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RESEARCH ARTICLES

p-Adic Welch Bounds and p-Adic Zauner Conjecture

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Abstract—Let p be a prime. For $d \in \mathbb{N}$, let $\mathbb{Q}_p^d$ be the standard d-dimensional p-adic Hilbert space. Let $m \in \mathbb{N}$ and $\text{Sym}^m(\mathbb{Q}_p^d)$ be the p-adic Hilbert space of symmetric m-tensors. We prove the following result. Let $\{\tau_j\}_{j=1}^n$ be a collection in $\mathbb{Q}_p^d$ satisfying (i) $\langle \tau_j, \tau_j \rangle = 1$ for all $1 \leq j \leq n$ and (ii) there exists $b \in \mathbb{Q}_p$ satisfying $\sum_{j=1}^n \langle x, \tau_j \rangle \tau_j = bx$ for all $x \in \mathbb{Q}_p^d$. Then

$$\max_{1\leq j,k\leq n, j\neq k}\{|n|, |\langle \tau_j, \tau_k\rangle|^{2m}\} \geq \frac{|n|^2}{\left|\binom{d+m-1}{m}\right|}. \quad (0.1)$$

We call Inequality (0.1) as the p-adic version of Welch bounds obtained by Welch [*IEEE Transactions on Information Theory, 1974*]. Inequality (0.1) differs from the non-Archimedean Welch bound obtained recently by M. Krishna as one can not derive one from another. We formulate p-adic Zauner conjecture.

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Key words: *p-adic number field, p-adic Hilbert space, Welch bound, Zauner conjecture.*

1. INTRODUCTION

In 1974 Prof. L. Welch proved the following result [91].

Theorem 1.1. *[91] (**Welch Bounds**) Let $n > d$. If $\{\tau_j\}_{j=1}^n$ is any collection of unit vectors in $\mathbb{C}^d$, then*

$$\sum_{1\leq j,k\leq n} |\langle \tau_j, \tau_k\rangle|^{2m} = \sum_{j=1}^n\sum_{k=1}^n |\langle \tau_j, \tau_k\rangle|^{2m} \geq \frac{n^2}{\binom{d+m-1}{m}}, \quad \forall m \in \mathbb{N}.$$

In particular,

$$\sum_{1\leq j,k\leq n} |\langle \tau_j, \tau_k\rangle|^{2} = \sum_{j=1}^n\sum_{k=1}^n |\langle \tau_j, \tau_k\rangle|^{2} \geq \frac{n^2}{d}.$$

Further,

(Higher order Welch bounds) $$\max_{1\leq j,k\leq n, j\neq k} |\langle \tau_j, \tau_k\rangle|^{2m} \geq \frac{1}{n-1}\left[\frac{n}{\binom{d+m-1}{m}} - 1\right], \quad \forall m \in \mathbb{N}.$$

In particular,

(First order Welch bound) $$\max_{1\leq j,k\leq n, j\neq k} |\langle \tau_j, \tau_k\rangle|^{2} \geq \frac{n-d}{d(n-1)}.$$

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It is impossible to list all applications of Theorem 1.1. A few are: in the study of root-mean-square (RMS) absolute cross relation of unit vectors [75], frame potential [10, 15, 20], correlations [74], codebooks [31], numerical search algorithms [92, 93], quantum measurements [77], coding and communications [81, 86], code division multiple access (CDMA) systems [55, 56], wireless systems [72], compressed/compressive sensing [2, 7, 33, 36, 76, 84, 85, 87], 'game of Sloanes' [49], equiangular tight frames [82], equiangular lines [25, 35, 48, 65], digital fingerprinting [64] etc.

Theorem 1.1 has been improved/different proofs were given in [21, 26, 27, 32, 46, 73, 81, 88, 89]. In 2021 M. Krishna derived continuous version of Theorem 1.1 [59]. In 2022 M. Krishna obtained Theorem 1.1 for Hilbert C*-modules [57], Banach spaces [60] and non-Archimedean Hilbert spaces [58].

In this paper we derive p-adic Welch bounds (Theorem 2.5). We formulate p-adic Zauner conjecture (Conjecture 3.3).

Motivation: Following are some of the important connections of Welch bounds to other prominent areas of research which made us to consider Welch bounds in p-adic setting.

(I) Spherical t-designs have direct relation with Welch bounds (for instance, see Chapter 6 in [90]). Existence of spherical t-designs is known (see [79]) but their exact number is not known. Asymptotic bounds for spherical t-design are recently derived (see [16]).

(II) Welch bounds are essential in the study of equiangular lines (see Chapter 12 in [90]). Existence of equiangular lines having a prescribed angle in a given dimension is largely unknown (see [83]). Asymptotic bound for equiangular lines is recently derived (see [51]).

(III) Benedetto and Fickus (see [10]) characterized finite unit norm frames for finite dimensional Hilbert spaces using frame potential which has connection with Welch bounds (see Chapter 6 in [90]). This characterization motivated the development of so called Fundamental Inequality for Finite Frames (see [20]).

(IV) In compressive sensing, Welch bounds are required in the construction of matrices with small coherence which uses the inner product (see Chapter 5 in [36]).

(V) An easy observation associated with first order Welch bound is that it gives van Lint-Seidel relative bound for equiangular lines which was obtained by van-Lint and Seidel in a different method (without knowing Welch bounds) [62]. Equiangular lines have interaction with several areas which suggests the use of Welch bounds in different areas.

(VI) Recently it is noticed that using Welch bounds one can show that Johnson-Lindenstrauss lemma is optimal [61]. As it is well-known, Johnson-Lindenstrauss lemma has applications even in computer science. This makes the way of Welch bounds to computer science.

2. p-ADIC WELCH BOUNDS

We begin by recalling the notion of p-adic Hilbert space. We refer [1, 29, 30, 52, 53] for more on p-adic Hilbert spaces.

Definition 2.1. *[29, 30] Let $\mathbb{K}$ be a non-Archimedean valued field (with valuation $|\cdot|$) and $\mathcal{X}$ be a non-Archimedean Banach space (with norm $\|\cdot\|$) over $\mathbb{K}$. We say that $\mathcal{X}$ is a* ***p-adic Hilbert space*** *if there is a map (called as p-adic inner product) $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathbb{K}$ satisfying following.*

(i) *If $x \in \mathcal{X}$ is such that $\langle x, y \rangle = 0$ for all $y \in \mathcal{X}$, then $x = 0$.*

(ii) *$\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{X}$.*

(iii) *$\langle \alpha x, y + z \rangle = \alpha \langle x, y \rangle + \langle x, z \rangle$ for all $\alpha \in \mathbb{K}$, for all $x, y, z \in \mathcal{X}$.*

(iv) *$|\langle x, y \rangle| \leq \|x\|\|y\|$ for all $x, y \in \mathcal{X}$.*

Following is the standard example which we consider in the paper.

Example 2.2. *[52] Let p be a prime. For $d \in \mathbb{N}$, let $\mathbb{Q}_p^d$ be the standard p-adic Hilbert space equipped with the inner product*

$$\langle (a_j)_{j=1}^d, (b_j)_{j=1}^d \rangle := \sum_{j=1}^{d} a_j b_j, \quad \forall (a_j)_{j=1}^d, (b_j)_{j=1}^d \in \mathbb{Q}_p^d$$

and norm

$$\|(x_j)_{j=1}^d\| := \max_{1\leq j\leq d} |x_j|, \quad \forall (x_j)_{j=1}^d \in \mathbb{Q}_p^d.$$

Let $I_{\mathbb{Q}_p^d}$ be the identity operator on $\mathbb{Q}_p^d$. Note that $\mathbb{Q}_p^d$ is not a non-Archimedean Hilbert space as it does not satisfies Equation (2) in [58] (see Page 40, [69]). Following is the first important result of the paper.

Theorem 2.3. ***(First Order p-adic Welch Bound)*** *Let p be a prime and $n, d \in \mathbb{N}$. If $\{\tau_j\}_{j=1}^n$ is any collection in $\mathbb{Q}_p^d$ such that there exists $b \in \mathbb{Q}_p$ satisfying*

$$\sum_{j=1}^{n} \langle x, \tau_j \rangle \tau_j = bx, \quad \forall x \in \mathbb{Q}_p^d,$$

then

$$\max_{1\leq j,k\leq n, j\neq k} \left\{ \left|\sum_{l=1}^{n} \langle \tau_l, \tau_l \rangle^2\right|, |\langle \tau_j, \tau_k \rangle|^2 \right\} \geq \frac{1}{|d|} \left|\sum_{j=1}^{n} \langle \tau_j, \tau_j \rangle\right|^2.$$

In particular, if $\langle \tau_j, \tau_j \rangle = 1$ for all $1 \leq j \leq n$, then

$$\textit{(First order p-adic Welch bound)} \quad \max_{1\leq j,k\leq n, j\neq k} \{|n|, |\langle \tau_j, \tau_k \rangle|^2\} \geq \frac{|n|^2}{|d|}.$$

Proof. Define $S_\tau : \mathbb{Q}_p^d \ni x \mapsto \sum_{j=1}^n \langle x, \tau_j \rangle \tau_j \in \mathbb{Q}_p^d$. Then

$$bd = \operatorname{Tra}(bI_{\mathbb{Q}_p^d}) = \operatorname{Tra}(S_\tau) = \sum_{j=1}^{n} \langle \tau_j, \tau_j \rangle,$$

$$b^2 d = \operatorname{Tra}(b^2 I_{\mathbb{Q}_p^d}) = \operatorname{Tra}(S_\tau^2) = \sum_{j=1}^{n} \sum_{k=1}^{n} \langle \tau_j, \tau_k \rangle \langle \tau_k, \tau_j \rangle.$$

Therefore

$$\begin{aligned}
\left|\sum_{j=1}^{n} \langle \tau_j, \tau_j \rangle\right|^2 &= |\operatorname{Tra}(S_\tau)|^2 = |bd|^2 = |d||b^2 d| = |d| \left|\sum_{j=1}^{n} \sum_{k=1}^{n} \langle \tau_j, \tau_k \rangle \langle \tau_k, \tau_j \rangle\right| \\
&= |d| \left|\sum_{l=1}^{n} \langle \tau_l, \tau_l \rangle^2 + \sum_{j,k=1, j\neq k}^{n} \langle \tau_j, \tau_k \rangle \langle \tau_k, \tau_j \rangle\right| \\
&\leq |d| \max \left\{ \left|\sum_{l=1}^{n} \langle \tau_l, \tau_l \rangle^2\right|, \left|\sum_{j,k=1, j\neq k}^{n} \langle \tau_j, \tau_k \rangle \langle \tau_k, \tau_j \rangle\right| \right\} \\
&\leq |d| \max \left\{ \left|\sum_{l=1}^{n} \langle \tau_l, \tau_l \rangle^2\right|, \max_{1\leq j,k\leq n, j\neq k} |\langle \tau_j, \tau_k \rangle \langle \tau_k, \tau_j \rangle| \right\}
\end{aligned}$$

$$= |d| \max_{1\leq j,k\leq n, j\neq k}\left\{\left|\sum_{l=1}^{n}\langle \tau_l, \tau_l\rangle^2\right|, |\langle \tau_j, \tau_k\rangle\langle \tau_k, \tau_j\rangle|\right\}$$

$$= |d| \max_{1\leq j,k\leq n, j\neq k}\left\{\left|\sum_{l=1}^{n}\langle \tau_l, \tau_l\rangle^2\right|, |\langle \tau_j, \tau_k\rangle|^2\right\}.$$

Whenever $\langle \tau_j, \tau_j\rangle = 1$ for all $1 \leq j \leq n$,

$$|n|^2 \leq |d| \max_{1\leq j,k\leq n, j\neq k}\{|n|, |\langle \tau_j, \tau_k\rangle|^2\}.$$

We next derive higher order p-adic Welch bounds. For this we need the concept of vector space of symmetric tensors. Given a vector space $\mathcal{V}$ of dimension d, let $\mathcal{V}^{\otimes m}$ be the vector space of m-tensors. A vector

$$\sum_{j=1}^{n} x_{j,1}\otimes\cdots\otimes x_{j,m} \in \mathcal{V}^{\otimes m}$$

is said to be symmetric if for every bijection $\sigma : \{1,\dots,m\} \to \{1,\dots,m\}$, we have

$$\sum_{j=1}^{n} x_{j,\sigma(1)}\otimes\cdots\otimes x_{j,\sigma(m)} = \sum_{j=1}^{n} x_{j,1}\otimes\cdots\otimes x_{j,m}.$$

Set of all symmetric m-tensors will form a vector space, denoted by $\text{Sym}^m(\mathcal{V})$. Following result will give dimension of this space.

Theorem 2.4. *[14, 23] If $\mathcal{V}$ is a vector space of dimension d and $Sym^m(\mathcal{V})$ denotes the vector space of symmetric m-tensors, then*

$$dim(Sym^m(\mathcal{V})) = \binom{d+m-1}{m}, \quad \forall m \in \mathbb{N}.$$

Theorem 2.5. *(**Higher Order p-adic Welch Bounds**) Let p be a prime and $n, d, m \in \mathbb{N}$. If $\{\tau_j\}_{j=1}^n$ is any collection in $\mathbb{Q}_p^d$ such that there exists $b \in \mathbb{Q}_p$ satisfying*

$$\sum_{j=1}^{n}\langle x, \tau_j^{\otimes m}\rangle \tau_j^{\otimes m} = bx, \quad \forall x \in Sym^m(\mathbb{Q}_p^d),$$

then

$$\max_{1\leq j,k\leq n, j\neq k}\left\{\left|\sum_{l=1}^{n}\langle \tau_l, \tau_l\rangle^{2m}\right|, |\langle \tau_j, \tau_k\rangle|^{2m}\right\} \geq \frac{1}{\left|\binom{d+m-1}{m}\right|}\left|\sum_{j=1}^{n}\langle \tau_j, \tau_j\rangle^m\right|^2.$$

In particular, if $\langle \tau_j, \tau_j\rangle = 1$ for all $1 \leq j \leq n$, then

$$\textbf{(Higher order } p\textbf{-adic Welch bound)} \quad \max_{1\leq j,k\leq n, j\neq k}\{|n|, |\langle \tau_j, \tau_k\rangle|^{2m}\} \geq \frac{|n|^2}{\left|\binom{d+m-1}{m}\right|}.$$

Proof. Define $S_\tau : \text{Sym}^m(\mathbb{Q}_p^d) \ni x \mapsto \sum_{j=1}^{n}\langle x, \tau_j^{\otimes m}\rangle \tau_j^{\otimes m} \in \text{Sym}^m(\mathbb{Q}_p^d)$. Then

$$b\dim(\mathrm{Sym^m}(\mathbb{Q}_\mathrm{p}^\mathrm{d})) = \mathrm{Tra}(bI_{\mathrm{Sym}^m(\mathbb{Q}_p^d)}) = \mathrm{Tra}(S_\tau) = \sum_{j=1}^{n}\langle \tau_j^{\otimes m}, \tau_j^{\otimes m}\rangle,$$

$$b^2\dim(\mathrm{Sym^m}(\mathbb{Q}_\mathrm{p}^\mathrm{d})) = \mathrm{Tra}(b^2 I_{\mathrm{Sym}^m(\mathbb{Q}_p^d)}) = \mathrm{Tra}(S_\tau^2) = \sum_{j=1}^{n}\sum_{k=1}^{n}\langle \tau_j^{\otimes m}, \tau_k^{\otimes m}\rangle\langle \tau_k^{\otimes m}, \tau_j^{\otimes m}\rangle.$$

Therefore by using Theorem 2.4 we get

$$\begin{aligned}
\left|\sum_{j=1}^{n}\langle\tau_j,\tau_j\rangle^m\right|^2 &= \left|\sum_{j=1}^{n}\langle\tau_j^{\otimes m},\tau_j^{\otimes m}\rangle\right|^2 = |\operatorname{Tra}(S_\tau)|^2 = \left|b\dim(\mathrm{Sym^m}(\mathbb{Q}_p^d))\right|^2\\
&= \left|\dim(\mathrm{Sym^m}(\mathbb{Q}_p^d))\right|\left|b^2\dim(\mathrm{Sym^m}(\mathbb{Q}_p^d))\right|\\
&= \left|\dim(\mathrm{Sym^m}(\mathbb{Q}_p^d))\right|\left|\sum_{j=1}^{n}\sum_{k=1}^{n}\langle\tau_j^{\otimes m},\tau_k^{\otimes m}\rangle\langle\tau_k^{\otimes m},\tau_j^{\otimes m}\rangle\right|\\
&= \left|\binom{d+m-1}{m}\right|\left|\sum_{j=1}^{n}\sum_{k=1}^{n}\langle\tau_j^{\otimes m},\tau_k^{\otimes m}\rangle\langle\tau_k^{\otimes m},\tau_j^{\otimes m}\rangle\right|\\
&= \left|\binom{d+m-1}{m}\right|\left|\sum_{j=1}^{n}\sum_{k=1}^{n}\langle\tau_j,\tau_k\rangle^m\langle\tau_k,\tau_j\rangle^m\right|\\
&= \left|\binom{d+m-1}{m}\right|\left|\sum_{l=1}^{n}\langle\tau_l,\tau_l\rangle^{2m}+\sum_{j,k=1,j\neq k}^{n}\langle\tau_j,\tau_k\rangle^m\langle\tau_k,\tau_j\rangle^m\right|\\
&\leq \left|\binom{d+m-1}{m}\right|\max\left\{\left|\sum_{l=1}^{n}\langle\tau_l,\tau_l\rangle^{2m}\right|,\left|\sum_{j,k=1,j\neq k}^{n}\langle\tau_j,\tau_k\rangle^m\langle\tau_k,\tau_j\rangle^m\right|\right\}\\
&\leq \left|\binom{d+m-1}{m}\right|\max\left\{\left|\sum_{l=1}^{n}\langle\tau_l,\tau_l\rangle^{2m}\right|,\max_{1\leq j,k\leq n,j\neq k}|\langle\tau_j,\tau_k\rangle^m\langle\tau_k,\tau_j\rangle^m|\right\}\\
&\leq \left|\binom{d+m-1}{m}\right|\max_{1\leq j,k\leq n,j\neq k}\left\{\left|\sum_{l=1}^{n}\langle\tau_l,\tau_l\rangle^{2m}\right|,|\langle\tau_j,\tau_k\rangle^m\langle\tau_k,\tau_j\rangle^m|\right\}\\
&= \left|\binom{d+m-1}{m}\right|\max_{1\leq j,k\leq n,j\neq k}\left\{\left|\sum_{l=1}^{n}\langle\tau_l,\tau_l\rangle^{2m}\right|,|\langle\tau_j,\tau_k\rangle|^{2m}\right\}.
\end{aligned}$$

Whenever $\langle\tau_j,\tau_j\rangle = 1$ for all $1\leq j\leq n$,

$$|n|^2 \leq \left|\binom{d+m-1}{m}\right|\max_{1\leq j,k\leq n,j\neq k}\{|n|,|\langle\tau_j,\tau_k\rangle|^{2m}\}.$$

Remark 2.6. *Conditions given in the Theorem 2.5 says that the operator S_τ in the proof of Theorem 2.5 is diagonalizable. Thus Theorem 2.5 is restrictive as the hypothesis is stronger than that of Theorem 2.3 in [58]. However, note that the field $\mathbb{Q}_p$ does not satisfies the Equation (2) in [58] (see [69]) and hence neither the results in this paper can be derived from the results in [58] nor the results in [58] can be derived from the results in this paper.*

Remark 2.7. *Theorems 2.3 and 2.5 hold by replacing $\mathbb{Q}_p^d$ by a d-dimensional p-adic Hilbert space over any non-Archimedean (complete) valued field (such as $\mathbb{C}_p$).*

3. p-ADIC ZAUNER CONJECTURE AND OPEN PROBLEMS

Using Theorem 2.3 we ask the following question.

Question 3.1. ***Given a prime p, for which $(d,n)\in\mathbb{N}\times\mathbb{N}$, there exist vectors $\tau_1,\dots,\tau_n\in\mathbb{Q}_p^d$ satisfying the following.***

(i) $\langle \tau_j, \tau_j \rangle = 1$ ***for all*** $1 \leq j \leq n$.

(ii) ***There exists*** $b \in \mathbb{Q}_p$ ***satisfying***

$$\sum_{j=1}^{n} \langle x, \tau_j \rangle \tau_j = bx, \quad \forall x \in \mathbb{Q}_p^d.$$

(iii)

$$\max_{1 \leq j,k \leq n, j \neq k} \{|n|, |\langle \tau_j, \tau_k \rangle|^2\} = \frac{|n|^2}{|d|}.$$

We can formulate a strong form of Question 3.1 as follows.

Question 3.2. ***Given a prime*** p***, for which*** $(d, n) \in \mathbb{N} \times \mathbb{N}$***, there exist vectors*** $\tau_1, \dots, \tau_n \in \mathbb{Q}_p^d$ ***satisfying the following.***

(i) $\langle \tau_j, \tau_j \rangle = 1$ ***for all*** $1 \leq j \leq n$.

(ii) ***There exists*** $b \in \mathbb{Q}_p$ ***satisfying***

$$\sum_{j=1}^{n} \langle x, \tau_j \rangle \tau_j = bx, \quad \forall x \in \mathbb{Q}_p^d.$$

(iii)

$$\max_{1 \leq j,k \leq n, j \neq k} \{|n|, |\langle \tau_j, \tau_k \rangle|^2\} = \frac{|n|^2}{|d|}.$$

(iv) $\|\tau_j\| = 1$ ***for all*** $1 \leq j \leq n$.

Why Question 3.2 is different than Question 3.1? Reason is that unlike non-Archimedean Hilbert spaces, in p-adic Hilbert spaces, norm is not defined as $\sqrt{|\langle \cdot, \cdot \rangle|}$. A particular case of Question 3.1 is the following p-adic version of Zauner conjecture (see [3–6, 11–13, 37, 40, 47, 54, 59, 63, 71, 78, 95] for Zauner conjecture in Hilbert spaces, [57] for Zauner conjecture in Hilbert C*-modules, [60] for Zauner conjecture in Banach spaces and [58] for Zauner conjecture in non-Archimedean Hilbert spaces).

Conjecture 3.3. ***(***p***-adic Zauner Conjecture) Let*** p ***be a prime. For each*** $d \in \mathbb{N}$***, there exist vectors*** $\tau_1, \dots, \tau_{d^2} \in \mathbb{Q}_p^d$ ***satisfying the following.***

(i) $\langle \tau_j, \tau_j \rangle = 1$ ***for all*** $1 \leq j \leq d^2$.

(ii) ***There exists*** $b \in \mathbb{Q}_p$ ***satisfying***

$$\sum_{j=1}^{d^2} \langle x, \tau_j \rangle \tau_j = bx, \quad \forall x \in \mathbb{Q}_p^d.$$

(iii)

$$|\langle \tau_j, \tau_k \rangle|^2 = |d|, \quad \forall 1 \leq j, k \leq d^2, j \neq k.$$

Question 3.2 gives the following Zauner conjecture.

Conjecture 3.4. ***(***p***-adic Zauner Conjecture - strong form) Let*** p ***be a prime. For each*** $d \in \mathbb{N}$***, there exist vectors*** $\tau_1, \dots, \tau_{d^2} \in \mathbb{Q}_p^d$ ***satisfying the following.***

(i) $\langle \tau_j, \tau_j \rangle = 1$ ***for all*** $1 \leq j \leq d^2$.

(ii) ***There exists*** $b \in \mathbb{Q}_p$ ***satisfying***

$$\sum_{j=1}^{d^2} \langle x, \tau_j \rangle \tau_j = bx, \quad \forall x \in \mathbb{Q}_p^d.$$

(iii)

$$|\langle \tau_j, \tau_k \rangle|^2 = |d|, \quad \forall 1 \leq j, k \leq d^2, j \neq k.$$

(iv) $\|\tau_j\| = 1$ ***for all*** $1 \leq j \leq d^2$.

We recall the definition of Gerzon's bound which allows us to remember companions to Welch bounds in Hilbert spaces.

Definition 3.5. *[49] Given $d \in \mathbb{N}$, define* ***Gerzon's bound***

$$\mathcal{Z}(d, \mathbb{K}) := \begin{cases} d^2 & \text{if} \quad \mathbb{K} = \mathbb{C} \\ \frac{d(d+1)}{2} & \text{if} \quad \mathbb{K} = \mathbb{R}. \end{cases}$$

Theorem 3.6. *[18, 24, 45, 49, 66, 70, 80, 92] Define $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $m := \dim_{\mathbb{R}}(\mathbb{K})/2$. If $\{\tau_j\}_{j=1}^n$ is any collection of unit vectors in $\mathbb{K}^d$, then*

(i) ***(Bukh-Cox bound)***

$$\max_{1 \leq j,k \leq n, j \neq k} |\langle \tau_j, \tau_k \rangle| \geq \frac{\mathcal{Z}(n-d, \mathbb{K})}{n(1 + m(n-d-1)\sqrt{m^{-1} + n - d}) - \mathcal{Z}(n-d, \mathbb{K})} \quad \text{if} \quad n > d.$$

(ii) ***(Orthoplex/Rankin bound)***

$$\max_{1 \leq j,k \leq n, j \neq k} |\langle \tau_j, \tau_k \rangle| \geq \frac{1}{\sqrt{d}} \quad \text{if} \quad n > \mathcal{Z}(d, \mathbb{K}).$$

(iii) ***(Levenstein bound)***

$$\max_{1 \leq j,k \leq n, j \neq k} |\langle \tau_j, \tau_k \rangle| \geq \sqrt{\frac{n(m+1) - d(md+1)}{(n-d)(md+1)}} \quad \text{if} \quad n > \mathcal{Z}(d, \mathbb{K}).$$

(iv) ***(Exponential bound)***

$$\max_{1 \leq j,k \leq n, j \neq k} |\langle \tau_j, \tau_k \rangle| \geq 1 - 2n^{\frac{-1}{d-1}}.$$

Theorem 3.6 and Theorem 2.3 give the following problem.

Question 3.7. ***Whether there is a p-adic version of Theorem 3.6? In particular, does there exists a version of***

(i) p***-adic Bukh-Cox bound?***

(ii) p***-adic Orthoplex/Rankin bound?***

(iii) p***-adic Levenstein bound?***

(iv) ***p-adic Exponential bound?***

We already wrote that Welch bounds have applications in study of equiangular lines. We wish to formulate equiangular line problem for p-adic Hilbert spaces. For the study of equiangular lines in Hilbert spaces we refer [8, 9, 17, 19, 28, 38, 39, 41, 44, 50, 51, 62, 67, 68, 94], quaternion Hilbert spaces we refer [34], octonion Hilbert spaces we refer [22], finite dimensional vector spaces over finite fields we refer [42, 43], for Banach spaces we refer [60] and for non-Archimedean Hilbert spaces we refer [58].

Question 3.8. ***(p-adic Equiangular Line Problem) Let p be a prime. Given $a \in \mathbb{Q}_p$, $d \in \mathbb{N}$ and $\gamma > 0$, what is the maximum $n = n(p, a, d, \gamma) \in \mathbb{N}$ such that there exist vectors $\tau_1, \dots, \tau_n \in \mathbb{Q}_p^d$ satisfying the following.***

(i) $\langle \tau_j, \tau_j \rangle = a$ ***for all*** $1 \leq j \leq n$.

(ii) $|\langle \tau_j, \tau_k \rangle|^2 = \gamma$ ***for all*** $1 \leq j, k \leq n, j \neq k$.

In particular, whether there is a p-adic Gerzon bound?

Question 3.8 can be easily lifted to formulate question of p-adic regular s-distance sets.

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CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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