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Two-Dimensional Macroscopic Model of Traffic Flows

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Abstract—The problem of simulating city and highway traffic flows is considered. Existing simulation techniques are reviewed in brief. A two-dimensional model of a synchronized traffic flow based on continuum approach and similar to kinetically consistent difference schemes is developed. Test problems are used to check the model.

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INTRODUCTION

Traffic jam is becoming increasingly important nowadays. Urban streets and their adjacent highways become a continuous traffic flow during rush hours, with vehicles often moving slower than pedestrians. Improving the traffic situation generally involves costly measures that can turn out to be inefficient. Mathematical simulation of traffic flows can help both analyze the existing transport systems and develop new ones.

At present, there are two main concepts for the simulation of traffic flows. The so-called "microscopic" models treat vehicles as separate particles that interact according to certain laws [1–11]. "Macroscopic" or hydrodynamic models represent another type of model. These treat traffic flows as compressible fluids [12–14], leaving separate vehicles out. Note that there are also kinetic (mesoscopic) models based on Boltzmann-like kinetic equations [15–18] that are an intermediate stage between the two previous types of models.

Most of the models developed so far are one-dimensional and do not deal with the way parameters are distributed across the road. In this work, we construct a two-dimensional mathematical model of multilane traffic that allows calculating the flows for the real geometry of roads.

We consider the so-called synchronized traffic flow when distances between vehicles are compatible with their dimensions and their speeds are far from free motion. Under these conditions and assuming that the traffic flow is considered for long road intervals (ten times greater than the vehicle length), we can use continuum approach to obtain equations similar to gas dynamics equations. We introduce the following concepts:

—the density of vehicles $\rho(x, y, t)$ is a function continuously depending on coordinates and time and measured in the number of vehicles in a lane per unit length;

—velocities $u(x, y, t)$ and $v(x, y, t)$ of vehicles along and across the road, respectively; both are continuous functions, with the velocity u measured in kilometers per hour, and v measured in lanes per hour.

In constructing the model, we use an assumption, which is the basis for deriving the quasi gas-dynamic system of equations [20, 21], i.e., additional mass flux that ensures the smoothness of the solution at such distances that are minimal and, therefore, a more detailed description would be pointless.

The terms describing the human will make traffic flow dynamics equations differ from gas dynamics equations. The model implements the following driving strategies:

—driving at a safe speed under the given conditions;

—changing to a faster or less dense lane;

—reaching the destination (target) such as exiting the highway.

MULTILANE TRAFFIC DESCRIPTION

Traffic flow models are conventionally one-dimensional, describing vehicles moving in one lane. To take into account the neighboring lanes (multilane traffic), some models can, for instance, use the corresponding sources on the right-hand sides of equations [18, 19]. The challenge of developing a fully two-dimensional model is that it is impossible to generalize a one-dimensional model for the two-dimensional case in the usual way since motions along and across the road are not equivalent.

First, we consider vehicles moving along the road. Drivers usually try to move at an optimal speed under the given conditions; drivers can accelerate or decelerate. We use the model similar to the one-dimensional model from [13], where the acceleration is proportional to the difference between the optimal and current velocities of the vehicle and inversely proportional to the relaxation time. This is the "follow-the-leader" case.

The accelerating/decelerating force is

$$f = a\rho, \tag{1}$$

the acceleration is

$$a = \frac{u_{\mathrm{eq}} - u}{T}, \tag{2}$$

the equilibrium speed is

$$u_{\mathrm{eq}} = u_f\left(1 - \frac{\rho}{\rho_{\mathrm{jam}}}\right), \tag{3}$$

where u_f is the speed of the free motion of vehicles, ρ_{jam} is the density, at which the vehicles stop moving ("traffic jam"), and the relaxation time is

$$T = t_0\left(1 + \frac{r\rho}{\rho_{\mathrm{jam}} - r\rho}\right), \tag{4}$$

where t_0 and r are the phenomenological constants, the speed limit is

$$0 \le u \le u_{\max}, \tag{5}$$

where $u_{\max}$ is the maximum allowed speed limit.

The analogue of pressure shows the density gradient influencing the traffic flow

$$P = \nu_x \frac{\rho^{\beta_x}}{\beta_x}, \tag{6}$$

where ν_x and β_x are phenomenological constants.

Now, we consider vehicles moving across the road. In this work, we propose a transverse motion model, in which vehicles change to the faster or less dense lane while driving towards their destination.

To reduce the time it takes to reach the destination, drivers try to change to the faster lane. When the density is low, the speed in the lane does not matter since the vehicle can move at any speed the driver likes. Thus, we have the factor ρ in the formula for v_u

$$v_u = k_u \rho \frac{\partial u}{\partial y}, \tag{7}$$

where k_u is the phenomenological constant.

The next term is for the change to a less dense lane; the higher the vehicle's speed, the faster this transition

$$v_d = -k_d u \frac{\partial \rho}{\partial y}, \tag{8}$$

where k_d is the phenomenological constant.

The driver is not only moving alone the road, he also has some target such as a turning he needs to make. To represent the target in the model, we use the point (x_t, y_t) specified for each place on the road $x_t = x_t(x, y)$, $yt = y_t(x, y)$. The desire to pursue the target is inversely proportional to the time left to reach it, but is not greater than unity:

$$w_t = \min\left(\frac{k_t}{T_t}, 1\right) = \min\left(k_t \frac{u}{x_t - x}, 1\right), \tag{9}$$

where k_t is the phenomenological constant. For the motion towards the target (the transverse velocity), we have

$$v_t = w_t u \frac{y_t - y}{x_t - x} = \min\left(k_t \frac{u}{x_t - x}, 1\right) u \frac{y_t - y}{x_t - x}. \tag{10}$$

The transverse velocity of motion $v(x, y, t)$ is the sum of components (7), (8), and (10).

ANALOGY WITH KINETICALLY CONSISTENT DIFFERENCE SCHEMES

We consider the Knudsen number Kn that is the ratio between the reference scale of the medium and the reference scale of the flow. In hydrodynamics, $\mathrm{Kn} < 10^{-3}$; however, for traffic flow $\mathrm{Kn} \sim 0.1$, which is the reason we use kinetically consistent difference schemes (KCDS) [20] in this work, as they work well within the wide range of Knudsen numbers.

One of the assumptions that form the basis for KCDS is additional mass flux that ensures a smooth solution at reference distances of the medium. For instance, in gas dynamics, the free path of the molecule is treated as such a distance. For traffic flows, the reference distance differs along and across the road. It is the distance $\delta(u)$ between the vehicles along the road for the velocity u, and the reference distance across the road is one lane.

In addition, we introduce the minimal time scale to meet the approximation of the continuous medium. We take the time interval of crossing the given point of the road by several vehicles as such a time.

$$\tau_x \sim \frac{\delta(u)}{u}, \quad \tau_y \sim \frac{1}{v}. \tag{11}$$

To make simplify the model, we treat τ_x and τ_y as constants.

We introduce an additional flux W_x in the right-hand side of the equation of continuity to ensure smoothing along the road

$$W_x = \frac{\tau_x}{2} \frac{\partial}{\partial x}(\rho u^2 + P). \tag{12}$$

The diffusion flux associated with the transverse motion of vehicles is

$$W_y = \frac{\tau_y}{2}\left(\frac{\partial \rho v^2}{\partial y} + \nu_y \rho^{\beta_y} \frac{\partial \rho}{\partial y}\right). \tag{13}$$

There are smoothing terms in the momentum equation as well.

TWO-DIMENSIONAL SYSTEM OF EQUATIONS FOR TRAFFIC FLOW DYNAMICS

Generalizing assumptions (1)–(13), we obtain a system of equations for the traffic flow dynamics

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\tau_x}{2} \frac{\partial}{\partial x}(\rho u^2 + P)\right) + \frac{\partial}{\partial x}\left(\frac{\tau_x}{2} \frac{\partial}{\partial y}(\rho u v)\right)$$

$$+ \frac{\partial}{\partial y}\left(\frac{\tau_y}{2}\left(\frac{\partial}{\partial y}(\rho v^2) + \nu_y \rho^{\beta_y} \frac{\partial \rho}{\partial y}\right)\right) + \frac{\partial}{\partial y}\left(\frac{\tau_y}{2} \frac{\partial}{\partial x}(\rho u v)\right) - \frac{\partial}{\partial x}\left(\frac{\tau_x}{2} f\right);$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} + \frac{\partial \rho u v}{\partial y} = f - \frac{\partial}{\partial x} P + \frac{\partial}{\partial x}\left(\frac{\tau_x}{2} \frac{\partial}{\partial x}(\rho u^3 + 3Pu)\right) + \frac{\partial}{\partial x}\left(\frac{\tau_x}{2} \frac{\partial}{\partial y}(\rho u^2 v)\right)$$

$$+ \frac{\partial}{\partial y}\left(\frac{\tau_y}{2}\left(\frac{\partial}{\partial y}(\rho u v^2) + \nu_y \rho^{\beta_y} \frac{\partial}{\partial y}(\rho u)\right)\right) + \frac{\partial}{\partial y}\left(\frac{\tau_y}{2} \frac{\partial}{\partial x}(\rho u^2 v)\right) - \frac{\partial}{\partial x}(\tau_x f u);$$

$$v = k_u \rho \frac{\partial u}{\partial y} - k_d u \frac{\partial \rho}{\partial y} + \min\left(k_t \frac{u}{x_t - x}, 1\right) u \frac{y_t - y}{x_t - x}.$$

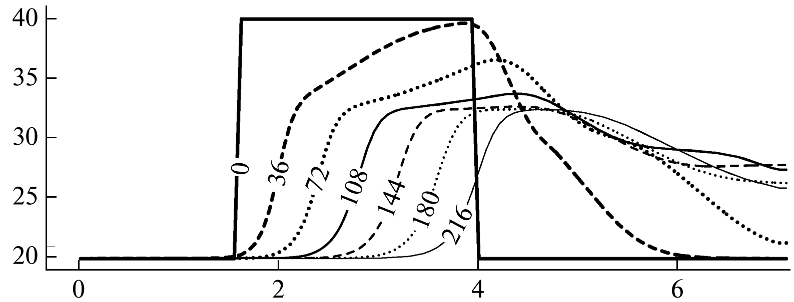

Fig. 1. Evolution of a small jump in density. In the profiles, time is shown in seconds.

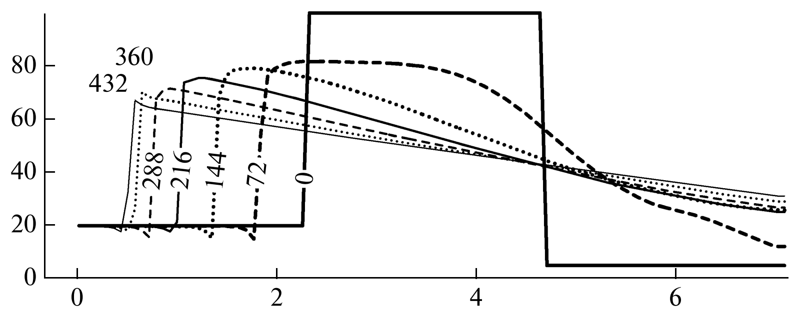

Fig. 2. Evolution of a great jump in density. In the profiles, time is shown in seconds.

MODEL CALIBRATION ISSUE

The model possesses a great number of coefficients and terms that can be chosen arbitrarily. In addition, it is critical to take into account the anisotropy and statistical and experimental data. We performed calculations for the following values of parameters

$$\nu_x = 60, \quad \nu_y = 4, \quad \beta_x = 2, \quad \beta_y = 0, \quad \tau_x = 2\times 10^{-3}, \quad \tau_y = 3\times 10^{-4},$$

$$t_0 = 50, \quad r = 0.95, \quad \rho_{\text{jam}} = 120, \quad u_f = 120, \quad u_{\max} = 90,$$

$$k_u = 0.045, \quad k_d = 0.045, \quad \text{and} \quad k_t = 0.005.$$

These parameters were taken from different estimates and works of other researchers.

NUMERICAL IMPLEMENTATION

One of the advantages of the proposed model is its simple numerical implementation based on approximating by conservative finite difference schemes. Dissipative terms in the right-hand sides of the equations result in additional computational advantages. In this work, we use explicit and semi-implicit methods based on the finite differences of the second-order space approximation.

This allows us to make good use of the new two-dimensional model for simulating flows on roads consisting of any number of lanes. The known one-dimensional multilane traffic models possess high computational complexity for a great number of lanes.

TEST PROBLEMS

Quasi-one-dimensional flow. First, we consider quasi-one-dimensional traffic flows. Figure 1 shows the evolution of a small density "step." We can see that the step moves forward and fades out.

Then, we increase the density of the step. The model nonlinearity results in a qualitatively different behavior in this case (Fig. 2). The step is moving backward until its density falls below some certain value (65 vehicles/km per lane in our case). Then, the step starts moving forward. The propagation of such density jumps often leads to jam.

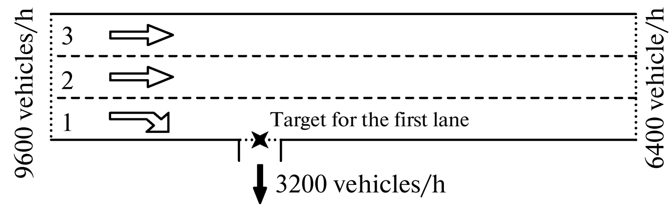

Fig. 3. The plan of a road having a side exit.

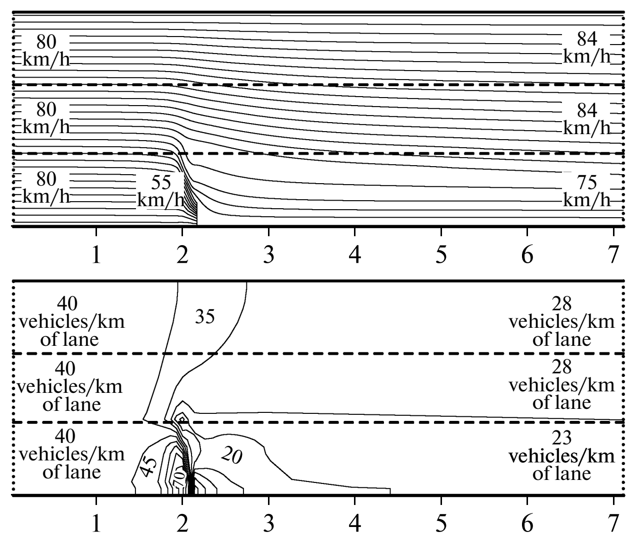

Fig. 4. Simulated results for exit: the speed (above) and the density (below).

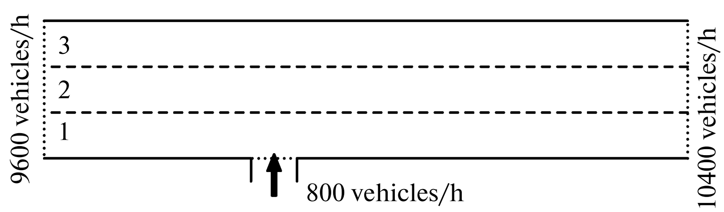

Fig. 5. The plan of a road having a side entry.

Exit. Now, we pass to two-dimensional problems. We consider the case of a three-lane road with an exit (Fig. 3). Suppose all vehicles from the first lane must drive to the exit. We can see (Fig. 4) that the density grows right before the exit. Then, vehicles are distributed uniformly across the road with a lower density and higher speed, which is in good correspondence with the observations.

Here, it is important to note that the model does not take into account the cases when, for instance, only some of the vehicles from some place of the road should drive to the exit. The reason for this restriction is that the target can be set for all vehicles at the given point of the road. To allow vehicles to have different targets at one point, we need to consider a multiphase flow. This is a subject for further study.

Entry. We consider the source of vehicles (the entry). There can be two different modes of motion for the road configuration shown in Fig. 5. The first case is the source of a sufficiently low flow rate (800 vehicles per hour). The density behind the entry is higher than in front of it (Fig. 6).

If we increase the number of vehicles entering the highway from 800 to 2800 vehicles per hour, we obtain a qualitatively different result (Fig. 7). The stream lines show that the entry acts as an obstacle on the road. It causes the density to increase in front of it, with this increase propagating to the beginning of the simulated part of the road. Finally, the density behind the source is less than the density in front of it.

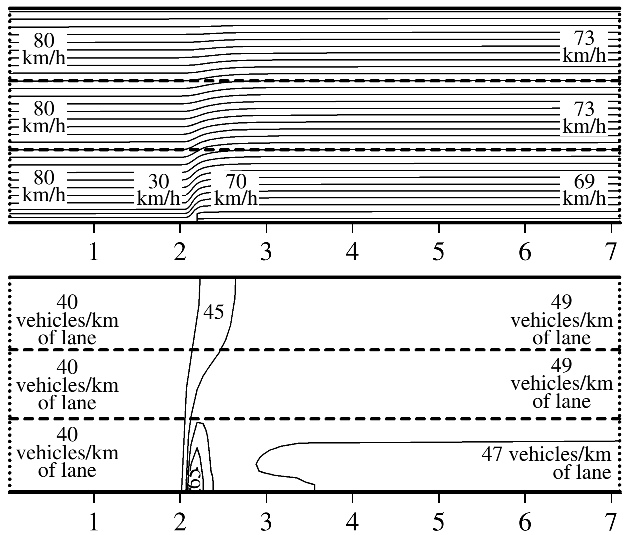

Fig. 6. Simulated results for entry: the speed (above) and the density (below).

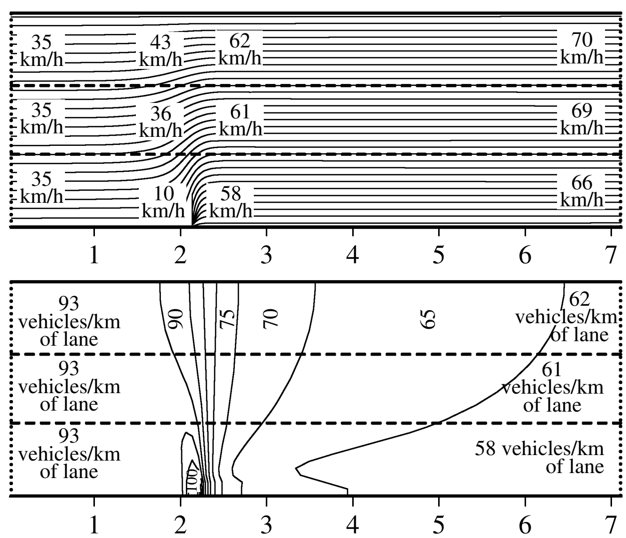

Fig. 7. Simulated results for entry, for a large vehicular flow: the speed (above) and the density (below).

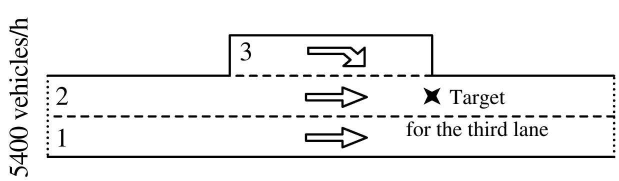

Fig. 8. The plan of a road having a local widening.

Local widening of the road. We consider the road configuration shown in Fig. 8. The density of vehicles falls at the wide part of the road (Fig. 9) but when the traffic flow shrinks back from three to two lanes, the speed falls significantly. Thus, the total time required to travel along the simulated part of the road grows as compared to the road with no widening.

If we increase the density of the entering flow from 30 to 40 vehicles/km of the lane, the vehicles will start gathering on the road (Fig. 10). The initial condition $\rho = 40$ on the road and $\rho \approx 0$ for the widening.

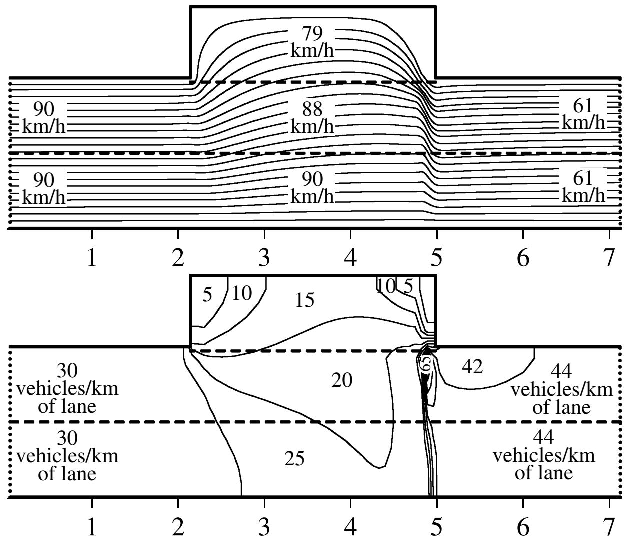

Fig. 9. Simulated results for widening: the speed (above) and the density (below).

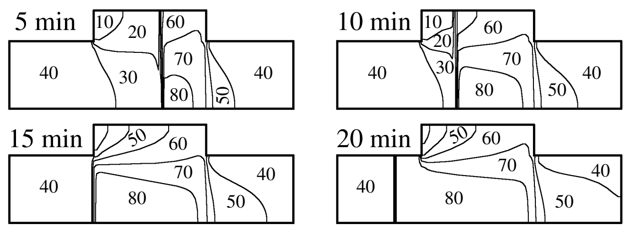

Fig. 10. Simulated results for widening for a large vehicular flow. Figures stand for the density of vehicles.

We can see that in this case the domain with high density is propagating to the left. This is the worst case when a temporary widening of the road makes its traffic capacity decrease significantly.

CONCLUSIONS

The proposed model is designed to describe multilane motion in a two-dimensional setting using continuum approach.

The developed model, algorithms, and software will help predict the motion of synchronized traffic flows and lead to recommendations for preventing traffic jam, analyze the influence of the geometric characteristics of the road, and solve many other problems of traffic flow dynamics.

In future, we plan to extend the model to describe multiphase motion and the interaction of roads (such as junctions).

The model can be adapted to describe the motion of people in crowded places.

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