific set of problems of mathematical physics, it is necessary first to turn to the possibility of formalizing the basic subject of investigation.

Thus, we will base ourselves on a strict definition of the general coherent structure on Poisson varieties and introduce according to it a definition of the coherent structure of the hydrodynamic type. For this, we will consider at first an abstract evolution equation of the $\partial \mathfrak{f}/\partial t = \widehat{SM}(\mathfrak{f})\delta H(\mathfrak{f})$ type, where $H(\mathfrak{f})$ is an autonomous (Hamilton) functional over the space of the states $\mathscr{F} \ni \mathfrak{f}$, $\delta H(\mathfrak{f})$ is the variation derivative H, which is an element of the space cotangent to space $\mathscr{F}$: $\delta H(\mathfrak{f}) \in T^*_{\mathfrak{f}}\mathscr{F}$, $\widehat{SM}(\mathfrak{f})$: $T^*_{\mathfrak{f}}\mathscr{F} \to T_{\mathfrak{f}}\mathscr{F}$ is the *structural mapping*, an operator determining the skew-symmetric and satisfying the Jacobi Poisson bracket

$$[K_1, K_2](\mathfrak{f}) \equiv \langle \delta K_1(\mathfrak{f}), \widehat{SM}(\mathfrak{f})\delta K_2(\mathfrak{f})\rangle,$$

where $\langle .,. \rangle$ is the internal product in the space of states. If the operator $\widehat{SM}(\mathfrak{f})$ is inconvertible, then its nucleus is not empty and $\widehat{SM}(\mathfrak{f})\delta C(\mathfrak{f}) \equiv 0$, and $C(\mathfrak{f})$ is the *Casimir function* satisfying condition $[H, C] = 0$. The system in the latter case will be the Poisson system (a generalization of the Hamiltonian system).

The Euler-Helmholtz equation in the two-dimensional case $\partial\omega/\partial t = \widehat{SM}(\omega)\delta H(\omega)$ or, explicitly,

$$\partial\omega/\partial t + \widehat{\Lambda}_2 \nabla_2 \psi \cdot \nabla_2 \omega = 0,$$

(where $\widehat{\Lambda}_2$ is the operator of the simplectic structure, a 2×2 matrix) is an example of the Poisson system with the functional $H = 0.5\int \psi\omega dx$ of the energy of the medium's motion in a limited area with the Dirichlet conditions on the boundary and Poisson bracket:

$$[K_1, K_2](\omega) = \langle \delta K_1(\omega), \widehat{SM}(\delta K_2(\omega))\rangle, \quad \widehat{SM} \equiv -\nabla_2\omega \cdot \widehat{\Lambda}_2 \nabla_2,$$

$$\widehat{\Lambda}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad v = \widehat{\Lambda}_2 \nabla_2 \psi.$$

The variation derivative $\delta H(\omega)$ can be calculated proceeding from the earlier introduced form of the Lamb energy $H = H(v) = 0.5\int v^2 dx$: by definition, the first variation $\delta H[\omega; \delta\omega] = \langle \delta H/\delta\omega, \delta\omega\rangle \equiv \int (\delta H/\delta\omega) \cdot \delta\omega dx$; however, on the other hand,

$$\delta H[\omega; \delta\omega] = \int v \delta v dx = \int (\nabla_2 \times \psi)\delta v dx = \int \psi \cdot (\nabla_2 \times \delta v) dx = \int \psi \cdot \delta\omega dx.$$

Thus, $\delta H(\omega) = \psi$. Thus, in the standard record, the second term in the left-hand part of the Euler-Helmholtz equation (EHE) given above in the explicit form is $(\partial\psi/\partial x_1)(\partial\omega/\partial x_2) - (\partial\psi/\partial x_2)(\partial\omega/\partial x_1)$.

If the task of finding the energy extremum in the presence of additional conditions of the type $\mathrm{extr}_\omega\{H(\omega) | I_1(\omega) = \Im_1; \ldots; I_s(\omega) = \Im_s\}$, where $I_{k=\overline{1,s}}$ are additional integrals of motion, consisting of involution ($[I_m, I_n] = 0$, $m, n = \overline{1,s}$), then its solutions will be the *states of relative equilibrium (SRE)* $\Omega(\Im_1, \ldots, \Im_s)$ (determining multiplicities, invariant for motion). In the study of the vortical flow characteristics, it is possible to use integrals of energy, circulation, enstrophy, angular momentum, and linear moment:

$$H(\omega) = \frac{1}{2}\int \psi\omega dx, \quad \Gamma(\omega) = \int \omega dx, \quad W(\omega) = \frac{1}{2}\int \omega^2 dx,$$

$$P(\omega) = \frac{1}{2}\int |x|^2 \omega dx, \quad L(\omega) = (L_1(\omega), L_2(\omega)) = \left(\int x_1\omega dx, \int x_2 \omega dx\right).$$

In particular, in order to obtain the Rankin vortex, it is necessary to solve the problem for minimization with limitations $\min_\omega\{H(\omega) | P(\omega) = \Im_1^{(0)}, \Gamma(\omega) = \Im_2^{(0)}\}$. As a result we have

$$\Omega_{\text{Rank.}}(\mathrm{i}_1^{(0)}) = (\Im_2^{(0)})^2/(4\pi\Im_1^{(0)})\chi(R = \sqrt{4\pi\Im_1^{(0)}/\Im_2^{(0)}}),$$

where $\chi(R)$ is the function-indicator of the circular area $\{x||x| \le R\}$ (moreover, the *function of the energy value* is $H(\mathfrak{I}_1^{(0)},\mathfrak{I}_2^{(0)}) = (\mathfrak{I}_2^{(0)})^2/(16\pi) - (\mathfrak{I}_2^{(0)})^2/(8\pi)\ln(4\mathfrak{I}_1^{(0)}/\mathfrak{I}_2^{(0)})$ and the solution rotates as an aggregate with angular velocity $\omega_{\text{Rank.}} = -\mathfrak{I}_2^{(0)}/(8\pi\mathfrak{I}_1^{(0)})$).

Consideration of the relative equilibrium states for the vortical NSE can in the overwhelming number of the cases of interest be reduced to the analysis of the stability of the inviscid SRE families. For the NSE in the form $\omega = (-\nabla_2\omega \cdot \widehat{\Lambda}_2\nabla_2)\delta H(\omega) + \nu\Delta_2\omega$ (in the flat D area, and the natural boundary condition $\omega = 0$ is given on the ∂D boundary), we assume initially $\nu = 0$ and consider the variation problem with limits, for example, on the enstrophy $\text{extr}_\omega\{H(\omega)|W(\omega) = \mathfrak{I}^{(0)}\}$ having an infinite SRE family SRE $\mathcal{R}_k$ $(k = 0,1,\ldots)$, and the state $\omega^\dagger \in \mathcal{R}_k$ when and only when $-\Delta_2\omega^\dagger = \mu_k\omega^\dagger$, $\{\mu_k\}_{k=0,1,\ldots}$ are corresponding values of the discrete spectrum of the Laplace operator multiplied by −1 in area D (the Euler-Lagrange equation coincides with this problem with its eigenvalues); moreover, $\mu_0 > 0$, $\mu_m < \mu_n$ at $m < n$ $(m,n \in \mathbb{N})$. All the $\mathcal{R}_k$ families are invariant relative to the vortical NSE, but the SRE's amplitude is nonstationary: for any $\omega^\dagger \in \mathcal{R}_{k=0,1,\ldots}$ their evolution is described by the equality $\omega(t) = \omega^\dagger\exp(-\nu\mu_k t)$; moreover, $dH/dt \le -2\nu\mu_0 H$ and $dW/dt \le -2\nu\mu_0 W$. The $\mathcal{R}_0$ family is an attractor for the NSE; at the same time, all the remaining families for $k \ge 1$ are unstable.

Now we provide the basic definition of coherence in a system that is based on the ideology of the variation of functionals in the Poisson systems [57, 58]: as the *coherent structure* (CS) for a (generalized) hydrodynamic system will be the generalized state of equilibrium (*elementary* CS) or a totality ordered in a certain way of a generalized SRE (the CS component) of the problem for variation of the specified Casimir invariant (or of energy, thermodynamic potential, etc.) at the given values of some set of other invariants that is the stationary solution of the Euler equation. *Generalized hydrodynamic systems* refer to dynamic systems whose evolution is described by equations coinciding with the hydrodynamics equations (of Euler, Navier-Stokes, Lamb, etc.), taking into account the formal substitution for dependent variables and additional terms, which actually do not change the structure of the equations (potential volume forces, Coriolis forces, terms accounting for effective dissipation, turbulent viscosity, etc.). Such systems have been studied for a long time, in particular, with respect to plasmodynamics—based on a drift approximation (the Poisson drift equations), applied to the theory of the quasi-geostrophic models of the atmosphere, based on the analysis of baroclinical flows behaviors, etc.

4. DYNAMICS AND KINETICS OF THE ONSAGER VORTICES

Consider the distribution of the Onsager pointed vortices, located at points $r_k = (x_k, y_k)$ with intensities (circulations) $\{\gamma_k\}$ $(k = \overline{1,N}, N \ge 2)$ of the limited one-connection area D (the area is $\text{meas}D \equiv \mathfrak{S}$), on a plane. We determine in the standard way the volume of the phase area of the N vortex system: $\int_{D^N}\prod_j^N dr_j = \mathfrak{S}^N$. Acting according to the ideology of the microcanonic description of the system of N vortices, we introduce the density of states $\phi(E) = \int_{D^N}\delta(E - H_N(r_1,\ldots,r_N))\prod_j^N dr_j$, determining the volume of the phase space per unit of energy E (here $H_N(r_1,\ldots,r_N) = -(4\pi)^{-1}\sum_{k=1}^N\sum_{j=1,j\ne k}^N\gamma_k\gamma_j\ln|r_k - r_j|$ is a function of the Hamilton–Raus N-vortex system, we ignore the influence of the boundary effects). Determine the N-vortex function of the system's distribution

$$\Psi_N(r_1,\ldots,r_N) = (\phi(E))^{-1}\delta(E - H_N(r_1,\ldots,r_N)),$$

in the presence of the normalization condition of Ψ_N as

$$\int_{D^N}\Psi_N(r_1,\ldots,r_N)\prod_j^N dr_j = 1.$$

We introduce the volume of the phase space, which corresponds to the energies of the H_N system, smaller than some given energy E: $\mathcal{V}_{ph.}(E) = \int_{E_{\min}}^E\phi(E')dE'$ ($E_{\min}$ is the minimal energy of the vortical system). This volume monotonically increases from some minimal value $\mathcal{V}_{ph.}^{(0)} > 0$ up to $\mathfrak{S}^N$ at the increase of energy E in the interval $[E_{\min}, E_\infty)$ $(E_\infty \to +\infty$ at the approximation of the pair of vortices $|r_k - r_j| \to 0)$.

Moreover, $\phi(E) = d\mathcal{V}_{ph.}(E)/dE$ will have the maximum (according to the known Rolle theorem) at a certain $E = E_m$ (at $E \to \infty$ we have $\phi(E) \to 0$ to provide the convergence of the interval in the right-hand part of the definition $\mathcal{V}_{ph.}(E) \le \mathfrak{S}^N$ given above).

The equilibrium evolution of the statistical system is described by the mean-field equation (Joyce Montgomery or UDM), linking the local vorticity and the function of current: $\Delta_2\psi = -\langle\omega\rangle$.

The appearance of large structures in a real vortical flow, which can be compared to those in the real ones in the spectral energy interval, can be explained and described through the use of the subequilibrial statistical mechanics of the vortices.

Consider the bounded system $\mathcal{D}$ of area G, occupied by the vortices (circulations γ_i will be thought of as those that have the same sign); as the coordinates x,y are canonically conjugate, the measure of the corresponding phase space of the N-vortical system is finite and presentable as $G^N = \iint\prod_j^N dx_j dy_j$. Moreover, the density of the energy states is equal to $\mu(E) = \int\delta(E - H(x_1, y_1, \ldots, x_N, y_N))\prod_{j=1}^N dx_j dy_j$; and the connection with the equilibrium N-vortical function is $F_N(r_1, \ldots, r_N) = (\mu(E))^{-1}\delta(E - H(x_1, y_1, \ldots, x_N, y_N))$ ($\int F_N \prod_{j=1}^N dx_j dy_j = 1$). The volume of the phase space responsible for energies H, not exceeding the assigned value E, is $\varphi(E) = \int_{E_m}^{E}\mu(E)dE$.

Function $\varphi(E)$ monotonically increases from zero to $\varphi_{\max} = G^N$, with the corresponding increase of the argument from $E = E_m$ to $E = +\infty$. Thus, $\mu(E) = d\varphi(E)/dE$ has the local maximum at $E = E_0$. For the microcanonical ensemble, the entropy S and temperature T are found in the following way: $S = \ln\mu(E)$ and $T = dE/dS$. Obviously, for $E > E_0$, function $S(E)$ decreases and, therefore, in this interval the temperature of the system is negative ($T < 0$). Thus, the qualitative picture of the vortices' behavior depending on E in this area is clear: it is the creation of vortical clusters with a concentration that is found in the significant dependence on the temperature field, and the density of the vortices in them increases with the increase of the energy of states, starting with E_0. In the case $E < E_0$, the temperature is $T > 0$, and the vortices will be accumulated near the area walls.

The dynamics of point-like vortices on a plane can be described by the Euler equations $d(\partial L/\partial\dot{\xi}_k)/dt = \partial L/\partial\xi_k$ ($\xi_k = \{\eta_k, \bar{\eta}_k\}$, $\eta_k = x_k + iy_k$), obtained through variation of action $J_L \equiv \int_0^t L(\xi, \dot{\xi})dt$ on the Lagrange-Chapman function:

$$L(\eta_j, \bar{\eta}_j, \dot{\eta}_j, \bar{\eta}_j) = \frac{1}{2i}\sum_{k=1}^{N}\gamma_k(\bar{\eta}_k\dot{\eta}_k - \eta_k\bar{\eta}_k) - \frac{1}{2}\sum_{k=1}^{N}\sum_{j=1}^{N}\gamma_k\gamma_j\ln((\eta_k - \eta_j)(\bar{\eta}_k - \bar{\eta}_j)).$$

These equations of motion $\dot{\eta}_k = (2\pi i)^{-1}\sum_j{}'\gamma_j/(\bar{\eta}_k - \bar{\eta}_j)$ ($k, j = \overline{1, N}$) in a vortical system can be presented by the Legendre transformation in the Hamiltonian form (for simplicity for equal circulations γ): $\gamma\dot{x}_j = \partial H/\partial y_j$, $\gamma\dot{y}_j = -\partial H/\partial x_j$, where $H(r_j|\gamma) = \gamma^2\sum_{j<k}U_{jk}$ is the Hamiltonian-Kirchhoff function of the system, $U_{jk}(r_j, r_k)$ is the potential of interaction of the vortices; and $U_{jk} = U_{jk}^{(0)} \equiv -(2\pi)^{-1}\ln|r_j - r_k|$.

The Poisson bracket corresponding to this (non-dissipative) Hamiltonian system is

$$\{f_1(r), f_2(r)\} = \sum_{i=1}^{N}\gamma^{-1}\left(\frac{\partial f_1}{\partial x_i}\frac{\partial f_2}{\partial y_i} - \frac{\partial f_1}{\partial y_i}\frac{\partial f_2}{\partial x_i}\right).$$

Defining in the standard way (similarly to the N-partial), the N-vortical function of the distribution $F_N(r_1, \ldots, r_N; t)$ as the density of the probability that at time moment t the ith vortex is located at the point r_i ($i = \overline{1, N}$), we consider the corresponding Liouville equation for F_N:

$$\frac{\partial F_N}{\partial t} + \sum_{i=1}^{N} v_i\frac{\partial F_N}{\partial r_i} = 0.$$

This equation can be presented as $\partial F_N/\partial t = \hat{\Lambda} F_N$, where $\hat{\Lambda}$ is the Liouville operator of the system of N vortices on the plane

$$\hat{\Lambda} = \{H,.\} = -\sum_{i<j}^{N} v_{(i)}^{(j)} \left(\frac{\partial}{\partial r_i} - \frac{\partial}{\partial r_j} \right) = \sum_{i<j}^{N} \hat{\Lambda}_{ij},$$

where $\hat{\Lambda}_{ij} \equiv -v_{(i)}^{(j)} \partial / \partial r_i - v_{(j)}^{(i)} \partial / \partial r_j$. The reduced k-vortical ($k < N$) functions of distribution are found through relationship

$$F_k(r_1, \ldots, r_k; t) = \int F_N(r_1, \ldots, r_k, r_{k+1}, \ldots, r_N; t) dr_{k+1} dr_{k+2} \ldots dr_N.$$

Consecutive integration of the Liouville equation by variables $r_{k+1}, \ldots, r_N$ ($1 \le k < N$) leads to a chain of meshing equations of the BBGKI type of the kind of

$$\frac{\partial F_k(r_1, \ldots, r_k; t)}{\partial t} = \sum_{i<j} \hat{\Lambda}_{ij} F_k(r_1, \ldots, r_k; t) + (N-k) \sum_{i=1}^{k} \int \hat{\Lambda}_{i,k+1} F_{k+1}(r_1, \ldots, r_{k+1}; t) dr_{k+1}.$$

Introducing k-vortical correlation functions for the given chain of equations and using the Ursell-Mayer decompositions

$$F_2(r_1, r_2) = F_1(r_1) F_1(r_2) + G_2(r_1, r_2),$$

$$F_3(r_1, r_2, r_3) = F_1(r_1) F_1(r_2) F_1(r_3) + F_1(r_1) G_2(r_2, r_3) + F_1(r_2) G_2(r_1, r_3) + F_1(r_3) G_2(r_1, r_2) + G_3(r_1, r_2, r_3), \ \ldots,$$

it is possible to obtain within $N \to \infty$, on the condition of the ignorance of the $(k+1)$th order correlation function, a closed combination of k kinetic equations; moreover, $F_1 \sim 1$, $\left| v_{(i)}^{(j)} \right| \sim \gamma \sim 1/N$, $G_2 \sim 1/N$ and in general $G_k \sim 1/N^{k-1}$. In particular, for $k = 1$ and $k = 2$, we obtain

$$\frac{\partial F_1(r_1; t)}{\partial t} = -(N-1) \int \hat{\Lambda}_{12} F_1(r_1; t) F_1(r_2; t) dr_2 = -\langle v \rangle_{F_1} \frac{\partial F_1(r_1; t)}{\partial r_1},$$

$$\langle v \rangle_{F_1} = (N-1) \int v_{(1)}^{(2)} F_1(r_2; t) dr_2,$$

$$\frac{\partial F_1(r_1; t)}{\partial t} + \langle v \rangle_{F_1} \frac{\partial F_1(r_1; t)}{\partial r_1} = (N-1) \int \hat{\Lambda}_{12} G_{12}(r_1, r_2; t) dr_2,$$

$$\frac{\partial G_2(r_1, r_2; t)}{\partial t} = \hat{\Lambda}_{12} (F_1(r_1; t) F_1(r_2; t) + G_2(r_1, r_2; t)) + \int [\hat{\Lambda}_{13} (F_1(r_1; t) G_2(r_2, r_3; t)) + \hat{\Lambda}_{23} (F_1(r_2; t) G_2(r_1, r_3; t))$$

$$+ (\hat{\Lambda}_{13} + \hat{\Lambda}_{23})(F_1(r_3; t) G_2(r_1, r_2; t))] dr_3 \equiv \widehat{\mathcal{N}}(F_1, G_2).$$

The next-to-last equation coincides formally with the Euler hydrodynamic equation and can be juxtaposed to the Vlasov equation (with the self-consistent field), used in the molecular-kinetic theory. The system of the two latter equations can be brought to one equation with the pseudo-collisional terms of different types. As tolerances establishing the explicit kind $\widehat{\mathcal{N}}(F_1, G_2)$, it is possible to include the collective effects, or the terms only of $1/N$ order, as well as transition (its absence) to the Markov limit $t \to \infty$ in the corresponding operator of the Volterra shift by the trajectory of the vortices.

5. APPEARANCE OF VORTICES IN THE SHIFT FLOW AS A CONSEQUENCE OF THE PROCESSES DESCRIBED IN THE TERMS OF MOLECULAR KINETICS

We consider problems associated with the appearance of vortical structures in a liquid in a shift flow (as the most physically evident process in the formation of coherence in a hydrodynamic flow) in terms of the molecular-kinetic theory.

The stationary Boltzmann equation in the n-dimensional ($n = \overline{1,3}$) case after nondimensionalization (using the characteristic values of the flow parameters) is

$$\widehat{\mathcal{G}}[f; \xi] \equiv \hat{D}_x^{[n]}[f] - \xi \left(\hat{L}[f] + \hat{N}[f, f] \right) = 0,$$

$$\hat{L}[f] = 2\omega^{-1/2} \hat{B}[\omega, \omega^{1/2} f], \quad \hat{N}[f, f] = \omega^{-1/2} \hat{B},$$

where $b(\theta, V) = 0.5 V d_0^2 \sin(2\theta)$ is the scatter index ellipsoid for solid spheres of radius $d_0/2$ and $\xi^{-1} = \mathrm{Kn} = \mathrm{Ma}/\mathrm{Re}$ is the Knudsen number (Ma and Re are the Mach and Reynolds numbers, respectively).

The collisional term $\hat{B}[f,f] \equiv \hat{L}[f] + \hat{N}[f,f]$: $X^{(1)} \times X^{(1)} \mapsto X^{(2)}$ is a bilinear integral operator, acting between the Cartesian pair of Banach spaces (BS) of vector functions $X^{(1)} \times X^{(1)}$ and BS $X^{(2)}$, $X^{(m)}\big|_{m=1,2} \subseteq C^{|m-2|}(\mathbb{R}^3_x; H)$, $H = L^2(\mathbb{R}^3_v\big|\omega)$ is the Hilbert space (HS) with scalar product $\langle \varpi_1, \varpi_2 \rangle = \int_{\mathbb{R}^3} \varpi_1(v)\varpi_2(v)dv$, $\hat{L}[.] = \widehat{K}[.] - \nu(v)\cdot(.)$ is the linear part of the collisional operator, $\widehat{K}[.]$ is the self-conjugate integral operator (in HS $H(\mathbb{R}^3_v)$), $\nu(v)$ is the frequency of collisions of the molecules-spheres $(0 < \nu(0) \le \nu(v) \le \infty)$, and $2\widehat{N}[.,.] = \partial^2(\widehat{B}[0;0]/\partial f^2)[.,.]$. The operator $\hat{L}$ is non-positive $(\langle f, \hat{L}[f]\rangle \le 0)$ and is of the Fredholm quality of the zero index $(\mathrm{ind}\,\hat{L} \equiv \dim \mathrm{Ker}\,\hat{L} - \dim \mathrm{coKer}\,\hat{L} = 0)$ with an $(n+2)$-dimensional kernel $\mathrm{Ker}\,\hat{L}$, consisting of functions $\tilde{\chi}_m$:

$$\tilde{\chi}_m\big|_{m=0}^{4} = \sqrt{\omega}\chi_m(v)\big|_{m=0}^{4}, \quad \chi_{m=0} = 1, \quad \chi_m = v_m\,(m = \overline{1,n}), \quad \chi_{m=n+1} = v^2.$$

The Frechet operator derivative, associated with this equation is $\partial\widehat{\mathcal{G}}[0;\xi]/\partial f \equiv \hat{J}(0;\xi) = v_1\partial/\partial x_1 - \xi\hat{L}$. To $\hat{J}[f;\xi] = 0$ we apply the Laplace transformation by coordinate x_1:

$$\hat{\Lambda}[f] = \int_0^{\infty} f(x)e^{-kx_1}dx_1 = \tilde{f}(k), \quad \hat{\Lambda}^{-1}[\tilde{f}] = \frac{1}{2\pi i}\int_{k^0-i\infty}^{k^0+i\infty} \tilde{f}(k)e^{kx_1}dk,$$

where $k^0(\in \mathbb{R}^1)$ is the parameter corresponding to the tolerable absciss of the line-contour in the (semi)plane of the transformation's convergence (in $\mathbb{C}_k$); we obtain $\hat{J}_k[\tilde{f};\xi] \equiv kv_1\tilde{f} - f(x_1 = 0) - \xi(\hat{K}[\tilde{f}] - \nu\tilde{f}) = 0$. We assume $f(x_1 = 0) = 0$, in order to get rid of the inhomogeneity of the equation (as will be seen below, this requirement is insignificant in the structure of the analysis).

In accordance with the property of the local stability of the index and the defect of the Φ-isomorphism $(\hat{L})$ with a relatively limited perturbance in the neighborhood $\Omega_\gamma(0)$ t. $k(\xi) = 0$, whose radius is determined by the value of the minimally reduced module $\gamma(\xi) = \xi/\|\hat{\underline{L}}^{-1}\|$ ($\hat{\underline{L}}$ is the narrowing of operator $\hat{L}$ by $X^{(1)} \ominus \mathrm{Ker}\,\hat{L}$, $\hat{\underline{L}}^{-1}\hat{L} = \hat{L}\hat{\underline{L}}^{-1} = \hat{I} - \hat{P}^{\chi}$, $\hat{\underline{L}}^{-1}\hat{P}^{\chi} = 0$, where $\hat{P}^{\chi}$ is a projector (in H) into a set of added invariants), set $k_{(j)}(\xi)v_1$ of the eigenvalues of the $(k_{(j)} \in \mathbb{C})$ operator $\hat{L}$ is three-dimensional $(j = \overline{0,2})$. The Eigen (generalized) functions of the operator beam $\hat{J}$ are $\tilde{\phi}_{(j)}(v_1;\xi)\exp(k_{(j)}(\xi)x_1)$, where $\mathrm{span}\{\tilde{\phi}_{(j)}(v_1;\xi)\}_{j=0}^{2} = \mathrm{Ker}\,\hat{J}_k$. We introduce endomorphisms $\widehat{P}(\xi)$: $X^{(1)} \mapsto X^{(1)}$, $\widehat{Q}(\xi)$: $X^{(2)} \mapsto X^{(2)}$, whose action in the corresponding BS is determined by equalities

$$\hat{P}(\xi)[^{(1)}f] = \sum_{j=0}^{2}\tilde{\phi}_{(j)}\langle v_1\widetilde{\psi}_{(j)}, {}^{(1)}f\rangle/\widetilde{\Phi}, \quad \hat{Q}(\xi)[^{(2)}f] = \sum_{j=0}^{2}v_1\tilde{\phi}_{(j)}\langle\widetilde{\psi}_{(j)}, {}^{(2)}f\rangle/\widetilde{\Phi},$$

where $\tilde{\psi}_{(j)}(v) = \hat{\underline{L}}^{-1}[v_1\tilde{\phi}_{(j)}]$ and $\widetilde{\Phi} \equiv \sum_{j=0}^{2}\langle v_1\widetilde{\psi}_{(j)}, \tilde{\phi}_{(j)}\rangle$ $({}^{(m)}f \in X^{(m)},\ m = 1,2)$. We have the opportunity of the following formal decomposition of function $f \in X^{(1)}$: $f = \widehat{P}(\xi)[f] + (\hat{I} - \widehat{P}(\xi))[f]$, where $\widehat{P}(\xi)[f] = \sum_{j=0}^{2}h_{(j)}(x_1;\xi)\tilde{\phi}_{(j)}(v_1;\xi)$, and

$$h_{(j)}(x_1;\xi) = (\widetilde{\Phi})^{-1}\langle v_1\widetilde{\psi}_{(j)}, f\rangle, \quad (\hat{I} - \widehat{P}(\xi))[f] \equiv g(x_1, v;\xi) = \sum_{j=0}^{2} g_{(j)}(x_1, v;\xi);$$

here $h_{(j)}(x_1) \in C^1(\mathbb{R}^1)$, $g_{(j)}(x_1, v_1) \in C(\mathbb{R}^1; (\hat{I} - \widehat{P}(\xi))X^{(1)})$. The commutative properties of operators $\widehat{P}, \widehat{Q}$ are non-trivial: $\hat{Q}(\xi)\hat{L} = \hat{L}\hat{P}(\xi) = kv_1\hat{P}(\xi)$, $\hat{Q}(\xi)\hat{J}_k = \hat{J}_k\hat{P}(\xi)$, from which we have

$$\hat{Q}(\xi)\hat{D}_x^{[1]} = \hat{D}_x^{[1]}\hat{P}(\xi), \quad \hat{Q}(\xi)\hat{J}(.;\xi) = \hat{J}(.;\xi)\hat{P}(\xi).$$

Application to both parts of the equation $\hat{J}[f;\xi] = \xi\hat{N}[f;f]$ of operators $\hat{Q}(\xi)$ and $\hat{I} - \hat{Q}(\xi)$ allows us to obtain a system of equations of the Lyapunov-Schmidt branching method ($j = \overline{0,2}$):

$$\widehat{\mathcal{LS}}_1[h_{(j)};g] \equiv \frac{dh_{(j)}}{dx_1} + k_{(j)}(\xi)h_{(j)} - \frac{2\xi}{\tilde{\Phi}}\left(\sum_{s=0}^{2} h_{(s)}^2 \langle\tilde{\psi}_{(j)}, \hat{N}[\tilde{\phi}_{(s)}, \tilde{\phi}_{(s)}]\rangle\right.$$

$$\left. + \sum_{s=0}^{2}\sum_{i=s+1}^{2} h_{(s)}h_{(i)}\langle\tilde{\psi}_{(j)}, \hat{N}[\tilde{\phi}_{(s)}, \tilde{\phi}_{(i)}]\rangle + \sum_{s=0}^{2} h_{(s)}\langle\tilde{\psi}_{(j)}, \hat{N}[g, \tilde{\phi}_{(s)})]\rangle + \langle\tilde{\psi}_{(j)}, \hat{N}[g,g]\rangle\right) = 0,$$

$$\widehat{\mathcal{LS}}_2[h_{(j)};g] \equiv \hat{D}_x^{[1]} g - \xi\hat{L}[g] - \xi(\hat{I} - \hat{Q}(\xi))\hat{N}\left[\sum_{j=0}^{2} h_{(j)}\tilde{\phi}_{(j)} + g, \sum_{j=0}^{2} h_{(j)}\tilde{\phi}_{(j)} + g\right] = 0.$$

The first one of the equations given above, considered as a totality of differential-functional equations, can be viewed as a system of autonomous combined propulsion units (CPUs) of the Riccati type with the coefficients expressed through the values of the functions of a certain (above-mentioned) kind on the second Frechet derivative of the collinional term CPU $\hat{N}[.,.]$.

The second Lyapunov-Schmidt equation can be rewritten as follows: $\hat{J}[g;\xi] = (\hat{I} - \hat{Q}(\xi))\xi\hat{N}[f[h,g;\phi], f[h,g;\phi]] \equiv W(x_1, v)$. As the polar features of the generalized resolvent $k_{(j)}(\xi)$ of the linearized Boltzmann operator are not included in it, there is a possibility to introduce for consideration a limited pseudo-reverse operation $\underline{\hat{J}}^{-1}[.;\xi]$: $(\hat{I} - \hat{Q}(\xi))X^{(2)} \mapsto (\hat{I} - \hat{P}(\xi))X^{(1)}$. Its construction is based on the theory of holomorphic semigroups of operators. We find the action of $\underline{\hat{J}}^{-1}$ by the composition of the reverse Laplace-Bochner D_0-transformation and the pseudo-operator $(\underline{\hat{J}}_k)^{-1}$

$$\underline{\hat{J}}^{-1}[.;\xi] := \frac{1}{2\pi i}\int_{k^0 - i\infty}^{k^0 + i\infty} \exp(kx_1)(\underline{\hat{J}}_k)^{-1}[.;\xi]dk.$$

The concrete presentation of the semigroup depends on the v_1 sign, so it is necessary to consider two families of subgroups $\hat{\mathcal{T}}^{\pm}(x_1)$. For the cases $v_1 \lessgtr 0$, the ways of integration of $\Gamma^{\pm}(\mu)$ in the integral of the right-hand part of the above formula are allowable by the Jordan lemma deformations of lines $k = k^0 \gtrless 0$, intersecting the axis $\Re(k)$ in the interval $]-k^{(-)};0[$ (at $v_1 > 0$) or $]0;k^{(+)}[$ (at $v_1 < 0$, $k^{(\pm)} = -\lim_{v_1 \to \pm\infty} \nu(v_1)/v_1$) and never intersecting the sectors of the k-complex area with the respectively negative and positive components of the generalized core spectrum $\sigma_{ess}(\hat{J}_k)$:

$$g(x_1, v; \xi) = \begin{cases} \displaystyle\int_0^{x_1} \hat{\mathcal{T}}^{+}(x_1 - x_1')W(x_1', v)dx_1', \ v_1 > 0, \\ \displaystyle\int_{x_1}^{\infty} \hat{\mathcal{T}}^{-}(x_1 - x_1')(W(x_1', v) - W(\infty, v))dx_1' + g(\infty, v_1), \ v_1 < 0, \end{cases}$$

$$\left\|\hat{\mathcal{T}}^{\pm}(x_1)[W(v)]\right\|_z < \exp(k^{(\pm)}x_1)\|W(v)\|_{z-1}, \quad \|W\|_z = \operatorname{vrai\,max}\left|(1 + v^2)^{z/2}W(v)\right|.$$

The nonlinear operator is

$$\widehat{\mathcal{LS}}_2[h_{(j)};g] : C^1(\mathbb{R}_+) \times BPG_z \mapsto BQG_{z-1},$$

where $BPG_z \subseteq C(\mathbb{R}^1; (\hat{I} - \hat{P})B_z(\mathbb{R}^3))$, $BQG_{z-1} \subseteq C(\mathbb{R}^1; (\hat{I} - \hat{Q})B_{z-1}(\mathbb{R}^3))$, and $B_z(\mathbb{R}^3)$ are BS functions $\varpi(v)$ with the norm $\|\varpi\|_z$. From what was said above, it is evident that both the Frechet derivative $\partial\widehat{\mathcal{LS}}_2(0,0)/\partial g$ and the operator that is the reverse of it $(\partial\widehat{\mathcal{LS}}_2(0,0)/\partial g)^{-1} \equiv (\underline{\hat{J}}[.;\xi])^{-1}$ are limited; therefore, based on the theorem of the implicit function, it is possible to introduce (in the only possible way) a limited, continuously greatly differentiated mapping $g[h(x)]$ (functional on three-dimensional sets of functions $h_{(j)}(x_1;\xi)$, and $g[0] = 0$). Formal renormalization $h_{(j)}(x_1) = \eta\bar{h}_{(j)}(x_1)$ and $g(x_1, v) = \eta^2\bar{g}(x_1, v)$

$(j = \overline{0,2})$, where $\eta \in \mathbb{R}^1$ is a scale parameter (for example, $\eta = \eta_0 = \min_{j=\overline{0,2}} |k_{(j)}(\xi)|$), leads to a change of operator $\widehat{\mathcal{LS}}_2$ by its homotopical expansion $\widehat{\widehat{\mathcal{LS}}}_2[(\eta, \bar{h}_{(j)}); \bar{g}]$: $(\mathbb{R}^1 \times C^1(\mathbb{R}_+)) \times BPG_z \mapsto BQG_{z-1}$. Consider the solution corresponding to the second equation of the branching system at the above-mentioned normalization of the equation in the neighborhood $\Omega_\rho(0; \bar{g}^\circ; \bar{h}_j^\circ) = \{|\eta| \le \rho; \|\bar{g} - \bar{g}^\circ\|_{BPG_z} \le \rho; \|\bar{h}_{(j)} - \bar{h}_{(j)}^\circ\|_{C^1} \le \rho\}$, where $\bar{h}_{(j)}^\circ$ is some (a priori known) function with the end norm, but $\bar{g}^\circ$ is determined by the relationship $\bar{g}^\circ(x_1, v) := (\hat{\underline{J}}[.;\xi])^{-1}((\bar{h}_{(j)}^\circ(x_1))^2 \xi(\hat{I} - \hat{Q})\widehat{N}\left[\sum_{i=0}^{2} \bar{\phi}_{(i)}, \sum_{i=0}^{2} \bar{\phi}_{(i)}\right])$.

In this neighborhood $(\partial \widehat{\widehat{\mathcal{LS}}}_2[(0, \bar{h}_{(j)}^\circ); \bar{g}^\circ] / \partial \bar{g})^{-1} \equiv (\hat{\underline{J}}[.;\xi])^{-1}$ and based on the above-mentioned theorem on the implicit function, $\bar{g}$ is the functional on dependent variables η, $\bar{h}_{(j)}(x_1)$: $\bar{g} = \bar{g}[\eta; \bar{h}_{(j)}]$. It is possible to show that the iteration scheme is realizable for obtaining functions $\bar{h}_{(j)}$ and $\bar{g}[\eta; \bar{h}_{(j)}]$, using as the initial approximation, for example, $\bar{h}_{(j)}^\circ = \bar{h}_{(j)}^{\text{hydro}}(x_1)$ (the functions obtained from the hydrodynamic calculations) and $\bar{g}^\circ$ in the earlier given form through a pseudo-reverse operator $(\hat{\underline{J}}[.;\xi])^{-1}(\ldots)$ (the radius of convergence of scheme ρ).

The former equation of the Lyapunov-Schmidt system $\widehat{\mathcal{LS}}_1[h_j; g] = 0$ that is of a global character, depending on the configuration variable, can be written as follows:

$$\frac{dh_j}{dx_1} = \sum_{k=0}^{2} \left({}^{[2]}b_{k,j} h_k^2 + \sum_{i=k+1}^{2} {}^{[1,1]}b_{k,j,i} h_k h_i + {}^{[1]}b_{k,j} h_k \right) + {}^{[0]}b_{0,j}[g], \quad j = \overline{0,2},$$

where

$${}^{[2]}b_{k,j} = 2\xi \langle \widetilde{\psi}_{(j)}, \hat{N}[\tilde{\phi}_{(k)}, \tilde{\phi}_{(k)}] \rangle / \widetilde{\Phi}, \quad {}^{[1,1]}b_{k,j,i} = 2\xi \langle \widetilde{\psi}_{(j)}, \hat{N}[\tilde{\phi}_{(k)}, \tilde{\phi}_{(i)}] \rangle / \widetilde{\Phi},$$

$${}^{[1]}b_{k,j} = \xi(k_{(j)}(\xi) + 2\langle \widetilde{\psi}_{(j)}, \hat{N}[g, \tilde{\phi}_{(k)}] \rangle / \widetilde{\Phi}), \quad {}^{[0]}b_{0,j}[g] = \xi \langle \tilde{\psi}_{(j)}, \hat{N}[g, g] \rangle / \widetilde{\Phi}.$$

Having studied the properties of this equation, we obtain the conditions for the appearance of cortices in the laminar flows. They consist of the possibility of presenting the spectrum of the matrix of the Riccati linearized system as a decomposition into eigenvalues lying on an imaginary axis (complex conjugated), and a set of eigenvalues with a negative imaginary part. In real physical systems, such a structure can be obtained in the consideration of the Boltzmann type of equation system (in the simplest case in the consideration of the Lorentz type of gas). Thus, the appearance of vortical structures can be explained by the Hopf theory of bifurcations, for which it is necessary to include the Langevin collisional term into the right-hand part of the kinetic equations. Moreover, the operator spectrum associated with the modified equation shifts and a intersection (with a possible overlapping) takes place of the zones of decay of their imaginary eigenvalues.

6. CONCLUSIONS

The optimal procedures for the construction of the kinetic theory of vortical objects should be oriented toward universality in applications and describe the possibilities of (1) the use of an additive presentation of the cluster function of the distribution of the vortical system (through the medium of the group densities of the probability of metastable elementary vortical clusters of different topologies); (2) the application for designing the corresponding hierarchy of the BBGKI type (a) of the desingularized elementary vortices (including those in the coarse-grained presentation, taking into account the changed statistics) and (b) the totalities of a certain kind of Onsager vortices that are the states of relative equilibrium and approximation, in the sense of the similarity of certain integral characteristics, the pseudo-local Rankin, Kirchhoff, Taylor, and other totalities of a certain kind (clusters of the vortical crystals' type [59]; (3) division of the dynamics of the mesostructures into external and internal, and the endolevel evolution is described in terms of the geodesic motion of the elementary vortices, and that of the exo-level, relative to their clusters, adapted to specific needs (including those of specially designed Lagrange varieties). The developed formalism of the double-level coherence (with the establishment of a certain correspondence of the properties of the meso- and macrostructures) is applicable to both the solution of applied problems and the analysis of the theoretical aspects in the dynamics of the coherent motion of a medium. Moreover, the given formalism has its conceptual justification in the theory of the Back-Cohen super-statistics [60], stating, in particular, that with the presence of scale segregation in the structural design of dynamic systems, the corresponding statistics and typoforms of the stationary states may change in an extremely fundamental way.

However, the analysis of the coherent macrostructures has specifics of its own (related directly to the basis properties of the given objects, following directly from the definition); thus, we propose formulating the procedures of their description based on the fully conditioned approaches or by the hierarchy of the kinetic equations (not directly using the theory of random processes). It seems reasonable to specially note that, generally speaking, the existence of vortex-like structures of different types is not a prerogative for exclusively gas-hydrodynamic flows (among which it is quite reasonable to specially mention some of the seemingly exotic, boundary, subspecies, for whose analysis it is quite possible to use the concepts of shallow water, (quasi-)geographical flows, and Rossby waves). Due to the similarity of the control equations, it is possible to expect a possibility for the implementation of this type of objects also in the theory of the helium-II movement, magnetic hydrodynamics, the dynamics of star clusters, and plasma dynamics (in the use of the apparatus of Vlasov's equation). In particular, in the electronic hydrodynamics (in the formal redenomination of variables: taking charge for the local circulation, the magnetic field strength for the current function), it is possible to obtain equations of the Charney-Hasegawa-Mima-Petviashvili type, whose solutions are vortex-like (and are accessible for observation in the course of the experiments); the presence of the filamentation of the thread-like structures is natural in the computation experiments for high-current electron-proton beams, including the relativistic and quantum-statistical dynamics of a high-flow liquid. However paradoxical the above-mentioned similarity of the control equations may seem, it is one of the causes for the relevance of the development of a suitable kinetic apparatus; the phenomenological corrections, ordinarily used by most of the authors in modeling vortical motions in hydrodynamic problems, cannot be directly transferred to the cases with fundamentally different problems of magnetic hydrodynamics in terms of the physical context, plasma dynamics, etc., and, besides, a change in the set of macroparametrical variables, even in the common hydrodynamics, plasma dynamics, etc., may lead to a fundamental change in the character of the flow (for example, that of Taylor-Couette and Benard). Thus, the above-mentioned corrections cease to effectively perform their role and cannot guarantee the correctness of the obtained results. At the same time, the construction and development of the theory of kinetics of vortical structures, in particular, those that are close in a sense to CS-states, allow the direct study of physically different processes, and the critical effects in the change of the macroparameters of the medium can here be taken into account in a more consistent way, outside the bounds of the limiting phenomenological tolerances (in particular, by the method of the bifurcation theory for solutions of nonlinear equations at the points of the discrete and pseudo-discrete spectra). The genesis and development/decay of the coherent state in the macrostructures need to be taken into account in the description of the change in the topology system in its entirety (the local intracluster transitions of the states in relative equilibrium to close geometrical configurations are, generally speaking, determined not only by the interaction with the nearest neighbors but also by the collective effects due to the influence of the neighboring mesostructures, although in a number of cases there is a possibility for their disconnection in scale). One of the procedures of the description of these processes that appears rational is the introduction of metrics, self-consistent with the dynamics of the coherent macrostructure metrics, on the cotangent stratification over the multiplicity of the external coordinate system (of the clusters) of vortices. Thus, it seems reasonable to turn to a detailed study of the problem (generally speaking, including a whole complex of problems) for constructing the basis for the corresponding physical geometry (Lagrangian or Hamiltonian). The geometrical methods of this kind can evidently be used not only for establishment of the Hamiltonian hierarchical structure of the hydrodynamic flow with the subsequent division of the latter into cluster multiparticle subsets but also for the generalization of the development theory for Riemannian multiformities. For this, it is necessary to also consider the Lagrangian spaces with the corresponding to the nonlinear connectivities and effective metrics. The choice of the proper method for the elevation of the tensor structure of the configuration space leads to the metrics of the Sasaki, Cheeger-Gromoll, etc., type. With the use of the analogs of the Christoffel coefficients of these metrics, it is possible to design kinetic equations with the properties given earlier, which can provide for the a priori given choice of their properties. This appears reasonable for modeling both the multiparticle dynamics on both multiplicities with nonlinear curvature and for taking account of the effects of the influence of the braking radiation, self-action, etc. (for example, in a self-gravitating beaming plasma or dust cloud), and for inclusion of the effects of friction, viscosity and in the general losses of energy, characteristic, in particular, of the NSE and modified Euler equation with the effective account for viscosity.

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