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RESEARCH ARTICLES

Approximation by Vallée-Poussin Type Means of Vilenkin-Fourier Series

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Abstract—We estimate the degree of approximation by linear means of Vallée-Poussin type of Vilenkin-Fourier series in classical Lebesgue spaces and in a space of generalized continuous functions. These results generalize ones obtained by I. Blahota and G.Gat for means of Walsh-Fourier series.

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1. INTRODUCTION

Let $\mathbf{P} = \{p_i\}_{i=1}^{\infty}$ be a sequence of natural numbers such that $2 \le p_i \le N$. We put $m_0 = 1$, $m_n = p_1 p_2 \dots p_n$ for $n \in \mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{Z}(p_i) = \{0, 1, \dots, p_i - 1\}$, $i \in \mathbb{N}$. Every number $x \in [0, 1)$ can be written as

$$x = \sum_{j=1}^{\infty} x_j/m_j, \quad x_j \in \mathbb{Z}(p_j), \quad j \in \mathbb{N}. \tag{1.1}$$

The expansion (1.1) is unique, if for $x = k/m_l$, $k, l \in \mathbb{N}$, we take the representation with finite number of $x_j \neq 0$.

Every $k \in \mathbb{Z}_+ = \{0, 1, ...\}$ can be expressed uniquely in the form

$$k = \sum_{j=1}^{\infty} k_j m_{j-1}, \quad k_j \in \mathbb{Z}(p_j), \quad j \in \mathbb{N}. \tag{1.2}$$

For $x \in [0, 1)$ and $k \in \mathbb{Z}_+$ with expansions (1.1) and (1.2) we define the function

$$\chi_k(x) = \exp\left(2\pi i \sum_{j=1}^{\infty} x_j k_j/p_j\right) = \prod_{j=1}^{\infty} (\exp(2\pi i x_j/p_j)^{k_j}.$$

As usually, the space $L^p[0, 1)$, consists of all measurable on $[0, 1)$ functions such that $\|f\|_p = \left(\int_0^1 |f(x)|^p \, dx\right)^{1/p}$. It is well known that Vilenkin system $\{\chi_n(x)\}_{n=0}^{\infty}$ is orthonormal and complete in $L^p[0, 1)$, $1 \le p < \infty$ (see [7, § 1.5]). Therefore, we define the Vilenkin-Fourier coefficients and partial Vilenkin-Fourier sums of $f \in L^1[0, 1)$ by formula

$$\hat{f}(j) = \int_0^1 f(x)\overline{\chi_j(x)} \, dx, \ \ j \in \mathbb{Z}_+, \ \ S_n(f)(x) = \sum_{k=0}^{n-1} \hat{f}(k)\chi_k(x), \ \ n \in \mathbb{N}.$$

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Let $G(\mathbf{P})$ be the group with elements $\widetilde{x} = (x_1, x_2, \dots)$, $x_j \in \mathbb{Z}(p_j)$, and addition $\widetilde{x} \oplus \widetilde{y} = \widetilde{z}$, where $z_j = x_j + y_j \pmod{p_j}$, $j \in \mathbb{N}$. The inverse operation $\widetilde{x} \ominus \widetilde{y}$ is defined in a similar way.

The function $\lambda_{\mathbf{P}}(\widetilde{x}) = \sum_{j=1}^{\infty} x_j/m_j$ maps $G(\mathbf{P})$ onto $[0,1]$. It is not bijective since elements of the type $x = k/m_l$, $k, l \in \mathbb{N}$, $k < m_l$, have two different prototypes. If for such x we set $x_j = [m_j x] \pmod{p_j}$, $j \in \mathbb{N}$, then we define inverse mapping $\lambda_{\mathbf{P}}^{-1}$ by $\lambda_{\mathbf{P}}^{-1}(x) = (x_1, \dots, x_l, 0, 0, \dots)$. For other $x \in [0,1)$ there exists the unique element $\widetilde{x} \in G(\mathbf{P})$ such that $\lambda_{\mathbf{P}}(\tilde{x}) = x$. Then we set $\lambda_{\mathbf{P}}^{-1}(x) = \widetilde{x}$.

We can define a generalized distance $\rho(x,y) = \lambda_{\mathbf{P}}(\lambda_{\mathbf{P}}^{-1}(x) \ominus \lambda_{\mathbf{P}}^{-1}(y))$ and an addition $x \oplus y = \lambda_{\mathbf{P}}(\lambda_{\mathbf{P}}^{-1}(x) \oplus \lambda_{\mathbf{P}}^{-1}(y))$ on $[0,1)$. The last operation is not defined if $\lambda_{\mathbf{P}}^{-1}(x) \oplus \lambda_{\mathbf{P}}^{-1}(y) = \widetilde{z}$, where $z_j = p_j - 1$ for all $j \geq j_0$. If $x \in [0,1)$ is fixed, then $x \oplus y$ is defined for a.e. $y \in [0,1)$ (more precisely, $x \oplus y$ is not defined for countable set of y). The operation $x \ominus y$ is introduced in a similar way.

For $f, g \in L^1[0,1)$ the convolution $f * g$ is defined by

$$f * g(x) = \int_0^1 f(x \ominus t) g(t)\, dt = \int_0^1 f(t) g(x \ominus t)\, dt.$$

From the Fubini theorem it follows that $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$. It is easy to see that $S_n(f) = f * D_n$, $D_n(x) = \sum_{j=0}^{n-1} \chi_j(x)$, $n \in \mathbb{N}$.

For $f \in L^p[0,1]$, $1 \leq p < \infty$, we introduce a modulus of continuity

$$\omega^*(f,\delta)_p = \sup_{0<h<\delta} \|f(x \oplus h) - f(x)\|_p, \quad \delta \in [0,1].$$

Instead of $L^\infty[0,1)$ we consider the space $C^*[0,1) = C^*(\mathbf{P}, [0,1))$ consisting of measurable on $[0,1)$ functions $f(x)$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for $x, y \in [0,1)$ with $\rho(x,y) < \delta$, the inequality $|f(x) - f(y)| < \varepsilon$ holds. The space $C^*[0,1)$ with uniform norm $\|f\|_\infty = \sup_{x \in [0,1)} |f(x)|$ is a Banach space. The modulus of continuity for $f \in C^*[0,1)$ is

$$\omega(f,\delta)_\infty = \sup_{\rho(x,y)<\delta} |f(x) - f(y)|.$$

It is known that $\{\omega^*(f, 1/m_n)_p\}_{n=0}^{\infty}$ can be an arbitrary nonincreasing sequence tending to zero (see [1, Ch. 2] in the cases $p = 1, 2, \infty$ and [6] for all $1 \leq p \leq \infty$).

Let $\mathcal{P}_n = \{f \in L^1[0,1) : \widehat{f}(j) = 0, j \geq n\}$, $n \in \mathbb{N}$. Then for $f \in L^p[0,1)$, $1 \leq p < \infty$, or $f \in C^*[0,1)$ in the case $p = \infty$ we define the best approximation by Vilenkin polynomials as $E_n(f)_p = \inf_{g \in \mathcal{P}_n} \|f - g\|_p$. By $\tau_n(f) \in \mathcal{P}_n$ we denote the unique polynomial of best approximation such that $\|f - \tau_n(f)\|_p = E_n(f)_p$.

By definition, if $\omega(t)$ is nondecreasing and continuous on $[0,1]$, $\omega(0) = 0$, then $\omega \in \Phi$. If $\omega \in \Phi$ and

$$\int_0^\delta t^{-1} \omega(t)\, dt \leq C\omega(\delta), \quad \delta \in (0,1),$$

then ω belongs to the Bary class B; if $\omega \in \Phi$, $\alpha > 0$ and

$$\delta^\alpha \int_\delta^1 t^{-\alpha-1} \omega(t)\, dt \leq C\omega(\delta), \quad \delta \in (0,1),$$

the ω belongs to the class Bary-Stechkin class B_α. For $\omega \in \Phi$ and $1 \leq p < \infty$ we consider a Hölder type class

$$H_p^\omega[0,1) = \{f \in L^p[0,1) : \omega(f,\delta)_p = O(\omega(\delta)),\ \delta \in [0,1]\}.$$

Let $A = (a_{n,k})_{n,k=1}^{\infty}$ be a lower triangle matrix of complex numbers and let the A-transform of $\{S_n(f)(x)\}_{n=1}^{\infty}$ be given by $T_n(f)(x) = \sum_{k=1}^{n} a_{n,k} S_k(f)(x)$. In the present paper we consider a Vallée-Poussin type means

$$T_{n,m}(f)(x) = \sum_{k=m}^{n} a_{n,k} S_k(f)(x), \quad m, n \in \mathbb{N}, \quad m \leq n, \tag{1.3}$$

where

$$a_{n,k} \geq 0, \quad n, k \in \mathbb{N}, \quad \sum_{k=m}^{n} a_{n,k} = 1. \tag{1.4}$$

Also we consider two conditions on coefficients a_{nk} generalizing monotonicity properties.

$$\sum_{k=m}^{n-1} |a_{n,k} - a_{n,k+1}| =: \sum_{k=m}^{n-1} |\Delta a_{n,k}| \leq K a_{nm}, \quad n \in \mathbb{N}. \tag{1.5}$$

$$\sum_{k=m}^{n-1} |\Delta a_{n,k}| \leq K a_{nn}, \quad n \in \mathbb{N}, \tag{1.6}$$

where K is independent of n and m. Close conditions were used by Leindler [9] for linear means of trigonometric Fourier series. Chandra [5] and later Leindler [9] considered linear means of trigonometric Fourier series as linear combinations of corresponding partial sums. This approach do not use estimates of Lebesgue constants which were applied in earlier works. Similar to [5] and [9] results for linear means of Vilenkin-Fouier series were proved by Iofina and Volosivets in [8].

In the paper of Blahota and Gát [4] the means of type (1.3) were considered for Walsh-Fourier series (i.e. for $\{\chi_k\}_{k=0}^{\infty}$ in the case $p_j \equiv 2$). They used the condition of monotonicity of $\{a_{nk}\}_{k=m}^{n}$. The aim of our paper is to generalized the results from [4] to the case of Vilenkin systems with bounded generating sequence $\mathbf{P}$ and more general conditions (1.5) and (1.6). Our proofs are more simple and brief than ones in [4].

2. AUXILIARY PROPOSITIONS

Lemma 2.1. *For $n \in \mathbb{N}$ one has $\int_0^1 D_n(x)\,dx = 1$ and $|D_n(x)| \leq n$, $x \in [0, 1)$. On the other hand, $D_{m_n}(x) = m_n$ for $x \in [0, m_n^{-1})$ and $D_{m_n}(x)$ vanishes for $x \in [m_n^{-1}, 1)$.*

Proof. The first statement of Lemma follows from the equality $\int_0^1 \chi_n(x)\,dx = 0$, $n \in \mathbb{N}$ (see [7, § 1.5]), the second one is obvious. Third is proved in [7, § 1.5]). □

Lemma 2.2 is proved in [1, Ch. 4,§ 3].

Lemma 2.2. *If $n \in \mathbb{N}$ and $x \in (0, 1)$, then $|D_n(x)| \leq N/x$, where $p_n \leq N$ for all $n \in \mathbb{N}$.*

Lemma 2.3. *Let $n \in \mathbb{N}$, $F_n(x) = \sum_{k=1}^{n} D_k(x)/n$.*

(i) If $n \in [m_{s-1}, m_s) \cap Z$, $s \in \mathbb{N}$, then

$$|nF_n(x)| \leq C_1 \sum_{\nu=0}^{s-1} m_\nu \sum_{i=\nu}^{s-1} \left(D_{m_i}(x) + \sum_{l=0}^{p_{\nu+1}-1} D_{m_i}(x \oplus l/m_{\nu+1}) \right). \tag{2.1}$$

(ii) For all $x \in (0, 1)$ and $n \in \mathbb{N}$ the inequality $|nF_n(x)| \leq C_2 x^{-2}$ is valid.

(iii) For all $n \in \mathbb{N}$ we have $\|F_n\|_1 \leq C_3$.

All constants C_1, C_2, C_3 are independent of n and x.

Proof. Assertions (i) and (iii) are proved by Pal and Simon in [10] (for (iii) see also [1, Ch. 4, § 10]). In the case of (ii) we take $x \in [m_{r+1}^{-1}, m_r^{-1})$, $r \in \mathbb{Z}_+$. If ν from the right-hand side of (2.1) is greater than r, then $i \geq \nu > r$. Therefore, we have $m_i^{-1} \leq m_{r+1}^{-1}$ and $l/m_{i+1} < 1/m_i \leq 1/m_{r+1}$ for all $l \in \mathbb{Z}(p_{i+1})$. We conclude that for such ν and $i \geq \nu$ both numbers x and $x \oplus l/m_{i+1}$ belong to $[m_{r+1}^{-1}, m_r^{-1})$ for all

$l \in \mathbb{Z}(p_{i+1})$ and that by Lemma 2.1 the equality $D_{m_i}(x) = D_{m_i}(x \oplus l/m_{i+1}) = 0$ holds. Thus, in the right-hand side of (2.1) we take only $\nu \le r$ and $i \le r$ to obtain by Lemma 2.1

$$|nF_n(x)| \le C_1 \sum_{\nu=0}^{r} m_\nu \sum_{i=\nu}^{r} (N+1) m_i \le C_1(N+1) \sum_{\nu=0}^{r} m_\nu \sum_{i=0}^{r} m_i \le$$

$$\le 4C_1 m_r^2 (N+1) \le 4C_1(N+1)x^{-2}.$$

The statement of (ii) is proved. □

Lemma 2.4 is the famous Watari-Efimov inequality (see [7, Ch. 10,§ 10.5].

Lemma 2.4. *Let $f \in L^p[0,1)$, $1 \le p < \infty$, or $f \in C^*[0,1)$ $(p = \infty)$. Then*

$$2^{-1}\omega^*(f, m_n^{-1})_p \le E_{m_n}(f)_p \le \|f - S_{m_n}(f)\|_p \le \omega^*(f, m_n^{-1})_p, \quad n \in \mathbb{Z}_+.$$

Lemma 2.5. *Let n, $m = m(n)$ and $\{a_{nk}\}_{n,k=1}^{\infty}$ satisfy the condition* (1.5)*. Then $a_{nn} \le C a_{nm}$, where C depends only on K. If they satisfy* (1.6)*, then $a_{nn} \ge C a_{nm}$.*

Proof. If (1.5) holds, then we write

$$a_{nn} - a_{nm} = \sum_{k=m}^{n-1} (a_{n,k+1} - a_{n,k}) \le \sum_{k=m}^{n-1} |a_{n,k+1} - a_{n,k}| \le K a_{nm},$$

i.e. $a_{nn} \le (K+1)a_{nm}$. The second statement of Lemma is proved in the same way. □

Lemma 2.6. *Let $1 < p < \infty$. Then the operators S_n are bounded in $L^p[0,1)$ and for $f \in L^p[0,1)$ we have*

$$\|f - S_n(f)\|_p \le C E_n(f)_p, \quad n \in \mathbb{N}.$$

Proof. The first statement of Lemma 2.6 was proved in 1976 independently by Schipp, Simon and Young (see [11]). The second statement follows from the first one by a standard procedure (see, e.g., [3, Ch. 7, § 20]). □

3. MAIN RESULTS

Theorem 3.1. *Let $f \in L^1[0,1)$, $n, m = m(n)$ are natural numbers and $\{a_{jk}\}_{j,k=1}^{\infty}$ satisfy the conditions* (1.4) *and* (1.6)*. If $a_{nn} = O(n^{-1})$, $n \in \mathbb{N}$, and $r \in Z_+$ is defined by the condition $m \in [m_r, m_{r+1})$, then*

$$\|f - T_{m,n}(f)\|_1 \le C\omega^*(f, m_r^{-1})_1. \tag{3.1}$$

Proof. Using (1.4) and the equality $S_k(\tau_m(f)) = \tau_m(f)$ for $k \ge m$ we obtain

$$f - T_{mn}(f) = f - \tau_m(f) - \sum_{k=m}^{n} a_{nk} S_k(f - \tau_m(f)),$$

where $\tau_m(f) \in \mathcal{P}_m$ is the polynomial of best approximation of order m for f in $L^1[0,1)$. Applying Lemma 2.3 (iii), summation by parts, equality $\sum_{j=1}^{k} D_j = kF_k$ and the convolution inequality $\|h * g\|_1 \le \|h\|_1 \|g\|_1$, $h, g \in L^1[0,1)$, we have

$$\|f - T_{mn}(f)\|_1 \le \|f - \tau_m(f)\|_1$$

$$+ \left\| \sum_{k=m}^{n} a_{nk} D_k * (f - \tau_m(f)) \right\|_1 \le E_m(f)_1$$

$$+C_1\left\|\sum_{k=m}^{n-1}\Delta a_{nk}kF_k+na_{nn}F_n-(m-1)a_{nm}F_{m-1}\right\|_1\|f-\tau_m(f)\|_1$$

$$\leq E_m(f)_1+C_1\left(\sum_{k=m}^{n-1}k|\Delta a_{nk}|+na_{nn}+(m-1)a_{nm}\right)E_m(f)_1. \tag{3.2}$$

By the condition (1.6) and Lemma 2.5 we find that

$$\sum_{k=m}^{n-1}k|\Delta a_{nk}|+na_{nn}+(m-1)a_{nm}$$

$$\leq C_2na_{nn}+na_{nn}+(m_1)a_{nm}\leq C_3na_{nn}\leq C_4$$

and $\|f-T_{mn}(f)\|_1\leq(C_1C_4+1)E_m(f)_1$. Since

$$E_m(f)_1\leq E_{m_r}(f)_1\leq\omega^*(f,m_r^{-1})_1$$

be Lemma 2.4, we obtain (3.1) □

Theorem 3.2. *Let $f\in L^1[0,1)$, $n,m=m(n)$ be natural numbers, r be defined as in Theorem 1 and $\{a_{jk}\}_{j,k=1}^{\infty}$ satisfy the conditions (1.4) and (1.5). If $a_{nm}=O(n^{-1})$, $n\in\mathbb{N}$, and $r\in Z_+$ is defined by the condition $m\in[m_r,m_{r+1})$, then (3.1) holds.*

Proof. We have (3.2) again. By the condition (1.5) and Lemma 2.5 we find that

$$\sum_{k=m}^{n-1}k|\Delta a_{nk}|+na_{nn}+(m-1)a_{nm}\leq\sum_{k=m}^{n-1}n|\Delta a_{nk}|+na_{nn}+na_{nm}$$

$$\leq C_1na_{nm}+na_{nn}+na_{nm}\leq C_2$$

and $\|f-T_{mn}(f)\|_1\leq C_3E_m(f)_1$. As in the proof of Theorem 1, we deduce (3.1). □

Corollary 3.3. *Under conditions of Theorem 1 or Theorem 2 the inequality $\|f-T_{mn}(f)\|_1\leq CE_m(f)_1$ holds.*

Corollary 3.4. *(i) Let $f\in L^1[0,1)$, $n,m=m(n)\in\mathbb{N}$, r be defined as in Theorem 1, $\{a_{jk}\}_{j,k=1}^{\infty}$ satisfy the condition (1.4) and $\{a_{nk}\}_{k=m}^{n}$ be nondecreasing for every $n\in\mathbb{N}$. If If $a_{nn}=O(n^{-1})$, $n\in\mathbb{N}$, then (3.1) holds.*

(ii) Let $f\in L^1[0,1)$, $n,m=m(n)\in\mathbb{N}$, r be defined as in Theorem 1, $\{a_{jk}\}_{j,k=1}^{\infty}$ satisfy the condition (1.4) and $\{a_{nk}\}_{k=m}^{n}$ be nonincreasing for every $n\in\mathbb{N}$. If $a_{nm}=O(n^{-1})$, $n\in\mathbb{N}$, then (3.1) holds.

Remark 3.5. *In the case $p_i\equiv p$ the number r in (3.1) is $[\log_p m]$, where $[x]$ is the integer part of x. The result of Corollary 3.4 (i) for $p_i\equiv 2$ is Theorem 4.1 from [4], while the result of Corollary 3.4 (ii) for $p_i\equiv 2$ coincides with one of Theorem 4.2 in [4]. The theorem 4.3 in the same paper [4] is contained in Theorem 4.2 since for $2^l\leq m<n<2^{l+1}$ the conditions $a_{nm=O(m^{-1})}$ and $a_{nm}=O(n^{-1})$ are equivalent.*

The analogues of Theorem 3.1 and 3.2 are valid for $f\in C^*[0,1)$. We combine them into

Theorem 3.6. *(i) Let $f\in C^*[0,1)$, $n,m=m(n)$ are natural numbers, r be as in Theorem 3.1 and $\{a_{jk}\}_{j,k=1}^{\infty}$ satisfy the conditions (1.4) and (1.6). If $a_{nn}=O(n^{-1})$, $n\in\mathbb{N}$, then*

$$\|f-T_{m,n}(f)\|_\infty\leq C\omega^*(f,m_r^{-1})_\infty. \tag{3.3}$$

(ii) Let $f\in C^*[0,1)$, $n,m=m(n)$ are natural numbers, r be as in Theorem 3.1 and $\{a_{jk}\}_{j,k=1}^{\infty}$ satisfy the conditions (1.4) and (1.5). If $a_{nm}=O(n^{-1})$, $n\in\mathbb{N}$, then (3.3) holds.

Proof. We repeat the arguments of proofs of Theorems 3.1 and 3.2 and use the almost obvious convolution inequality $\|h * g\|_\infty \le \|h\|_\infty \|g\|_1$ for $h \in C^*[0,1)$, $g \in L^1[0,1)$. □

In $L^p[0,1)$, $1 < p < \infty$, we obtain a more sharp result.

Theorem 3.7. *(i) Let $1 < p < \infty$, $f \in L^p[0,1)$, $n, m = m(n)$ are natural numbers, r be as in Theorem 3.1 and $\{a_{jk}\}_{j,k=1}^\infty$ satisfy the condition* (1.4). *Then*

$$\|f - T_{m,n}(f)\|_p \le C_1 \sum_{k=m}^n a_{nk} E_k(f)_p \le C_2 \omega^*(f, m_r^{-1})_p. \tag{3.4}$$

Proof. By Lemma 2.6 we have

$$\|f - T_{mn}(f)\|_p = \left\| \sum_{k=m}^n a_{nk}(f - S_k(f)) \right\|_p \le \sum_{k=m}^n a_{nk} \|f - S_k(f)\|_p$$

$$\le C_1 \sum_{k=m}^n a_{nk} E_k(f)_p \le C_1 E_m(f)_p \sum_{k=m}^n a_{nk} = C_1 E_m(f)_p.$$

As in the proof of Theorem 3.1 we obtain (3.4). □

If we consider a function from a generalized Hölder class, then we can sharpen the estimates of Theorems 3.1 and 3.2.

Theorem 3.8. *Let $f \in L^1[0,1)$, $n, m = m(n)$ are natural numbers and $\{a_{jk}\}_{j,k=1}^\infty$ satisfy the conditions* (1.4) *and* (1.6)*, $\omega \in B \cap B_1$ and $f \in H_1^\omega[0,1)$. Then*

$$\|f - T_{m,n}(f)\|_1 \le C(1 + n a_{nn}) \omega(n^{-1})_1. \tag{3.5}$$

Proof. By Lemma 2.1, (1.4), summation by parts and generalized Minkowski inequality we have

$$\|f - T_{m,n}(f)\|_1 = \left\| \int_0^1 (f(\cdot \ominus t) - f(\cdot)) \sum_{k=m}^n a_{nk} D_k(t)\, dt \right\|_1$$

$$\le \int_0^1 \|f(\cdot \ominus t) - f(\cdot)\|_1$$

$$\times \left| \sum_{k=m}^{n-1} \Delta a_{nk} k F_k(t) + n a_{nn} F_n(t) - (m-1) a_{nm} F_{m-1}(t) \right| dt. \tag{3.6}$$

It is known that $\omega \in B_1$ satisfies the Δ_2-condition $\omega(2t) \le C_1 \omega(t)$, $t \in [0, 1/2]$. Let $\omega^*(f,t)_1 = \omega^*(f,1)_1$ for $t \ge 1$. Then we have for $t \in [0,1)$

$$\|f(\cdot \ominus t) - f(\cdot)\|_1 \le \omega^*(f, 2t)_1 \le C_2 \omega(t)$$

and

$$\|f - T_{m,n}(f)\|_1 \le C_3 \left(\int_0^{1/n} + \int_{1/n}^1 \right) \omega(t) \left| \sum_{k=m}^n a_{nk} D_k(t) \right| dt = I_1 + I_2.$$

By Lemmas 2.2, 2.1 and the condition $\omega \in B$ we have

$$I_1 \le C_3 \int_0^{1/n} \omega(t) \left| \sum_{k=m}^n a_{nk} D_k(t) \right| dt$$

$$\leq C_4 \int_0^{1/n} \frac{\omega(t)}{t} \sum_{k=m}^{n} a_{nk}\, dt \leq C_5 \omega(n^{-1}). \tag{3.7}$$

On the other hand, by Lemmas 2.3 (ii), 2.5, (3.6),(1.6) and the condition $\omega \in B_1$, we find that

$$I_2 \leq C_3 \int_{1/n}^{1} \omega(t) \frac{C_5}{t^2} \left(\sum_{k=m}^{n-1} |\Delta a_{nk}| + a_{nn} + a_{nm} \right) dt$$

$$\leq C_6 a_{nn} \int_{1/n}^{1} \frac{\omega(t)}{t^2}\, dt \leq C_7 a_{nn} n \omega(n^{-1}). \tag{3.8}$$

Combining (3.7) and (3.8) we obtain (3.9). □

Theorem 3.9 can be proved similar to Theorem 3.8.

Theorem 3.9. *Let $f \in L^1[0,1)$, $n, m = m(n)$ are natural numbers and $\{a_{jk}\}_{j,k=1}^{\infty}$ satisfy the conditions* (1.4) *and* (1.5)*, $\omega \in B \cap B_1$ and $f \in H_1^{\omega}[0,1)$. Then*

$$\|f - T_{m,n}(f)\|_1 \leq C(1 + n a_{nm}) \omega(n^{-1})_1. \tag{3.9}$$

Remark 3.10. *Similar to Theorems 3.8 and 3.9 results are valid in $L^p[0,1)$, $1 < p < \infty$, and $C^*[0,1)$.*

The following examples show that for some concrete ω Theorems 3.8 and 3.9 are more sharp than Theorems 3.1 and 3.2. Let $\omega(t) = t^{\alpha}$, $0 < \alpha < 1$ (i.e. $\omega \in B \cap B_1$), $j \in \mathbb{N}$, $m = j$, $n = 2^j > j$ and $a_{nk} = (2^j - j + 1)^{-1}$ for $j \leq k \leq 2^j$. Then Theorems 3.1 and 3.2 give $\|f - T_{j,2^j}(f)\|_1 = O(j^{-\alpha})$, $j \in \mathbb{N}$, for $f \in H_1^{\omega}[0,1)$, while by Theorems 3.8 and 3.9 one can obtain $\|f - T_{j,2^j}(f)\|_1 = O(2^{-j\alpha})$, $j \in \mathbb{N}$, since $2^j a_{2^j,k} = 2^j/(2^j - j + 1) \leq 2$ for $j \leq k \leq 2^j$.

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CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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