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RESEARCH ARTICLES

Rigidity and Unlikely Intersection for Stable p-Adic Dynamical Systems

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Abstract—Berger asked the question "To what extent the preperiodic points of a stable p-adic power series determines a stable p-adic dynamical system ?" In this work we have applied the preperiodic points of a stable p-adic power series in order to determine the corresponding stable p-adic dynamical system.

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1. INTRODUCTION AND MOTIVATION

Let K be the finite extension of the p-adic field $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$, and the unique maximal ideal $\mathfrak{m}_K$. We denote the units in $\mathcal{O}_K$ by $\mathcal{O}_K^*$. Let $\bar{K}$ be the algebraic closure of K and $\bar{\mathfrak{m}}_K$ be the integral closure of $\mathfrak{m}_K$ in $\bar{K}$. Let C_p be the p-adic completion of $\bar{K}$ and denote $\mathfrak{m}_{C_p} = \{z \in C_p \mid |z|_p < 1\}$.

In [6], Berger studied to what extent the torsion points $\mathrm{Tors}(F)$ of a formal group F over $\mathcal{O}_K$ determines the formal group. He proved that if $\mathrm{Tors}(F_1) \cap \mathrm{Tors}(F_2)$ is infinite then $F_1 = F_2$. He further asked the question, if $\mathcal{D}$ is a stable p-adic dynamical system, then:

"To what extent the preperiodic points $\mathrm{Preper}(\mathcal{D})$ determines $\mathcal{D}$?"

In this work, we have answered this question by proving our main Theorem 3.8 in Section 3. We have also provided an alternate proof of it following some examples in Section 4. The proofs relies on the following tools:

(a) The first proof uses the correspondence between $\mathrm{Tors}(F)$ and $\mathrm{Preper}(\mathcal{D})$.

(b) The alternate proof uses the following two facts:

(i) Galois correspondence of a stable p-adic dynamical system $\mathcal{D}$. Indeed, we proved that given any stable p-adic dynamical system $\mathcal{D}$ over $\mathcal{O}_K$, there exists a $\sigma \in \mathrm{Gal}(\bar{K}/K)$ and a stable series $w(x) \in \mathcal{D}$ such that $\sigma(x) = w(x)$, $\forall\, x \in \mathrm{Preper}(\mathcal{D})$.

(ii) Rigidity of power series on open unit disk $\mathfrak{m}_{C_p}$. We say that a subset $\mathcal{Z} \subset \mathfrak{m}_{C_p}$ is Zariski dense in $\mathfrak{m}_{C_p}$ if every power series $h(x) \in \mathcal{O}_K[[x]]$ that vanishes on $\mathcal{Z}$ is necessarily equal to zero. A subset $\mathcal{Z} \subset \mathfrak{m}_{C_p}$ is Zariski dense in $\mathfrak{m}_{C_p}$ if and only if it is infinite.

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2. p-ADIC DYNAMICAL SYSTEM AND SOME RESULTS

In this section, we recall some preliminaries, and prove some helpful results:

Definition 2.1. *[5] A (discrete) dynamical system consists of a set Γ and a function $\gamma : \Gamma \to \Gamma$. Its dynamics is indeed the study of the behavior of the points in Γ by repeatedly applying γ on the points of Γ, i.e., we study the iterates of γ. If we consider the n^{th} iterate*

$$\gamma^{\circ n}(x) = \underbrace{\gamma \circ \gamma \circ \cdots \circ \gamma(x)}_{n \text{ iterates}},$$

then the orbit of $x \in \Gamma$ is defined by $O_\gamma(x) = \left\{x,\ \gamma(x),\ \gamma^{\circ 2}(x),\ \gamma^{\circ 3}(x), \cdots\right\}$.

(i) The point x is called periodic of period n if $\gamma^{\circ n}(x) = x$ for some $n \geq 1$.

(ii) If $\gamma(x) = x$, then x is a fixed point.

(iii) A point x is preperiodic if some iterate $\gamma^{\circ i}(x)$ is periodic i.e., there exists m, n such that $\gamma^{\circ m}(x) = \gamma^{\circ n}(x)$. In other words, x is preperiodic if its orbit $O_\gamma(x)$ is finite.

Definition 2.2. *[6] A stable p-adic dynamical system $\mathcal{D}$ over $\mathcal{O}_K$ is a collection of p-adic power series in $\mathcal{O}_K[[x]]$ without constant term such that the power series commutes with each other under formal composition. A power series f in $\mathcal{D}$ is called stable if $f'(0)$ is neither 0 nor a root of 1. We say that $\mathcal{D} \subseteq x \cdot \mathcal{O}_K[[x]]$ is a stable p-adic dynamical system of finite height if the elements of $\mathcal{D}$ commute with each other under composition, and if $\mathcal{D}$ contains a stable series f such that $f'(0) \in \mathfrak{m}_K$ and $f(x) \not\equiv 0 \bmod \mathfrak{m}_K$ (i.e., f is of finite height) as well as a stable series u such that $u'(0) \in \mathcal{O}_K^\times$. The collection $\mathcal{D}$ can be made as large as possible in the sense that whenever a stable power series commutes with any member of $\mathcal{D}$, it belongs to $\mathcal{D}$. Such a collection $\mathcal{D}$ is the main object in p-adic dynamical systems [4].*

Example 2.3. *If F is a formal group law of finite height over $\mathcal{O}_K$, then the endomorphism ring $End_{\mathcal{O}_K}(F)$ of F is a stable p-adic dynamical system.*

Proposition 2.4. *For an invertible power series, preperiodic points are exactly the periodic points, i.e., fixed points of iterates of u.*

Proof. Let $u(x) \in x \cdot \mathcal{O}_K[[x]]$ be invertible. For any preperiodic point α of $u(x)$, there exists natural numbers m, n with $m > n$ such that $u^{\circ m}(\alpha) = u^{\circ n}(\alpha)$. Since $u(x)$ is invertible, $u^{\circ(-n)}(x)$ exists in $x \cdot \mathcal{O}_K[[x]]$ and hence $u^{\circ(m-n)}(\alpha) = \alpha$. □

If u is an invertible series over $\mathcal{O}_K$, then the preperiodic points of u are exactly the periodic points by Proposition 2.4. Now we remember that: the only periodic points of u are roots of $u^{\circ p^m}(x) - x$ for some $m \in \mathbb{N}$. The full ring $\mathbb{Z}_p$ acts on the invertible members of the dynamical system $\mathcal{D}$. For, the series $u^{\circ p^m}$ converge to the identity in the appropriate topology, and thus the map $\mathbb{Z} \to \mathcal{D}$ by $n \mapsto u^{\circ n}$ is continuous when $\mathbb{Z}$ has the p-adic topology, so extends to $\mathbb{Z}_p \to \mathcal{D}$. It follows from this that if $m \in \mathbb{Z}$ and $m = p^r n$ with $p \nmid n$, then the fixed points of $u^{\circ m}$ are the fixed points of $u^{\circ p^r}$. To be more precise, we consider the following two lemmas:

Lemma 2.5. *Let u be an invertible series in $\mathcal{O}_K[[x]]$. Then for every natural number $n \geq 0$, for any $\lambda \in \bar{K}$ with $v(\lambda) > 0$, if λ is a fixed point of u, then λ is also a fixed point of $u^{\circ p^n}$.*

Proof. Note that $u^{\circ 2}(\lambda) = u(u(\lambda)) = u(\lambda) = \lambda$. Thus by induction on n, the result follows. □

Lemma 2.6. *Let u be an invertible series in $\mathcal{O}_K[[x]]$. Then for every natural number $n \geq 0$, for any $\lambda \in \bar{K}$ with $v(\lambda) > 0$, if λ is a fixed point of u, then λ is also a fixed point of $u^{\circ p^n}$. More generally, for $z \in \mathbb{Z}_p$, λ is also a fixed point of $u^{\circ z}$.*

Proof. We recall that the map $\mathbb{Z} \to \mathcal{O}_K[[x]]$ by $n \mapsto u^{\circ n}$ is continuous when $\mathbb{Z}$ has the p-adic topology and $\mathcal{O}_K[[x]]$ has $(\mathfrak{m}_K,\ x)$-adic topology. This latter topology also has the property that if $\{U_i\}$, U are invertible series in $\mathcal{O}_K[[x]]$ with limit $\lim_i U_i = U$,then $\lim_i U_i(\lambda) = U(\lambda)$. That is, evaluation at λ is a continuous map from $\mathcal{O}_K[[x]]$ to $\{\xi \in \bar{K} : v(\xi) > 0\}$.

Now suppose that λ is a fixed point of u, and $z \in \mathbb{Z}_p$. There is a sequence of positive integers $\{z_i\}$ with limit z, and so λ is a fixed point of each $u^{\circ z_i}$, so that $u^{\circ z}(\lambda) = u^{\circ \lim_i z_i}(\lambda) = \lim_i (u^{\circ z_i}(\lambda)) = \lim_i \lambda = \lambda$. □

We define the following two sets:

$$\mathrm{Preper}(u) = \bigcup_n \{x \in \mathcal{O}_K | u^{\circ p^n}(x) = x\} = \text{all preperiodic points of an invertible series } u \in \mathcal{D}$$

$$T(f) = \bigcup_n \{x \in \bar{\mathfrak{m}}_K | f^{\circ n}(x) = 0\} = \text{all torsion points of a noninvertible series } f \in \mathcal{D}. \quad (2.1)$$

We note the following interesting result, which says that Preper($\mathcal{D}$) is independent of choices of stable series in $\mathcal{D}$:

Proposition 2.7. *[4] Let $u, f \in \mathcal{D}$ be invertible and noninvertible series, respectively. Then the set of roots of iterates of f is equal to the set periodic points of $u(x)$. That is, if $T(f)$ denotes the set of roots of iterates of f, then $T(f) = \mathrm{Preper}(u)$.*

3. THE MAIN RESULTS

We start with a conjecture.

Conjecture 3.1. *[7] If f and u are, respectively, two stable noninvertible and invertible power series in a stable p-adic dynamical system $\mathcal{D}$, then there exists a formal group F with coefficients in $\mathcal{O}_K$, two endomorphisms f_F and u_F of F, and a nonzero power series h such that $f \circ h = h \circ f_F$ and $u \circ h = h \circ u_F$. We call h to be the isogeny from f_F to f.*

Remark 3.2. *The conjecture 3.1 is proved in [7, Theorem. B] for $K = \mathbb{Q}_p$. This conjecture resembles to that one given by Lubin in [4] while [1, 2] and [3] proved several results in the support of Lubin's conjecture, which says, if a noninvertible series commutes with an invertible series, there is a formal group somehow in the background.*

For the above formal group F over $\mathcal{O}_K$, its endomorphism ring $\mathrm{End}_{\mathcal{O}_K}(F)$ is a stable p-adic dynamical system. We denote by $\mathrm{Tors}(F) = \bigcup_n T(n, f_F)$, the torsion points of F, where $T(n, f_F) = \{\alpha \in \bar{\mathfrak{m}}_K : f_F^{\circ n}(\alpha) = 0\}$. Then we have the following nice result:

Theorem 3.3. *[6] If F_1 and F_2 are two formal groups over $\mathcal{O}_K$ and if $\mathrm{Tors}(F_1) \cap \mathrm{Tors}(F_2)$=infinite, then $F_1 = F_2$.*

Definition 3.4. *Let $f(x)$ and $g(x)$ be two noninvertible stable power series over $\mathcal{O}_K$ without constant term. We call a power series $h(x) \in \mathcal{O}_K[[x]]$ an $\mathcal{O}_K$-isogeny of $f(x)$ into $g(x)$ if $h \circ f = g \circ h$. If $u(x)$ be any invertible series in $\mathcal{O}_K[[x]]$ then $u \circ h$ is also an $\mathcal{O}_K$-isogeny of f.*

Next we prove the following lemma.

Lemma 3.5. *Let $f(x)$ and $g(x)$ be two noninvertible stable power series over $\mathcal{O}_K$ each with finite Weierstrass degree. Let h be an isogeny of f into g, then h maps $T(f)$ into $T(g)$. Moreover, $h : T(f) \to T(g)$ is surjective.*

Proof. At first we will show that $h(0) = 0$. Since $g(h(0)) = h(f(0)) = h(0)$, $h(0)$ is a fixed point of $g(x)$. But $g(x)$ being noninvertible can have 0 as its only fixed point and hence $h(0) = 0$.

Now let $\alpha \in T_n(f) \subset T(f)$, then $g^{\circ n}(h(\alpha)) = h(f^{\circ n}(\alpha)) = h(0) = 0$. This implies $h(\alpha) \in T(g)$. This shows that h maps $T(f)$ to $T(g)$.

On the other hand, take any $\beta \in T_m(g) \subset T(g)$ for some natural number $m \in \mathbb{N}$ and let $\alpha \in \bar{\mathfrak{m}}_K$ such that $h(\alpha) = \beta$. We need to show that $\alpha \in T(f)$. For, $h(f^{\circ m}(\alpha)) = g^{\circ m}(h(\alpha)) = 0$ implies $f^{\circ m}(\alpha)$ is a root of $h(x)$ which is also true for all $n \geq m$. Since $h(x)$ can have only finitely many roots in $\bar{\mathfrak{m}}_K$, we must have

$$f^{\circ n+\tilde{n}}(\alpha) = f^{\circ n}(\alpha) \text{ for some } n,\ \tilde{n} \in \mathbb{N}.$$

This implies that $f^{\circ n}(\alpha)$ is a fixed point of $f^{\circ \tilde{n}}(x)$. Since $f^{\circ \tilde{n}}(x)$ is noninvertible, it has the only fixed point 0 and hence $f^{\circ n}(\alpha) = 0$. Thus $\alpha \in T_n(f) \subset T(f)$. Thus h is surjective. ☐

Definition 3.6. *We denote a stable p-adic dynamical system $\mathcal{D}$ by the package $(\mathcal{D}, f, u; F, f_F, u_F; h)$, where F is the background formal group of $\mathcal{D}$ with f_F, u_F noninvertible and invertible endomorphisms respectively, while u, f are invertible and noninvertible power series in $\mathcal{D}$ respectively, along with an isogeny map $h : f_F \to f$ as in Conjecture 3.1*

Now we will prove the uniqueness of the formal group F in Conjecture 3.1.

Proposition 3.7. *There exists a unique formal group F for each stable p-adic dynamical system $\mathcal{D}$ in the Conjecture 3.1.*

Proof. Let $\mathcal{D}$ be a stable p-adic dynamical system over $\mathcal{O}_K$ consisting of a noninvertible series f and an invertible series u. By Conjecture 3.1, there exists a formal group F over $\mathcal{O}_K$ with endomorphisms f_F, u_F and an isogeny h from f_F to f. We want to show that F is unique. If possible let there exists another formal group G over $\mathcal{O}_K$ with endomorphisms f_G, u_G and an isogeny, say, h' from f_G to f. By Lemma 3.5, we have the surjections $h : T(f_F) \to T(f)$ and $h' : T(f_G) \to T(f)$. Therefore for every $\alpha \in \text{Preper}(\mathcal{D})$ there exists some $\beta_1 \in \text{Tors}(F)$ and some $\beta_2 \in \text{Tors}(G)$ such that $h(\beta_1) = \alpha = h'(\beta_2)$. This shows that both Tors(F) and Tors(G) has infinitely many points in common and thus by the Theorem 3.3, we get $F = G$. ☐

We will now prove the main result of the paper.

Theorem 3.8. *If $(\mathcal{D}_1, f_1, u_1; F_1, f_{F_1}, u_{F_1}; h_1)$ and $(\mathcal{D}_2, f_2, u_2; F_2, f_{F_2}, u_{F_2}; h_2)$ are two dynamical systems over $\mathcal{O}_K$ such that Preper$(\mathcal{D}_1) \cap$ Preper$(\mathcal{D}_2)$ is infinite , then $\mathcal{D}_1 = \mathcal{D}_2$.*

Proof. By Lemma 3.5, the isogenies h_i defines surjective maps $h_i : T(f_{F_i}) \to T(f_i),\ i = 1, 2$. Thus for any $\beta_i \in T(f_i)$ there exists an $\alpha_i \in T(f_{F_i})$ such that $h_i(\alpha_i) = \beta_i$. We note that Tors$(F_1) \cap$ Tors(F_2) will have infinitely many points in common if Preper$(\mathcal{D}_1) \cap$ Preper$(\mathcal{D}_2)$=infinite, because the isogenies h_i maps $T(f_{F_i})$ into $T(f_i)$ by Lemma 3.5. But given that Preper$(\mathcal{D}_1) \cap$ Preper$(\mathcal{D}_2)$ is infinite, and hence Tors$(F_1) \cap$ Tors(F_2) is infinite. Therefore by Theorem 3.3, we conclude $F_1 = F_2$. Hence by the uniqueness property of Proposition 3.7, we must have $\mathcal{D}_1 = \mathcal{D}_2$. ☐

4. ALTERNATIVE PROOF OF THEOREM 3.8

In this section we give another proof of the main Theorem 3.8 which deserved to be included because of its beauty. We are indebted to the ideas of [6]. At first, we note the following beautiful result.

Theorem 4.1. *[6] Given a formal group F over $\mathcal{O}_K$ with torsion points Tors(F), there is a stable endomorphism u_F of F and $\sigma \in Gal(\bar{K}/K)$ such that*

$$\sigma(z) = u_F(z) \text{ for all } z \in Tors(F).$$

Now we prove a similar result for a stable p-adic dynamical system.

Theorem 4.2. *Let $\mathcal{D}$ be a stable p-adic dynamical system, then there exists a stable power series $w(x) \in \mathcal{D}$ and an $\sigma \in Gal(\bar{K}/K)$ such that*

$$\sigma(z) = w(z), \text{ for all } z \in Preper(\mathcal{D}). \tag{4.1}$$

Proof. By the Conjecture 3.1, if f and u are two stable noninvertible and invertible power series in $\mathcal{D}$, then there exists a formal group F with coefficients in $\mathcal{O}_K$, two endomorphisms f_F and u_F of F, and a nonzero power series h such that $f \circ h = h \circ f_F$ and $u \circ h = h \circ u_F$, where h is the isogeny from f_F to f.

By Lemma 3.5, h maps $T(f_F)$ into $T(f)$, and hence for every $\beta \in \mathrm{Tors}(F)$, we get $h(\beta) \in \mathrm{Preper}(\mathcal{D})$. Moreover, by Lemma 3.5, we see $h : T(f_F) \to T(f)$ is also surjective. Thus for every $\alpha \in \mathrm{Preper}(\mathcal{D})$ there exists some $\beta \in \mathrm{Tors}(F)$ such that $h(\beta) = \alpha$. From the Theorem 3.3, we have

$$\sigma(z) = u_F(z) \text{ for all } z \in \mathrm{Tors}(F). \tag{4.2}$$

Now it remains to show that we can replace u_F by an element $w \in \mathcal{D}$ in equation (3.1) such that $w \notin \mathrm{End}_{\mathcal{O}_K}(F)$. Applying the isogeny h both sides of equation (3.1) and using the relation $u \circ h = h \circ u_F$ from Conjecture 3.1, we get

$$\begin{aligned} &\sigma(z) = u_F(z) \text{ for all } z \in \mathrm{Tors}(F), \\ \Rightarrow\ & h(\sigma(z)) = (h \circ u_F)(z) \text{ for all } z \in \mathrm{Tors}(F), \\ \Rightarrow\ & \sigma(h(z)) = u(h(z) \text{ for all } z \in \mathrm{Tors}(F), \ (\because \sigma(h(z)) = h(\sigma(z))) & (4.3) \\ \Rightarrow\ & \sigma(\tilde{z}) = u(\tilde{z}), \text{ for all } \tilde{z} = h(z) \in \mathrm{Preper}(\mathcal{D}). & (4.4) \end{aligned}$$

The relation (4.4) follows from the relation (3.2) because h maps $T(f_F)$ into $T(f)$, by Lemma 3.5. Finally denoting $w(x) := u(x) \in \mathrm{Preper}(\mathcal{D})$, we are done. □

The following example describes a situation when we get a relation like (4.1).

Example 4.3. *Let $f(x) \in x \cdot \mathcal{O}_K[[x]]$ be a noninvertible and irreducible polynomial of degree 5 with set of zeros $\Theta := \{r_1, r_2, r_3, r_4, r_5\}$ such that the extension $K(\Theta) := K(r_1, r_2, r_3, r_4, r_5)$ is Galois with Galois group say, $Gal(K(\Theta)/K)$. Any $\tau \in Gal(K(\Theta)/K)$ permutes the elements of Θ. Define some $w(x) \in x \cdot \mathcal{O}_K[[x]]$ by $w(x) = x + s(x)f(x)$ for some $s(x) \in \mathcal{O}_K[[x]]$. We claim there exist some $\tau \in Gal(K(\Theta)/K)$ so that $\tau(r_i) = w(r_i)$ for all $r_i \in \Theta$.*
Case I: *Suppose $w(x)$ fixes one of r_i, then $g(x) - x$ has root r_i, and so $f(x) \mid (w(x) - x)$. In this case $w(r_i) = r_i$ for every $i = 1, 2, 3, 4, 5$, which implies $w(x)$ induces the identity permutation on the set Θ, that is, for $\tau = Id \in Gal(K(\Theta)/K)$ we have $\tau(z) = w(z)$ for all $z \in \Theta$.*
Case II: *Suppose $w(x)$ do not fix any of $r_i, i = 1, 2, 3, 4, 5$. Since the splitting field of $f(x)$ is of degree 5, either w induces a permutation $(r_1r_2r_3r_4r_5)$ or a permutation $(r_1r_2r_3)(r_4r_5)$. If w induces the permutation $(r_1r_2r_3)(r_4r_5)$, then $w^{\circ 2}$ induces the permutation of type $(r_1r_2r_3)(r_4)(r_5)$. This shows r_4 and r_5 are the fixed points and the permutation is not identity. So by the argument of Case I, this can not happen. Hence w induces the 5-cycle $(r_1r_2r_3r_4r_5)$. Therefore by repeated composition of w each r_1, r_2, r_3, r_4, r_5 can be expressed as a polynomial in r_1. In other words, the splitting field is $K(\Theta) = K[r_1]$ of degree 5. Now we claim that w induces the same permutation as a power of τ. Without loss of generality, choose the notation such that $\tau = (r_1r_2r_3r_4r_5)$. Now we have the following subcases:*
(i) if $w(r_1) = r_2$ then $\tau(w(r_1)) = \tau(r_2) \Rightarrow w(\tau(r_1)) = r_3 \Rightarrow w(r_2) = r_3$. Applying τ on both sides of $w(r_2) = r_3$, we get $w(r_3) = r_4$. Once again, applying τ on $w(r_3) = r_4$, we get $w(r_4) = r_5$. So indeed w induces τ.
(ii) if $w(r_1) = r_3$, similarly, w induces τ^2.
(iii) if $w(r_1) = r_4$, similarly, w induces τ^3.
(iv) if $w(r_1) = r_5$, similarly, w induces τ^4.
Finally, since there is a continuous surjection $\mathrm{Gal}(\bar{K}/K) \twoheadrightarrow \mathrm{Gal}(K(\Theta)/K)$, for the given τ there exist a $\sigma \in Gal(\bar{K}/K)$ such that $\sigma|_{K(\Theta)} = \tau$ so that $\sigma|_{K(\Theta)}(r_i) = w(r_i)$ for all $r_i \in \Theta$.

Lemma 4.4. *Let $\mathcal{D}$ be stable p-adic dynamical system over $\mathcal{O}_K$ and $I(x) \in x \cdot \mathcal{O}_K[[x]]$. If $I(z) \in Preper(\mathcal{D})$ for infinitely many z, then $I \in \mathcal{D}$.*

Proof. Since h maps $T(f_F)$ into $T(f)$ by Lemma 3.5, Theorem 4.2 implies there is a $\sigma \in \mathrm{Gal}(\bar{K}/K)$ and a $w \in \mathcal{D}$ such that $\sigma(z) = w(z)\ \forall\ z \in \mathrm{Preper}(\mathcal{D})$. If $z \in \mathrm{Preper}(\mathcal{D})$, then we have

$$\sigma(I(z)) = w(I(z)). \tag{4.5}$$

Since $\mathrm{Preper}(\mathcal{D})$ is stable under the action of $\mathrm{Gal}(\bar{K}/K)$, for all $z \in \mathrm{Preper}(\mathcal{D})$

$$\begin{aligned} \sigma(I(z)) &= I(\sigma(z)) = I(w(z)) \\ &\Rightarrow \sigma(I(z)) = I(w(z)) \end{aligned} \tag{4.6}$$

From equations (4.5) and (3.3), we have $w(I(z)) = I(w(z))\ \forall\ z \in \mathrm{Preper}(\mathcal{D})$. But since $\mathrm{Preper}(\mathcal{D})$ is infinite, by Zariski dense property, we get $w \circ I = I \circ w$. This shows $I \in \mathcal{D}$. □

Alternative proof of Theorem 3.8. By Theorem 4.2, there exists an element $\sigma \in \mathrm{Gal}(\bar{K}/K)$ and a stable power series w in $\mathcal{D}_1$ such that

$$\sigma(z) = w(z)\ \forall\ z \in \mathrm{Preper}(\mathcal{D}_1).$$

The set $\mathcal{Z} := \mathrm{Preper}(\mathcal{D}_1) \cap \mathrm{Preper}(\mathcal{D}_2)$ is stable under the action of $\mathrm{Gal}(\bar{K}/K)$, and hence for all $z \in \mathcal{Z}$, we have $\sigma(z) \in \mathcal{Z}$. Therefore $w(z) \in \mathcal{Z}$ because $\sigma(z) = w(z)\ \forall\ z \in \mathrm{Preper}(\mathcal{D}_1)$. Since $\mathcal{Z} \subset \mathrm{Preper}(\mathcal{D}_2)$ is infinite, by the Lemma 4.4, we get $w \in \mathcal{D}_2$. This forces to conclude $\mathcal{D}_1 = \mathcal{D}_2$. □

We have produced the following two situations towards justification of the Theorem 3.8.

Example 4.5. *We establish our argument rather contrapositively. We claim that there can not be two "different" stable p-adic dynamical systems $\mathcal{D}_1$ and $\mathcal{D}_2$ over $\mathcal{O}_K$ satisfying the statement of Theorem 3.8. For, if $\mathcal{D}_1 \neq \mathcal{D}_2$ satisfies Preper($\mathcal{D}_1$) $\cap$ Preper($\mathcal{D}_2$)=infinite. Then there exists two noninvertible series $f_1(x),\ f_2(x)$ respectively $\mathcal{D}_1,\ \mathcal{D}_2$ such that $f_1 \circ f_2 \neq f_2 \circ f_1$. But since Preper($\mathcal{D}_1$) $\cap$ Preper($\mathcal{D}_2$) is infinite, both $f_1 - f_2$ vanishes on the infinite set Preper($\mathcal{D}_1$) $\cap$ Preper($\mathcal{D}_2$). Thus by Zariski dense property, we have $f_1 = f_2$ and hence $f_1 \circ f_2 = f_1^{\circ 2} = f_2 \circ f_1$, which is a contradiction. Thus our claim is established.*

Example 4.6. *Consider the noninvertible series $f_F(x) = 3x + x^3$ over $\mathbb{Z}_3$, where $\mathbb{Z}_3$ is the ring of integers of the 3-adic field $\mathbb{Q}_3$. It is an endomorphism of a 1-dimensional Lubin-Tate formal group F over $\mathbb{Z}_3$. Our idea is to recoordinatize the endomorphism f_F and to form its condensation $\left(f_F\left(x^{\frac{1}{p-1}}\right)\right)^{p-1}$. Let us define a map $h(x) = x^2$ so that $h \circ f_F = f \circ h$, and hence h is an isogeny from f_F to f. Consider a stable p-adic dynamical system $\mathcal{D}_1$ over $\mathbb{Z}_3$ consisting of the noninvertible series $f(x) := \left(f_F\left(x^{\frac{1}{2}}\right)\right)^2 = 9x + 6x^2 + x^3$ and the invertible series $u(x) = 4x + x^2$. It can be checked that $f \circ u = u \circ f$. Here $\Theta_1 := \{0, +\sqrt{-3}, -\sqrt{-3}\} \subset T(f_F)$ and $\Theta_2 := \{0, -3, -3\} \subset T(f)$ are sets of zeros of f_F and f, respectively. We must note that according to construction* (2.1)*, the elements in $T(f_F)$ or $T(f)$ might not belong to $\mathbb{Q}_3$ but over some algebraic extension. Indeed, here $\sqrt{-3} \notin \mathbb{Q}_3$. We only have to make sure that the isogeny h maps the zeros of f_F into the zeros of f. In fact, here the isogeny h takes Θ_1 to Θ_2 because $h(0) = 0,\ h(\sqrt{-3}) = 3,\ h(-\sqrt{-3}) = 3$. Clearly $f(x)$ can not be an endomorphism of the formal group F (not even any formal group) because it has repeated root. So the choice $\mathcal{D}_1$ is nontrivial with background formal group F and compatible with respect to the statement of the Theorem 3.8, in other word, the dynamical systems in our theorem exists.*

The more difficult is to find another stable p-adic dynamical system $\mathcal{D}_2$ satisfying same criteria as $\mathcal{D}_1$. We earnestly hope that any example satisfying the statement to that of Theorem 3.3 would lead us to find $\mathcal{D}_2$. However, we can create an easier $\mathcal{D}_2$ as follows.

For, let us consider another Lubin-Tate formal group G over $\mathbb{Z}_3$ with noninvertible endomorphism g_G satisfying $g_G(x) \equiv 3^2x$ (mod degree 2) and $g_G(x) \equiv x^{3^2}$ (mod $3\mathbb{Z}_3$). Such a non-invertible endomorphism is $g_G(x) = 9x + 30x^3 + 27x^5 + 9x^7 + x^9$ which commutes with an invertible endomorphism $u_G(x) = 5x + 5x^2 + x^5$ such that $g_G(G(x, y)) = G(g_G(x), g_G(y))$ and $u_G(G(x, y)) = G(u_G(x), u_G(y))$. We have formed the condensation $g(x) = 81x + 540x^2 + 1386x^3 +$

$1782x^4 + 1287x^5 + 546x^6 + 135x^7 + 18x^8 + x^9$ *of the endomorphism* $g_G(x)$ *by the isogeny* $h_2(x) = x^2$ *such that* $h_2 \circ g_G = g \circ h_2$ *and* h_2 *maps the zeros of* g_G *into the zeros of* g. *Further,* g *has double roots* -3 *and hence it can not be an endomorphism of any formal group. We have created an invertible series* $\tilde{u}(x) = 25x + 50x^2 + 35x^3 + 10x^4 + x^5$ *such that* $g \circ \tilde{u} = \tilde{u} \circ g$. *Thus we construct* $\mathcal{D}_2$ *as a stable* p*-adic dynamical system consisting of the invertible series* $\tilde{u}$ *and the noninvertible series* g, *whose background formal group is* G.

Finally, since $Preper(\mathcal{D}_1)$ *and* $Preper(\mathcal{D}_2)$ *are independent of choices of stable series in* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *respectively, we can take* $Preper(\mathcal{D}_1) = Preper(u)$ *and* $Preper(\mathcal{D}_2) = T(g)$. *But,* $u \in \mathcal{D}_1$ *commutes with* $g \in \mathcal{D}_2$ *and hence* $Preper(\mathcal{D}_1) \cap Preper(\mathcal{D}_2)$ *is infinite. On the other hand,* f_F *commutes with* u_G *and hence* $Tors(F) \cap Tors(G)$ *is infinite, which implies* $F = G$. *The uniqueness property in Propsition 3.7 says* $\mathcal{D}_1 = \mathcal{D}_2$, *and this is indeed true by our construction.*

Remark 4.7. *The existence of the stable* p*-adic dynamical systems* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *in the above Example 4.6 supports the Conjecture 3.1.*

Remark 4.8. *The Theorem 3.3 deals with the category of stable* p*-adic dynamical systems which are endomorphisms of formal groups while the main Theorem 3.8 of this paper deals with and classifies larger category of stable* p*-adic dynamical systems.*

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