ISSN 2070-0466, p-Adic Numbers, Ultrametric Analysis and Applications, 2021, Vol. 13, No. 3, pp. 239–249. © Pleiades Publishing, Ltd., 2021.

RESEARCH ARTICLES

p-Adic Dynamical Systems of the Function ax^{-2}

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Received January 14, 2021; in final form, February 12, 2021; accepted February 12, 2021

Abstract—In this paper we study p-adic dynamical systems generated by the function $f(x) = \frac{a}{x^2}$ in the set of complex p-adic numbers. We find an explicit formula for the n-fold composition of f for any $n \geq 1$. Using this formula we give fixed points, periodic points, basin of attraction and Siegel disk of each fixed (periodic) point depending on parameters p and a.

DOI: 10.1134/S2070046621030067

Key words: *dynamical systems, fixed point, periodic point, Siegel disk, complex p-adic field.*

1. INTRODUCTION

Nowadays the theory of p-adic numbers is one of very actively developing area in mathematics. It has numerous applications in many branches of mathematics, biology, physics and other sciences (see for example [4, 7, 12] and the references therein).

In this paper we continue our study of p-adic dynamical systems generated by rational functions (see [1-10]) and references therein for motivations and history of p-adic dynamical systems).

Let us recall the main definition of the paper:

p-Adic numbers. Denote by (n, m) the greatest common divisor of the positive integers n and m.

Let $\mathbb{Q}$ be the field of rational numbers.

For each fixed prime number p, every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{n}{m}$, where $r, n \in \mathbb{Z}$, m is a positive integer, $(p, n) = 1$, $(p, m) = 1$.

The p-adic norm of x is given by

$$|x|_p = \begin{cases} p^{-r}, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$$

It has the following properties:

1) $|x|_p \geq 0$ and $|x|_p = 0$ if and only if $x = 0$,

2) $|xy|_p = |x|_p |y|_p$,

3) the strong triangle inequality

$$|x + y|_p \leq \max\{|x|_p, |y|_p\},$$

3.1) if $|x|_p \neq |y|_p$ then $|x + y|_p = \max\{|x|_p, |y|_p\}$,

3.2) if $|x|_p = |y|_p$ then for $p = 2$ we have $|x + y|_p \leq \frac{1}{2}|x|_p$ (see [12]).

The completion of $\mathbb{Q}$ with respect to p-adic norm defines the p-adic field which is denoted by $\mathbb{Q}_p$ (see [5]).

The algebraic completion of $\mathbb{Q}_p$ is denoted by $\mathbb{C}_p$ and it is called the set of *complex p-adic numbers*.

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For any $a \in \mathbb{C}_p$ and $r > 0$ denote

$$U_r(a) = \{x \in \mathbb{C}_p : |x - a|_p < r\}, \;\; V_r(a) = \{x \in \mathbb{C}_p : |x - a|_p \leq r\},$$

$$S_r(a) = \{x \in \mathbb{C}_p : |x - a|_p = r\}.$$

Dynamical systems in $\mathbb{C}_p$. To define a dynamical system we consider a function $f : x \in U \to f(x) \in U$, (in this paper $U = U_r(a)$ or $\mathbb{C}_p$) (see for example [6]).

For $x \in U$ denote by $f^n(x)$ the n-fold composition of f with itself (i.e. n times iteration of f to x):

$$f^n(x) = \underbrace{f(f(f \dots (f}_{n \, \text{times}}(x))) \dots).$$

For arbitrary given $x_0 \in U$ and $f : U \to U$ the discrete-time dynamical system (also called the trajectory) of x_0 is the sequence of points

$$x_0, x_1 = f(x_0), x_2 = f^2(x_0), x_3 = f^3(x_0), \dots \tag{1.1}$$

The main problem: Given a function f and initial point x_0 what ultimately happens with the sequence (1.1). Does the limit $\lim_{n\to\infty} x_n$ exist? If not what is the set of limit points of the sequence?

A point $x \in U$ is called a fixed point for f if $f(x) = x$. The point x is a periodic point of period m if $f^m(x) = x$. The least positive m for which $f^m(x) = x$ is called the prime period of x.

A fixed point x_0 is called an *attractor* if there exists a neighborhood $U(x_0)$ of x_0 such that for all points $x \in U(x_0)$ it holds $\lim\limits_{n\to\infty} f^n(x) = x_0$. If x_0 is an attractor then its *basin of attraction* is

$$\mathcal{A}(x_0) = \{x \in \mathbb{C}_p : \; f^n(x) \to x_0, \; n \to \infty\}.$$

A fixed point x_0 is called *repeller* if there exists a neighborhood $U(x_0)$ of x_0 such that $|f(x) - x_0|_p > |x - x_0|_p$ for $x \in U(x_0)$, $x \neq x_0$.

Let x_0 be a fixed point of a function $f(x)$. Put $\lambda = f'(x_0)$. The point x_0 is attractive if $0 < |\lambda|_p < 1$, *indifferent* if $|\lambda|_p = 1$, and repelling if $|\lambda|_p > 1$.

The ball $U_r(x_0)$ (contained in V) is said to be a *Siegel disk* if each sphere $S_\rho(x_0)$, $\rho < r$ is an invariant sphere of $f(x)$, i.e. if $x \in S_\rho(x_0)$ then all iterated points $f^n(x) \in S_\rho(x_0)$ for all $n = 1, 2 \dots$. The union of all Siegel disks with the center at x_0 is called *a maximum Siegel disk* and is denoted by $SI(x_0)$.

In Section 2 we consider the function $f(x) = \frac{a}{x^2}$ and study the dynamical systems generated by this function in $\mathbb{C}_p$. We give fixed points, periodic points, basin of attraction and Siegel disk of each fixed (and periodic) point.

2. THE FUNCTION a/x^2

Consider the dynamical system associated with the function $f : \mathbb{C}_p \to \mathbb{C}_p$ defined by

$$f(x) = \frac{a}{x^2}, \;\; a \neq 0, \;\; a \in \mathbb{C}_p, \tag{2.1}$$

where $x \neq 0$.

Denote by $\theta_{j,n}$, $j = 1, \dots, n$, the nth root of unity in $\mathbb{C}_p$, while $\theta_{1,n} = 1$.

This function has three fixed points x_k, $k = 1, 2, 3$, which are solutions to $x^3 = a$ in $\mathbb{C}_p$.

For these fixed points we have

$$x_k^3 = a \;\; \Rightarrow \;\; x_k = \theta_{k,3} a^{\frac{1}{3}} \;\; \Rightarrow \;\; |x_k^3|_p = |a|_p \;\; \Rightarrow \;\; |x_k|_p = \alpha \equiv (|a|_p)^{1/3}. \tag{2.2}$$

Thus $x_k \in S_\alpha(0)$, $k = 1, 2, 3$.

We have

$$f'(x) = \frac{-2a}{x^3} = \frac{-2}{x} \cdot f(x).$$

Therefore at a fixed point we get

$$f'(x_k) = \frac{-2}{x_k} \cdot f(x_k) = -2.$$

$$|f'(x_k)|_p = \begin{cases} 1/2, & \text{if } p = 2 \\ 1, & \text{if } p \geq 3 \end{cases}$$

Hence the fixed point x_k is an attractive for $p = 2$ and an indifferent for $p \geq 3$. Therefore the fixed point is never repeller.

We can explicitly calculate f^n.

Lemma 2.1. *For any $x \in \mathbb{C}_p \setminus \{0\}$ we have*

$$f^n(x) = a^{\frac{1}{3}(1-(-2)^n)} \cdot x^{(-2)^n}, \ \ n \geq 1.$$

Proof. We use induction over n. For $n = 1, 2$ the formula is clear. Assume it is true for n and show it for $n + 1$:

$$\begin{aligned} f^{n+1}(x) = f^n(f(x)) &= a^{\frac{1}{3}(1-(-2)^n)} \cdot (f(x))^{(-2)^n} \\ &= a^{\frac{1}{3}(1-(-2)^n)} \cdot (\frac{a}{x^2})^{(-2)^n} = a^{\frac{1}{3}(1-(-2)^{n+1})} \cdot x^{(-2)^{n+1}}. \end{aligned}$$

This completes the proof. □

Recall $\alpha = (|a|_p)^{1/3}$. For $r > 0$, take $x \in S_r(0)$, i.e., $|x|_p = r$. Then we have

$$|f^n(x)|_p = \left|a^{\frac{1}{3}(1-(-2)^n)} \cdot x^{(-2)^n}\right|_p = \alpha^{1-(-2)^n} \cdot r^{(-2)^n}, \ \ n \geq 1. \tag{2.3}$$

2.1. Dynamics on $\mathbb{C}_p \setminus S_\alpha(0)$

Lemma 2.2. *For α defined in (2.2) the following assertions hold:*

1. *The sphere $S_\alpha(0)$ is invariant with respect to f, (i.e., $f(S_\alpha(0)) \subset S_\alpha(0)$);*
2. *$f(U_\alpha(0)) \subset \mathbb{C}_p \setminus V_\alpha(0)$;*
3. *$f(\mathbb{C}_p \setminus V_\alpha(0)) \subset U_\alpha(0)$.*

Proof. 1. If $x \in S_\alpha(0)$, i.e., $|x|_p = \alpha$, then

$$|f(x)|_p = |\frac{a}{x^2}|_p = \frac{|a|_p}{\alpha^2} = \alpha.$$

2. If $x \in U_\alpha(0)$, i.e., $|x|_p < \alpha$, then

$$|f(x)|_p = |\frac{a}{x^2}|_p > \frac{|a|_p}{\alpha^2} = \alpha.$$

Therefore, $f(x) \in \mathbb{C}_p \setminus V_\alpha(0)$. Proof of the part 3 is similar. □

Lemma 2.3. *The function (2.1) does not have any periodic point in $\mathbb{C}_p \setminus S_\alpha(0)$.*

Proof. We know that all three fixed points belong to $S_\alpha(0)$. Let $x \in \mathbb{C}_p \setminus S_\alpha(0)$ be a m-periodic ($m \geq 2$) point for (2.1), i.e., x satisfies $f^m(x) = x$. Then it is necessary that $|f^m(x)|_p = |x|_p$. But for any $x \in \mathbb{C}_p \setminus S_\alpha(0)$ (i.e. $|x|_p = r \neq \alpha$), by (2.3) we get

$$|f^m(x)|_p = \alpha^{1-(-2)^m} \cdot r^{(-2)^m} = \alpha \cdot \left(\frac{r}{\alpha}\right)^{(-2)^m} \neq r, \ \forall r \neq \alpha. \tag{2.4}$$

Therefore, $f^m(x) = x$ is not satisfied for any $x \in \mathbb{C}_p \setminus S_\alpha(0)$. □

For given $r > 0$, denote

$$r_n = \alpha^{1-(-2)^n} \cdot r^{(-2)^n}.$$

Then by (2.3) one can see that the trajectory $f^n(x)$, $n \geq 1$ of $x \in S_r(0)$ has the following sequence of spheres on its route:

$$S_r(0) \to S_{r_1}(0) \to S_{r_2}(0) \to S_{r_3}(0) \to \dots$$

Now we calculate the limits of r_n.

Case of even n. From (2.3) it is easy to see that

$$\lim_{n\to\infty} |f^n(x)|_p = \lim_{n\to\infty} r_n = \begin{cases} 0, \ \text{if } \ r < \alpha \\ \alpha, \ \text{if } \ r = \alpha \\ +\infty, \ \text{if } \ r > \alpha. \end{cases}$$

Case of odd n. In this case we have

$$\lim_{n\to\infty} |f^n(x)|_p = \lim_{n\to\infty} r_n = \begin{cases} +\infty, \ \text{if } \ r < \alpha \\ \alpha, \ \text{if } \ r = \alpha \\ 0, \ \text{if } \ r > \alpha. \end{cases}$$

Summarizing above-mentioned results we obtain the following theorem:

Theorem 2.4. *Let α be defined by (2.2). Then*

1. *if $x \in U_\alpha(0)$ then*

$$\lim_{k\to\infty} f^{2k}(x) = 0, \quad \lim_{k\to\infty} |f^{2k-1}(x)|_p = +\infty.$$

2. *if $x \in S_\alpha(0)$ then $f^n(x) \in S_\alpha(0)$, $n \geq 1$.*
3. *if $x \in \mathbb{C}_p \setminus V_\alpha(0)$ then*

$$\lim_{k\to\infty} |f^{2k}(x)|_p = +\infty, \quad \lim_{k\to\infty} f^{2k-1}(x) = 0.$$

Remark 2.5. *Note that Theorem 2.4 is true for more general function: $f(x) = \frac{a}{x^q}$, where q is a natural number, $q \geq 2$. In this case $\alpha = |a|_p^{1/(q+1)}$. The case $q = 1$ is simple: in this case any point $x \in \mathbb{C}_p \setminus \{0\}$ is 2-periodic. That is $f(f(x)) = x$. Indeed,*

$$f(f(x)) = \frac{a}{\frac{a}{x}} = a \cdot \frac{x}{a} = x.$$

2.2. Dynamics on $S_\alpha(0)$

By Theorem 2.4 it remains to study the dynamical system of $f : S_\alpha(0) \to S_\alpha(0)$. Recall that all fixed points x_k, $k = 1, 2, 3$ are in $S_\alpha(0)$.

Lemma 2.6. *The distance between fixed points is*

$$|x_1 - x_2|_p = |x_1 - x_3|_p = |x_2 - x_3|_p = \begin{cases} \alpha, & \text{if } p \neq 3 \\ \frac{\alpha}{\sqrt{3}}, & \text{if } p = 3. \end{cases} \tag{2.5}$$

Proof. Since $x_i^3 = a$, $i = 1, 2, 3$, for $x_i \neq x_j$ we have

$$0 = x_i^3 - x_j^3 = (x_i - x_j)(x_i^2 + x_i x_j + x_j^2) \ \Rightarrow \ x_i^2 + x_i x_j + x_j^2 = 0$$

$$\Leftrightarrow \ (x_i - x_j)^2 = -3x_i x_j \ \Rightarrow \ |x_i - x_j|_p^2 = |3x_i x_j|_p.$$

From the last equality, using $|x_i|_p = |x_j|_p = \alpha$, we get (2.5). □

Take $x \in S_\alpha(0)$ such that $|x - x_1|_p = \rho$, i.e., $x = x_1 + \gamma$, with $|\gamma|_p = \rho$. Note that $\rho \leq \alpha$. Then by Lemma 2.1 we have

$$|f^n(x) - x_1|_p = |f^n(x) - f^n(x_1)|_p = \alpha^{1-(-2)^n} |x^{(-2)^n} - x_1^{(-2)^n}|_p. \tag{2.6}$$

Now we use the following formula

$$x^{2^n} - y^{2^n} = (x - y) \prod_{j=0}^{n-1} (x^{2^j} + y^{2^j}).$$

Then from (2.6) we get

$$|f^n(x) - x_1|_p = \alpha^{1-(-2)^n} \cdot \begin{cases} \rho \prod_{j=0}^{n-1} |(x_1 + \gamma)^{2^j} + x_1^{2^j}|_p, & \text{if } n \text{ is even} \\ \frac{\rho}{|xx_1|_p} \prod_{j=0}^{n-1} |(x_1 + \gamma)^{-2^j} + x_1^{-2^j}|_p, & \text{if } n \text{ is odd.} \end{cases} \tag{2.7}$$

We have

$$|(x_1 + \gamma)^{2^j} + x_1^{2^j}|_p = \left| 2x_1^{2^j} + \sum_{s=1} \binom{2^j}{s} x_1^{2^j - s} \gamma^s \right|_p = \begin{cases} |2|_p \alpha^{2^j}, & \text{if } \rho < \alpha \\ \leq |2|_p \alpha^{2^j}, & \text{if } \rho = \alpha. \end{cases} \tag{2.8}$$

Here we used that

$$\left| \binom{2^j}{s} \right|_p \leq \begin{cases} \frac{1}{2}, & \text{if } p = 2 \\ 1, & \text{if } p \geq 3. \end{cases}$$

Using (2.8) we get

$$|(x_1 + \gamma)^{-2^j} + x_1^{-2^j}|_p = \frac{|(x_1 + \gamma)^{2^j} + x_1^{2^j}|_p}{|(x_1 + \gamma)^{2^j} x_1^{2^j}|_p} = \begin{cases} |2|_p \alpha^{-2^j}, & \text{if } \rho < \alpha \\ \leq |2|_p \frac{1}{|(x_1+\gamma)^{2^j}|_p}, & \text{if } \rho = \alpha. \end{cases} \tag{2.9}$$

In **case of even** n, by (2.8) from (2.7), we get

$$|f^n(x) - x_1|_p = \alpha^{1-2^n} \cdot \rho \prod_{j=0}^{n-1} |(x_1 + \gamma)^{2^j} + x_1^{2^j}|_p$$

$$= \rho \cdot \alpha^{1-2^n} \cdot |2|_p^n \prod_{j=0}^{n-1} \alpha^{2^j} \cdot \begin{cases} 1, & \text{if } \rho < \alpha \\ \leq 1, & \text{if } \rho = \alpha \end{cases} = \rho \cdot |2|_p^n \cdot \begin{cases} 1, & \text{if } \rho < \alpha \\ \leq 1, & \text{if } \rho = \alpha. \end{cases} \tag{2.10}$$

Similarly, in **case of odd** n, by (2.9) from (2.7) we get

$$|f^n(x) - x_1|_p = \alpha^{1+2^n} \cdot \frac{\rho}{\alpha^2} \cdot |2|_p^n \prod_{j=0}^{n-1} \alpha^{-2^j} = \rho \cdot |2|_p^n \ \text{ if } \ \rho < \alpha. \tag{2.11}$$

The same formulas are also true for x_2 and x_3.

For fixed α (defined in (2.2)) and $t \in S_\alpha(0)$ denote

$$\mathcal{S}_{\rho,t} = S_\alpha(0) \cap S_\rho(t) = \{x \in S_\alpha(0) : |x-t|_p = \rho\}.$$

Thus we have proved the following lemma

Lemma 2.7. *Let $\rho < \alpha$. Then for any $x \in \mathcal{S}_{\rho,x_i}$ $(i = 1, 2, 3)$ we have*

- *if $p = 2$ then*

$$f^n(x) \in \mathcal{S}_{2^{-n}\rho,x_i}.$$

- *if $p \geq 3$ then*

$$f^n(x) \in \mathcal{S}_{\rho,x_i}, \ \ n \geq 1.$$

In particular, the set $\mathcal{S}_{\rho,x_i}$ is invariant with respect to f for any $\rho < \alpha$.

Denote

$$\mathcal{V}_{\rho,t} = \bigcup_{0 \leq r < \rho} \mathcal{S}_{r,t} = \{x \in S_\alpha(0) : |x-t|_p < \rho\}.$$

Lemma 2.8. *If $x \in \mathcal{S}_{\rho,x_i}$, for some $i = 1, 2, 3$, then:*

i. If ρ is such that

$$\rho < \begin{cases} \alpha, & \text{if } p \neq 3 \\ \frac{\alpha}{\sqrt{3}}, & \text{if } p = 3. \end{cases}$$

then

$$x \in \begin{cases} \mathcal{S}_{\frac{\alpha}{\sqrt{3}},x_j}, & \text{for } p = 3 \\ \mathcal{S}_{\alpha,x_j}, & \text{for } p \neq 3, \end{cases} \quad j \neq i.$$

ii. If $p = 3$ and $\rho \geq \frac{\alpha}{\sqrt{3}}$ then

$$x \in \begin{cases} \mathcal{V}_{\rho,x_j}, & \text{for } \rho = \frac{\alpha}{\sqrt{3}} \\ \mathcal{S}_{\rho,x_j}, & \text{for } \rho > \frac{\alpha}{\sqrt{3}}, \end{cases} \quad j \neq i.$$

Proof. For $x \in \mathcal{S}_{\rho,x_i}$, using property of p-adic norm and formula (2.5) we get

$$|x - x_j|_p = |x - x_i + x_i - x_j|_p = \begin{cases} \alpha, & \text{if } p \neq 3 \\ \frac{\alpha}{\sqrt{3}}, & \text{if } p = 3, \ \rho < \frac{\alpha}{\sqrt{3}} \\ \leq \rho, & \text{if } p = 3, \ \rho = \frac{\alpha}{\sqrt{3}} \\ \rho, & \text{if } p = 3, \ \rho > \frac{\alpha}{\sqrt{3}} \end{cases}$$

This completes the proof. □

Denote

$$\mathcal{U}_\alpha = \{x \in S_\alpha(0) : |x - x_1|_p = |x - x_2|_p = |x - x_3|_p = \alpha\}.$$

As a corollary of Lemma 2.8 we have

Lemma 2.9. *If $p \neq 3$ then $S_\alpha(0)$ has the following partition*

$$S_\alpha(0) = \mathcal{U}_\alpha \cup \bigcup_{i=1}^{3} \mathcal{V}_{\alpha,x_i}.$$

Lemma 2.10. *Let α be defined by (2.2). Then:*

1. *If $p = 2$ then the set $\mathcal{U}_\alpha$ is invariant with respect to f.*
2. *If $p \geq 3$ and $x \in \mathcal{U}_\alpha$ then one of the following assertions holds:*

2.a) There exists n_0 and $\mu_{n_0} < \alpha$ such that

$$f^n(x) \in \mathcal{U}_\alpha, \ \ \forall n \leq n_0,$$

$$f^n(x) \in \mathcal{S}_{\mu_{n_0}}(x_i), \ \ \forall n > n_0 \ \textit{for some} \ i = 1, 2, 3.$$

2.b) $f^n(x) \in \mathcal{U}_\alpha, \ \ \forall n \geq 1.$

Proof. 1. For any $x \in \mathcal{U}_\alpha$ we have

$$|f(x) - x_i|_p = \left|\frac{a}{x^2} - \frac{a}{x_i^2}\right|_p = |a|_p \left|\frac{(x_i - x)(x_i + x)}{x^2 x_i^2}\right|_p$$

$$= \alpha^3 \cdot \frac{\alpha|x + x_i|_p}{\alpha^4} = |x + x_i|_p = |x - x_i + 2x_i|_p = \begin{cases} \alpha, & \text{if } p = 2 \\ \mu_{1,i}, & \text{if } p \geq 3, \end{cases} \tag{2.12}$$

where $\mu_{1,i} \leq \alpha$. The part 1 follows from this equality.

2. If in (2.12) there exists i such that $\mu_{1,i} = |x + x_i|_p < \alpha$, then $f(x) \in \mathcal{S}_{\mu_{1,i},x_i}$. The set $\mathcal{S}_{\mu_{1,i},x_i}$ is invariant with respect to f. In case of all $\mu_{1,i} = \alpha$ we have $f(x) \in \mathcal{U}_\alpha$. Then we note that

$$|f^2(x) - x_i|_p = |f(x) - x_i + 2x_i|_p = \begin{cases} \alpha, & \text{if } p = 2 \\ \mu_{2,i} \leq \alpha, & \text{if } p \geq 3. \end{cases}$$

Thus we can repeat the above argument: if there exists i such that $\mu_{2,i} < \alpha$, then $f^2(x) \in \mathcal{S}_{\mu_{2,i},x_i}$ which is invariant with respect to f. If all $\mu_{2,i} = \alpha$ then $f^2(x) \in \mathcal{U}_\alpha$. Iterating this argument one proves the part 2. □

Lemma 2.11. *For $k \in \{1,2,3\}$, $j \in \{1,2,3\} \setminus \{k\}$ and fixed points x_k, x_j we have*

1. $x_j \notin \mathcal{V}_{\rho,x_k}$, *if and only if*

$$\rho \leq \begin{cases} \alpha, & \text{if } p \neq 3 \\ \frac{\alpha}{\sqrt{3}}, & \text{if } p = 3. \end{cases}$$

2. *if $p = 2$ then*

$$\mathcal{V}_{\alpha,x_j} \cap \mathcal{V}_{\alpha,x_k} = \emptyset, \ \text{for all } j,k \in \{1,2,3\}, \ j \neq k,$$

Proof. Follows from (2.5) and Lemma 2.7. □

Summarizing above mentioned results we get

Theorem 2.12. *If α is defined by (2.2). Then for the dynamical system generated by $f : S_\alpha(0) \to S_\alpha(0)$ given in (2.1) the following assertions hold.*

1. *If $p = 2$ then $\mathcal{A}(x_j) = \mathcal{V}_{\alpha,x_j}$, i.e.,*

$$\lim_{n\to\infty} f^n(x) = x_j, \ \text{for any } x \in \mathcal{V}_{\alpha,x_j}.$$

$$f^n(x) \in \mathcal{U}_\alpha, \ n \geq 1, \ \text{for all } x \in \mathcal{U}_\alpha.$$

2. *If $p \geq 3$ then*

$$SI(x_j) = \mathcal{V}_{\alpha,x_j}, \ j \in \{1,2,3\}.$$

Moreover,

$$SI(x_1) = SI(x_2) = SI(x_3), \ \text{if } p = 3.$$

$$SI(x_j) \cap SI(x_k) = \emptyset, \ \text{if } p > 3.$$

3. *If $p \geq 3$ and $x \in \mathcal{U}_\alpha$ then one of the following assertions holds*

3.a) *There exists n_0 and $\mu_{n_0} < \alpha$ such that*

$$f^n(x) \in \mathcal{U}_\alpha, \ \forall n \leq n_0,$$

$$f^n(x) \in \mathcal{S}_{\mu_{n_0}}(x_i), \ \forall n > n_0 \ \text{for some } i = 1,2,3.$$

3.b) $f^n(x) \in \mathcal{U}_\alpha, \ \forall n \geq 1.$

This theorem does not give behavior of $f^n(x) \in \mathcal{U}_\alpha$, $n \geq 1$, i.e., in the case when the trajectory remains in $\mathcal{U}_\alpha$ (that is when $p = 2$ and in the case part 3.b of Theorem 2.12). Since there is not any fixed point of f in $\mathcal{U}_\alpha$, below we are interested to periodic points of f in $\mathcal{U}_\alpha$: for a given natural $m \geq 2$ the m-periodic points of this set are solutions of the following system of equations

$$\begin{aligned} &f^m(x) = a^{\frac{1}{3}(1-(-2)^m)} \cdot x^{(-2)^m} = x, \\ &|x - x_1|_p = |x - x_2|_p = |x - x_3|_p = \alpha. \end{aligned} \tag{2.13}$$

Remark 2.13. *Note that in case $m = 2$, there is no any solution to the first equation of (2.13) (except fixed points). Therefore below we consider the case $m \geq 3$.*

Denote

$$M_m = \begin{cases} \left\{(j,p) : |\theta_{k,3} - \theta_{j,2^m-1}|_p = 1, \ \forall k = 1,2,3\right\} & \text{if } m \text{ is even}, \\ \left\{(j,p) : |\theta_{k,3} - \theta_{j,2^m+1}|_p = 1, \ \forall k = 1,2,3\right\} & \text{if } m \text{ is odd}. \end{cases}$$

Lemma 2.14. *The solutions of the system (2.13) in $\mathbb{C}_p$ are*

$$\hat{x}_j = a^{\frac{1}{3}} \cdot \begin{cases} \theta_{j,2^m-1}, & \text{if } m \text{ is even}, \\ 1/\theta_{j,2^m+1}, & \text{if } m \text{ is odd}, \end{cases} \tag{2.14}$$

where $(j,p) \in M_m$.

Proof. From (2.13) we get

$$\left(\frac{x}{a^{1/3}}\right)^{(-2)^m-1} = 1.$$

Which has solutions (2.14). The condition $(j,p) \in M_m$ is needed to satisfy the second equation of the system (2.13). □

Remark 2.15. *We note that:*

- *In the case $p = 2$, by part 1 of Theorem 2.12, it follows that all m-periodic points (except fixed ones) mentioned in (2.14) belong to $\mathcal{U}_\alpha$.*
- *In the case $m \geq 3$ and $p \geq 3$ it is not clear to see $M_m \neq \emptyset$. It is known that (see [2, Corollary 2.2.]) the equation $x^k = 1$ has $g = (k, p-1)$ different roots in $\mathbb{Q}_p$. Using this fact and assuming that $a \in \mathbb{Q}_p$ and $a^{\frac{1}{3}}$ exists in $\mathbb{Q}_p$, one can see how many periodic solutions of (2.13) exist in $\mathbb{Q}_p$. For example, if $p = 31$ then $t^3 = 1$ (with $t = \frac{x}{a^{1/3}}$) has $g = (3, 30) = 3$, i.e., all possible solutions in $\mathbb{Q}_p$ and for $m = 4$ the equation $t^{2^4-1} = 1$ has $g = (15, 30) = 15$ distinct solutions in $\mathbb{Q}_p$. Three of 15 solutions coincide with solutions of $t^3 = 1$, therefore remains 12 distinct solutions to satisfy the second equation of (2.13). For these solutions one can check the condition $M_m \neq \emptyset$.*

Lemma 2.16. *If x_* is a solution to (2.13) then*

$$x_* \ is \begin{cases} \text{attracting}, & \text{if } p = 2 \\ \text{indifferent}, & \text{if } p \geq 3. \end{cases}$$

Proof. We have

$$\left|(f^m)'(x_*)\right|_p = \left|(-2)^m \cdot a^{\frac{1}{3}(1-(-2)^m)} \cdot x_*^{(-2)^m-1}\right|_p$$

$$= \left|(-2)^m \cdot \frac{f^m(x_*)}{x_*}\right|_p = \begin{cases} 1/2^m, & \text{if } p = 2 \\ 1, & \text{if } p \geq 3. \end{cases}$$

This completes the proof. □

Consider a m-periodic point x_*. It is clear that this point is a fixed point for the function $\varphi(x) \equiv f^m(x)$. The point x_* generates m-cycle:

$$x_*, x^{(1)} = f(x_*), \dots, x^{(m-1)} = f^{m-1}(x_*).$$

Clearly, each element of this cycle is fixed point for function φ. We use the following

Theorem 2.17. *[2] Let x_0 be a fixed point of an analytic function $\varphi : U \to U$. The following assertions hold:*

1. *if x_0 is an attractive point of φ and if $r > 0$ satisfies the inequality*

$$Q = \max_{1 \le n < \infty} \left| \frac{1}{n!} \frac{d^n \varphi}{dx^n}(x_0) \right|_p r^{n-1} < 1$$

and $U_r(x_0) \subset U$ then $U_r(x_0) \subset \mathcal{A}(x_0)$;

2. *if x_0 is an indifferent point of φ then it is the center of a Siegel disk. If r satisfies the inequality*

$$S = \max_{2 \le n < \infty} \left| \frac{1}{n!} \frac{d^n \varphi}{dx^n}(x_0) \right|_p r^{n-1} < |\varphi'(x_0)|_p$$

and $U_r(x_0) \subset U$ then $U_r(x_0) \subset SI(x_0)$.

Lemma 2.16 suggests the following

Theorem 2.18.
- *If $p = 2$ then for any $m = 2, 3, \dots,$ the m-cycles are attractors and open balls with radius α are contained in the basins of attraction.*
- *If $p \ge 3$ then for any $m = 2, 3, \dots,$ every m-cycle is a center of a Siegel disk with radius α.*

Proof. Let x_* be a m-periodic point. Recall that $|x_*|_p = \alpha$. We use Theorem 2.17, by Lemma 2.1 we get:

$$\begin{aligned}
Q = \max_{1 \le n < \infty} \left| \frac{1}{n!} \frac{d^n \varphi}{dx^n}(x_*) \right|_p r^{n-1} &= \max_{1 \le n < \infty} \left| \frac{1}{n!} a^{\frac{1}{3}(1-(-2)^m)} \cdot \prod_{s=0}^{n-1} ((-2)^m - s) \cdot x_*^{(-2)^m - n} \right|_p r^{n-1} \\
&= \max_{1 \le n < \infty} \left| \frac{1}{n!} \cdot \prod_{s=0}^{n-1} ((-2)^m - s) \cdot \frac{x_*}{x_*^n} \right|_p r^{n-1} \\
&= \max_{1 \le n < \infty} \left| \frac{1}{n!} \cdot \prod_{s=0}^{n-1} ((-2)^m - s) \right|_p \left(\frac{r}{\alpha} \right)^{n-1} \\
&= \max_{1 \le n < \infty} \left(\frac{r}{\alpha} \right)^{n-1} \cdot \begin{cases} \left| \binom{2^m}{n} \right|_p, & \text{if } m - \text{even} \\ \left| \binom{2^m + n}{2^m} \right|_p, & \text{if } m - \text{odd} \end{cases} < 1.
\end{aligned} \tag{2.15}$$

If $r < \alpha$, this condition is satisfied. The second part is similar. □

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