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RESEARCH ARTICLES

A Beilinson-Bernstein Theorem for Analytic Quantum Groups. I*

Nicolas Dupré**

Universität Duisburg-Essen, Fakultät für Mathematik,
Thea-Leymann-Straße 9, D—45127 Essen, Germany

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Abstract—In this two-part paper, we introduce a p-adic analytic analogue of Backelin and Kremnizer's construction of the quantum flag variety of a semisimple algebraic group, when q is not a root of unity and $|q-1|<1$. We then define a category of λ-twisted D-modules on this analytic quantum flag variety. We show that when λ is regular and dominant and when the characteristic of the residue field does not divide the order of the Weyl group, the global section functor gives an equivalence of categories between the coherent λ-twisted D-modules and the category of finitely generated modules over $\widehat{U_q^\lambda}$, where the latter is a completion of the ad-finite part of the quantum group with central character corresponding to λ. Along the way, we also show that Banach comodules over the Banach completion $\widehat{\mathcal{O}_q(B)}$ of the quantum coordinate algebra of the Borel can be naturally identified with certain topologically integrable modules.

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1. INTRODUCTION

1.1. Background and Motivation

Let L be a complete discrete valuation field of mixed characteristic $(0,p)$, with discrete valuation ring R, uniformizer π and residue field k. We fix an element $q\in R^\times$ and assume that $q\equiv 1 \pmod{\pi}$ and that q is not a root of unity. Ardakov and Wadsley have recently started an ongoing program aiming to develop p-adic analytic analogues of D-modules in order to understand p-adic representation theory, see [2—5]. Their aim is to use p-adic analytic localisation results analogous to the classical theorem of Beilinson-Bernstein [9] in order to better understand locally analytic representations of p-adic groups, which were introduced by Schneider and Teitelbaum in a series of papers including [43—45]. There have also been other approaches at using localisation techniques to understand locally analytic representations, notably by Huyghe, Patel, Schmidt and Strauch [25, 38, 39, 41].

Let us briefly recall one of Ardakov and Wadsley's main results. Let $\mathbf{G}$ be a simply connected split semisimple algebraic group over R with R-Lie algebra $\mathfrak{g}$ and let X be its flag scheme $\mathbf{G}/\mathbf{B}$. In [3], they defined a family $(\widehat{\mathcal{U}_{n,L}})_{n\geq 0}$ of Banach completions of the enveloping algebra $U(\mathfrak{g}_L)$ of the L-Lie algebra $\mathfrak{g}_L:=\mathfrak{g}\otimes_R L$. Moreover, for a weight λ, they introduced a family $(\widehat{\mathcal{D}_{n,L}^\lambda})_{n\geq 0}$ of sheaves of completed deformed twisted crystalline differential operators on X. Their theorem then states:

Theorem 1.1 ([3]). *For any $n\geq 0$ and for λ regular and dominant, the global section functor gives an equivalence of categories between coherent sheaves of $\widehat{\mathcal{D}_{n,L}^\lambda}$-modules and finitely generated $\widehat{\mathcal{U}_{n,L}}$-modules with central character corresponding to λ.*

*The text was submitted by the author in English.

**E-mail: nicolas.dupre@uni-due.de

Our aim is to prove an analogue of the above Theorem when working with quantum groups, where for simplicity we only treat the case $n = 0$ in this paper. The study of quantum groups in a p-adic analytic setting was first proposed by Soibelman in [48], where he introduced quantum deformations of the algebras of locally analytic functions on p-adic Lie groups and of the corresponding distribution algebra. His ideas were also heavily influenced by the aforementioned work of Schneider and Teitelbaum. This paper of Soibelman then inspired a short note of Lyubinin [35] and also a different approach for GL_2 in [51]. Recently, there has also been a new approach at constructing p-adic analytic quantum groups using Nichols algebra in [47]. However, besides these, not much work has been done in this area. In [18], we constructed quantum analogues of the Arens–Michael envelope of $\mathfrak{g}_L$ and of the algebra of rigid analytic functions on the analytification of G_L, and proved that these were Fréchet–Stein algebras. We also constructed several Banach completions of those algebras, and some of these objects feature in this paper. Our hope is that more work will be done to pursue these efforts. The theory of quantum groups has strong links with the representation theory of algebraic groups in positive characteristic. We expect that a successful theory of p-adic analytic quantum groups would have similar links with the representation theory of p-adic groups, and we view our work as a first effort towards developing such a theory.

Recently, there has also been some work hinting at noncommutative analogues of rigid analytic geometry in [10]. In this light, we think that defining noncommutative analogues of analytic flag varieties as we do in this paper is interesting in its own right. It would be interesting to compare our constructions with their general framework.

1.2. Quantum Flag Varieties and Quantum D-Modules

The proof of Theorem 1.1 relied on the classical Beilinson–Bernstein theorem, and similarly we will use a quantum group analogue of it due to Backelin and Kremnizer [8][1]. We briefly recall their constructions. Let U_q be the quantized enveloping algebra of $\mathfrak{g}_L$. Let $\mathcal{O}_q$ be the quantized coordinate algebra of G_L, and let $\mathcal{O}_q(B)$ be the quotient Hopf algebra of $\mathcal{O}_q$ corresponding to a Borel subgroup of G_L. Backelin and Kremnizer then define the quantum flag variety to be the category $\mathcal{M}_{B_q}(G_q)$ of $\mathcal{O}_q(B)$-equivariant $\mathcal{O}_q$-modules. Specifically, an object of this category is an $\mathcal{O}_q$-module equipped with a right $\mathcal{O}_q(B)$-comodule structure such that $\mathcal{O}_q$-action map is a comodule homomorphism. In this language, the global section functor Γ is the functor of taking $\mathcal{O}_q(B)$-coinvariants. They then define the ring of quantum differential operators on G_L to be the smash product algebra $\mathcal{D}_q = \mathcal{O}_q \# U_q$, and a λ-twisted D-module becomes an object M of the quantum flag variety equipped with an additional $\mathcal{D}_q$-action such that the $\mathcal{O}_q(B)$-coaction and the action of the quantum Borel subalgebra $U_q^{\geq 0} \subset U_q \subset \mathcal{D}_q$ 'differ by λ' (here λ is an element of the character group T_P of the weight lattice). There is also a distinguished object $\mathcal{D}_q^\lambda$ which represents global sections in the category of λ-twisted D-modules. The precise definitions are made in Section 3. Their main theorem is that, when λ is regular and dominant, the global section functor gives an equivalence of categories between λ-twisted D-modules and modules over $\Gamma(\mathcal{D}_q^\lambda)$.

Nothing stops us from making completely analogous definitions using certain Banach completions $\widehat{\mathcal{O}_q}$, $\widehat{\mathcal{O}_q(B)}$ and $\widehat{\mathcal{D}_q}$ of these algebras (see section 1.3 below). That allows us to define what we call the analytic quantum flag variety as the category $\widehat{\mathcal{M}_{B_q}(G_q)}$ of $\widehat{\mathcal{O}_q(B)}$-equivariant Banach $\widehat{\mathcal{O}_q}$-modules, meaning that the objects of this category are Banach $\widehat{\mathcal{O}_q}$-modules which are also Banach $\widehat{\mathcal{O}_q(B)}$-comodules such that the $\widehat{\mathcal{O}_q}$-action map is a comodule homomorphism. We note that this category is not abelian. Instead it fits into Schneiders' framework of quasi-abelian categories [46]. In particular it has a derived category and, under suitable conditions, we can right derive left exact functors. The global section functor Γ here is also the functor of taking $\widehat{\mathcal{O}_q(B)}$-coinvariants, and we use this framework of quasi-abelian categories to make sense of the cohomology of Γ. We can then define λ-twisted D-modules to be objects $\mathcal{M}$ in $\widehat{\mathcal{M}_{B_q}(G_q)}$ which are equipped with an additional $\widehat{\mathcal{D}_q}$-action such that the $\widehat{\mathcal{O}_q(B)}$-coaction and the action of $\widehat{U_q^{\geq 0}}$ differ by λ. There is also a distinguished object $\widehat{\mathcal{D}_q^\lambda}$ which represents global sections. All the precise definitions will be made in the second part of the paper [19].

[1]We note that there exists a different approach to quantum D-modules and Beilinson–Bernstein by Tanisaki [49].

1.3. General Strategy

Let us briefly outline the argument used by Ardakov and Wadsley in [3] to prove that one gets an equivalence of categories in Theorem 1.1. We will employ essentially the same strategy.

1. They first work with integral versions of classical algebraic D-modules and show that large enough twists of coherent D-modules are acyclic and generated by their global sections. Using this, they then show that the category of coherent $\widehat{\mathcal{D}^{\lambda}_{n,L}}$-modules has a family of generators obtained from taking certain twists of $\widehat{\mathcal{D}^{\lambda}_{n,L}}$. In particular those are π-adic completions of algebraic D-modules.

2. The first step essentially reduces the problem to working with those coherent $\widehat{\mathcal{D}^{\lambda}_{n,L}}$-modules which can be 'uncompleted'. They then show that these are generated by their global sections. This uses the classical Beilinson-Bernstein theorem.

3. Finally, they show that completions of acyclic coherent D-modules are also acyclic. This uses technical facts about the cohomology of a projective limit of sheaves.

4. Once you know that coherent $\widehat{\mathcal{D}^{\lambda}_{n,L}}$-modules are acyclic and generated by their global sections, the result follows from standard general facts.

In order to adapt this, we are first required to work with integral forms of quantum groups and the corresponding integral quantum flag variety, see sections 2.2, 2.3 & 3.3. Specifically, there is an integral form $\mathcal{A}_q$ of $\mathcal{O}_q$ which was first defined by Andersen, Polo and Wen [1]. By taking $\mathcal{B}_q$ to be its image in the quotient Hopf algebra $\mathcal{O}_q(B)$, we are then able to define the category $\mathscr{C}_R$ of $\mathcal{B}_q$-equivariant $\mathcal{A}_q$-modules. We can also define an integral form $\mathcal{D}$ of the ring $\mathcal{D}_q$, and use it to define λ-twisted D-modules in $\mathscr{C}_R$ (here λ is an element of T_P^R, the character group over R of the weight lattice). These integral forms allow us to define the Banach completions we mentioned above by simply setting $\widehat{\mathcal{O}_q} := \widehat{\mathcal{A}_q} \otimes_R L$, $\widehat{\mathcal{O}_q(B)} := \widehat{\mathcal{B}_q} \otimes_R L$ and $\widehat{\mathcal{D}_q} := \widehat{\mathcal{D}} \otimes_R L$ respectively.

Unlike in the first step above, we are not able to show that large enough twists of coherent $\mathcal{D}$-modules are acyclic and generated by global sections, but we manage to show it for those which are annihilated by π. This turns out to be enough for the first two steps to work. We then have to develop the correct tools from noncommutative algebraic geometry in the category $\widehat{\mathcal{M}_{B_q}(G_q)}$ in order for the ideas used in the third step to even make sense.

1.4. Čech Complexes

To have a version of step 3 above, we need to work with the right sort of complexes, computing the cohomology of global sections, in order to apply the argument on the cohomology of a projective limit. To do so, it is convenient to work with proj categories. Indeed, the classical flag variety is isomorphic to $\mathrm{Proj}(\mathcal{O}(G/N))$, and Backelin-Kremnizer showed that $\mathcal{M}_{B_q}(G_q)$ is equivalent to $\mathrm{Proj}(\mathcal{O}_q(G/N))$ in the sense of Artin-Zhang [6]. We show that the integral quantum flag variety enjoys the same property. To obtain this result, one problem we ran into is that, while it is well-known that the algebra $\mathcal{O}_q$ is Noetherian, it isn't known in general whether its integral form $\mathcal{A}_q$ is also Noetherian (in type A, it is known to be true from Polo's appendix in [1]). That makes it non-trivial to define the objects which should play the role of coherent modules. Thankfully, we were able to prove that the integral form of $\mathcal{O}_q(G/N)$ is Noetherian, and using this we showed that the Noetherian objects in $\mathscr{C}_R$ are precisely those which are finitely generated over $\mathcal{A}_q$, see Theorem 3.22. Once this obstacle is cleared, the proof that we have a noncommutative projective scheme is essentially identical to the one in [8].

This result is essential because it allows us to define our promised complex which computes the cohomology of global sections for these integral forms. We think of this as a Čech-like complex. Using the Proj description of $\mathscr{C}_R$, one can in a suitable sense cover the category with analogues of the Weyl group translates of the big cell, see sections 3.9 & 3.10. The complexes are then obtained using general constructions from Rosenberg [40]. In [19], we will see that after taking π-adic completions, the objects of $\mathscr{C}_R$ are then naturally sent to another intermediate category, which we will unoriginally call $\widehat{\mathscr{C}_R}$ and which is in some sense an integral form of $\widehat{\mathcal{M}_{B_q}(G_q)}$. We will use the Weyl group localisations mentioned above to write down an analogue of our Čech-like complexes in this new integral category. After extending scalars, this will give us a Čech-like complex in the category $\widehat{\mathcal{M}_{B_q}(G_q)}$. This is the right object in order to apply the arguments from step 3.

1.5. Main Results

At several stages of this paper, we work with Banach comodules over $\widehat{\mathcal{O}_q(B)}$. We first give a more explicit description of these objects. We begin by defining what we call topologically integrable modules over a certain completion $\widehat{U^{\mathrm{res}}(\mathfrak{b})}$ of $U_q^{\geq 0}$, see section 4.3. Roughly, these are modules where the torus acts topologically semisimply and the positive part acts locally topologically nilpotently. The definition is partly inspired from work of Féaux de Lacroix [20], who developed a notion of semisimplicity for topological Fréchet modules (note that we already used the notion of topological semisimplicity in our previous work [18, Section 5]). Our first main result is then:

Theorem A. *The category* $\mathbf{Comod}(\widehat{\mathcal{O}_q(B)})$ *of Banach right* $\widehat{\mathcal{O}_q(B)}$*-comodules is canonically equivalent to the category of topologically integrable* $\widehat{U^{res}(\mathfrak{b})}_L$*-modules.*

This result allows for a more intuitive understanding of what these comodules are, and also draws further parallels between our constructions and standard notions that appear in p-adic representation theory. We note that Banach comodules over a Banach coalgebra have also been studied in a more general, categorical setting in [31].

Our next two results will be proved in [19] but we state them already. The first one states that the cohomology of Γ in $\widehat{\mathcal{M}_{B_q}(G_q)}$ can be computed using the Čech-like complexes described above:

Theorem B. *For any* $\mathcal{M} \in \widehat{\mathcal{M}_{B_q}(G_q)}$*, the standard complex* $\check{C}(\mathcal{M})$ *computes* $R\Gamma(\mathcal{M})$*.*

As a consequence of this, we will obtain that Γ has finite cohomological dimension (something which wasn't obvious beforehand!). Both of these are essential in order to obtain a Beilinson-Bernstein theorem, but we also think of them as interesting results in their own right. We view our analytic quantum flag variety as being in some sense a noncommutative analytic space, and these results make it feasible to work with it.

Finally, with all the above at hand, we are able to run the strategy from section 1.3 to obtain our version of Beilinson-Bernstein localisation. Before stating it, we need to introduce a few more notions. We call a D-module in $\widehat{\mathcal{M}_{B_q}(G_q)}$ coherent if it is finitely generated over $\widehat{\mathcal{D}_q}$. Moreover, U_q contains a subalgebra U_q^{fin}, called its finite part, which is the subalgebra of elements on which the adjoint action of U_q is locally finite. This has an integral form $U^{\mathrm{fin}} \subseteq U$ which contains the centre of U, and given λ we may form a quotient $U^\lambda = U^{\mathrm{fin}} \otimes_{Z(U)} R_\lambda$. Completing, we obtain an algebra $\widehat{U_q^\lambda} = \widehat{U^\lambda} \otimes_R L$ which is a Noetherian Banach algebra. Our Beilinson-Bernstein localisation then states:

Theorem C. *Suppose* $\lambda \in T_P^k$ *is regular and dominant, and assume that* p *is a very good prime for the root system of* $\mathfrak{g}$*. Then the functor* Γ *of global sections and the localisation functor* Loc_λ *are quasi-inverse equivalences of categories between the category* $\mathrm{coh}(\widehat{\mathcal{D}^\lambda_{B_q}(G_q)})$ *of* λ*-twisted coherent* $\widehat{\mathcal{D}_q}$*-modules on the analytic quantum flag variety and the category of finitely generated modules over* $D := \Gamma(\widehat{\mathcal{D}_q^\lambda})$*. Moreover, there is a surjective algebra homomorphism* $\widehat{U_q^\lambda} \to D$ *which is an isomorphism whenever* p *does not divide the order of the Weyl group.*

See [19] for the definitions of the localisation functor Loc_λ and for the definition of the set T_P^k, and see section 3.15 for the meaning of very good primes. We simply note here that the condition that p does not divide $|W|$ is automatically satisfied if p is larger than the Coxeter number.

Thus, we may think of the above Theorem as saying that the category $\widehat{\mathcal{M}_{B_q}(G_q)}$ is D-affine, and moreover we can identify this category with the category of finitely generated $\widehat{U_q^\lambda}$-modules under some reasonable condition on p. We note that in order to just get D-affinity without any statement on global sections, the condition on p can be weakened to say that it is a good prime, see [19] for the details.

1.6. Computation of Global Sections

We were made aware that there may be gaps in the proof of the computation of global sections in [8, Proposition 4.8], see [50, Remark 5.4]. We simply point out here that these potential issues do not affect our work as we never use their computation of global sections.

Firstly, in our proof of Theorem C, we will only use the D-affinity of quantum flag variety in order to obtain the corresponding result for Banach completions. And indeed, the equivalence of categories given by the global section functor from the category of λ-twisted D-modules on the quantum flag variety to the category of modules over the global sections of $\mathcal{D}_q^\lambda$ in [8] does not require the full computation of global sections. It does rely on a quantum analogue of the Beilinson-Bernstein 'key lemma', but that only needs for there to be a map $U_q^\lambda \to \Gamma(\mathcal{D}_q^\lambda)$ in order to interpret global sections of D-modules as modules over U_q^{fin}. That way one obtains a splitting of some particular maps at the level of global sections which can be used to prove that every D-module is acyclic and generated by its global sections, see the proof of [8, Theorem 4.12]. But we do not need to know that $U_q^\lambda \to \Gamma(\mathcal{D}_q^\lambda)$ is an isomorphism for that part of the Beilinson-Bernstein theorem to hold (this is also true classically).

Secondly, our computation of global sections via the homomorphism $\widehat{U_q^\lambda} \to D$ in [19] does not use the computation of global sections at the uncompleted level. Instead, our arguments go via reduction modulo π, where q becomes 1 and the situation becomes non-quantum. Therefore, what we crucially need is instead the computation of global sections of the sheaves of twisted crystalline differential operators on the flag variety in positive characteristic, obtained in [12, Proposition 3.4.1].

1.7. Conventions and Notation

Unless explicitly stated otherwise, the term "module" will be used to mean *left* module, and Noetherian rings are both left and right Noetherian. Also, all of our filtrations on modules or algebras will be positive and exhaustive unless specified otherwise. Following [3, Def 2.7], an R-submodule W of an L-vector space will be called a *lattice* if $V = LW$ and W is π-adically separated, i.e $\bigcap_{n\geq 0} \pi^n W = 0$. Given an R-module M, we denote by $\widehat{M}$ its π-adic completion and write $\widehat{M_L} := \widehat{M} \otimes_R L$.

Given an L-normed vector space X, we denote by X° its unit ball. Given a Banach algebra A, a Banach A-module M will always be assumed to have action map of norm at most 1, i.e M° will always be assumed to be an A°-module.

In a Hopf algebra H, we use Sweedler's notation for the comultiplication, i.e we write $\Delta(h) = \sum h_1 \otimes h_2$. All our comodules will be *right* comodules unless stated otherwise.

Finally, while we talked about R-group schemes and their corresponding Lie algebras in this introduction, quantum groups are defined purely in terms of the root system and are traditionally defined starting from complex Lie algebras and algebraic groups, regardless of what the base field is. This is the convention we follow as well. Hence we let $\mathfrak{g}$ be a complex semisimple Lie algebra. We fix a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ contained in a Borel subalgebra. We choose a positive root system and we denote the simple roots by $\alpha_1, \ldots, \alpha_n$. Let $C = (a_{ij})$ denote the Cartan matrix. We let G be the simply connected semisimple algebraic group corresponding to $\mathfrak{g}$, and we let B be the Borel subgroup corresponding to the positive root system, and let $N \subset B$ be its unipotent radical. Let $\mathfrak{b} = \mathrm{Lie}(B)$ and $= \mathrm{Lie}(N)$. Let W be the Weyl group of $\mathfrak{g}$, and let $\langle\, , \rangle$ denote the standard normalised W-invariant bilinear form on $\mathfrak{h}^*$. Let $P \subset \mathfrak{h}^*$ be the weight lattice and $Q \subset P$ be the root lattice. Let T_P denote the abelian group $\mathrm{Hom}_{\mathbb{Z}}(P, L^\times)$. We will use the additive notation for this group. Let d be the smallest natural number such that $\langle \mu, P\rangle \subset \frac{1}{d}\mathbb{Z}$ for all $\mu \in P$. Let $d_i = \frac{\langle \alpha_i, \alpha_i\rangle}{2} \in \{1,2,3\}$ and write $q_i := q^{d_i}$.

We make the following two assumptions. First, we assume that $q^{\frac{1}{d}}$ exists in R and that $q^{\frac{1}{d}} \equiv 1 \pmod{\pi}$. Then for each $\lambda \in P$, we have an associated element in T_P sending a given $\mu \in P$ to $q^{\langle \lambda, \mu\rangle}$, which we will also denote by λ. Secondly, we assume that $p > 2$ and, if $\mathfrak{g}$ has a component of type G_2, we furthermore restrict to $p > 3$. This ensures that p does not divide any non-zero entry of the Cartan matrix.

All the above algebraic groups and Lie algebras have k-forms, and we write $G_k, \mathfrak{g}_k, \ldots$ etc to denote them.

2. PRELIMINARIES ON QUANTUM GROUPS AND THEIR INTEGRAL FORMS

2.1. Quantized Enveloping Algebra

We begin by recalling basic facts about quantized enveloping algebras (see eg [15, Chapter I.6] for more details). For $n \in \mathbb{Z}$ and $t \in L$, we write $[n]_t := \frac{t^n - t^{-n}}{t - t^{-1}}$. We then set the quantum factorial numbers to be given by $[0]_t! = 1$ and $[n]_t! := [n]_t[n-1]_t \cdots [1]_t$ for $n \geq 1$. Then we set

$$\begin{bmatrix} n \\ i \end{bmatrix}_t := \frac{[n]_t!}{[i]_t![n-i]_t!}$$

when $n \geq i \geq 1$.

Definition 2.1. *The simply connected quantized enveloping algebra $U_q(\mathfrak{g})$ is defined to be the L-algebra with generators $E_{\alpha_1}, \ldots, E_{\alpha_n}, F_{\alpha_1}, \ldots, F_{\alpha_n}, K_\lambda, \lambda \in P$, satisfying the following relations:*

$$\begin{aligned}
&K_\lambda K_\mu = K_{\lambda+\mu}, \quad K_0 = 1,\\
&K_\lambda E_{\alpha_i} K_{-\lambda} = q^{\langle \lambda, \alpha_i\rangle} E_{\alpha_i}, \quad K_\lambda F_{\alpha_i} K_{-\lambda} = q^{-\langle \lambda, \alpha_i\rangle} F_{\alpha_i},\\
&[E_{\alpha_i}, F_{\alpha_j}] = \delta_{ij} \frac{K_{\alpha_i} - K_{-\alpha_i}}{q_i - q_i^{-1}},\\
&\sum_{l=0}^{1-a_{ij}} (-1)^l \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix}_{q_i} E_{\alpha_i}^{1-a_{ij}-l} E_{\alpha_j} E_{\alpha_i}^l = 0 \quad (i \neq j),\\
&\sum_{l=0}^{1-a_{ij}} (-1)^l \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix}_{q_i} F_{\alpha_i}^{1-a_{ij}-l} F_{\alpha_j} F_{\alpha_i}^l = 0 \quad (i \neq j).
\end{aligned}$$

We will also abbreviate $U_q(\mathfrak{g})$ to U_q when no confusion can arise as to the choice of Lie algebra $\mathfrak{g}$. We can define Borel and nilpotent subalgebras, namely $U_q^{\geq 0}$ is the subalgebra generated by all the $K's$ and the $E's$, and U_q^+ is the subalgebra generated by all the $E's$. Similarly, U_q^- is defined to be the subalgebra generated by all the $F's$. There is also a Cartan subalgebra given by $U_q^0 := L[K_\lambda : \lambda \in P]$, which is isomorphic to the group algebra LP. There is an algebra automorphism ω of U_q defined by $\omega(E_{\alpha_i}) = F_{\alpha_i}, \omega(F_{\alpha_i}) = E_{\alpha_i}$ and $\omega(K_\lambda) = K_{-\lambda}$.

Recall that U_q is a Hopf algebra with operations given by

$$\begin{array}{lll}
\Delta(K_\lambda) = K_\lambda \otimes K_\lambda & \varepsilon(K_\lambda) = 1 & S(K_\lambda) = K_{-\lambda}\\
\Delta(E_{\alpha_i}) = E_{\alpha_i} \otimes 1 + K_{\alpha_i} \otimes E_{\alpha_i} & \varepsilon(E_{\alpha_i}) = 0 & S(E_{\alpha_i}) = -K_{-\alpha_i}E_{\alpha_i}\\
\Delta(F_{\alpha_i}) = F_{\alpha_i} \otimes K_{-\alpha_i} + 1 \otimes F_{\alpha_i} & \varepsilon(F_{\alpha_i}) = 0 & S(F_{\alpha_i}) = -F_{\alpha_i}K_{\alpha_i}
\end{array}$$

for $i = 1, \ldots, n$ and all $\lambda \in P$. Then $U_q^{\geq 0}$ is a sub-Hopf algebra of U_q.

Also recall that there is a triangular decomposition

$$U_q \cong U_q^- \otimes_L U_q^0 \otimes_L U_q^+$$

and that $U_q^{\pm}$ have bases consisting of PBW type monomials. More specifically, if $\beta_1, \ldots, \beta_N$ are the positive roots, ordered in a particular way, then there are elements $E_{\beta_1}, \ldots, E_{\beta_N}$ of U_q^+ such that the set of all ordered monomials $E_{\beta_1}^{m_1} \cdots E_{\beta_N}^{m_N}$ forms a basis for U_q^+. We now let $F_{\beta_j} := \omega(E_{\beta_j})$ and the corresponding monomials in the F's will form a basis of U_q^-. The triangular decomposition immediately gives a PBW type basis for U_q, namely it consists of monomials of the form

$$M_{\boldsymbol{r},\boldsymbol{s},\lambda} := \boldsymbol{F}^{\boldsymbol{r}} K_\lambda \boldsymbol{E}^{\boldsymbol{s}}$$

where $\boldsymbol{r}, \boldsymbol{s} \in \mathbb{Z}_{\geq 0}^N$. We recall that the *height* of such a monomial is defined to be

$$\operatorname{ht}(M_{\boldsymbol{r},\boldsymbol{s},\lambda}) := \sum_{j=1}^{N} (r_j + s_j) \operatorname{ht}(\beta_j)$$

where $\operatorname{ht}(\beta) := \sum_{i=1}^n a_i$ for a positive root $\beta = \sum_i a_i \alpha_i$. This gives rise to a positive filtration on U_q defined by

$$F_i U_q := L\text{-span}\{M_{\boldsymbol{r},\boldsymbol{s},\lambda} : \operatorname{ht}(M_{\boldsymbol{r},\boldsymbol{s},\lambda}) \leq i\}.$$

This filtration can actually be extended to a multifiltration as follows. The associated graded algebra $U^{(1)} = \operatorname{gr} U_q$ can be seen to have the same presentation as U_q, with the exception that now all the E's commute with all the F's. Moreover it is isomorphic to U_q as a vector space. We can then make $U^{(1)}$ into a $\mathbb{Z}_{\geq 0}^{2N}$-filtered algebra, by assigning to each monomial $M_{\boldsymbol{r},\boldsymbol{s},\lambda}$ the degree $(r_1, \ldots, r_N, s_1, \ldots, s_N)$ where we impose the reverse lexicographic orderin ordering on $\mathbb{Z}_{\geq 0}^{2N}$. Denote the corresponding associated graded algebra of $U^{(1)}$ by $U^{(2N+1)}$. This algebra is known to be q-commutative over L (see [17, Proposition 10.1]). Here we say that an L-algebra A is q-commutative over a subalgebra B if it is finitely generated over B, say by $x_1, \ldots, x_m$, such that the x_i normalise B and for all $1 \leq i \leq j \leq m$ there are $n_{ij} \in \frac{1}{d}\mathbb{Z}$ such that $x_i x_j = q^{n_{ij}} x_j x_i$. We regord here a noncommutative analogue of Hilbert's basis theorem, which follows directly from [36, Theorem 1.2.10] and induction.

Lemma 2.2. *If A is q-commutative over B and B is Noetherian, then so is A.*

Hence we see that U_q is a Noetherian L-algebra.

2.2. Integral Forms of U_q

We now recall details about two integral forms that we will work with. First recall the notation:

$$E_{\alpha_i}^{(s)} := \frac{E_{\alpha_i}^s}{[s]_{q_i}!}, \quad F_{\alpha_i}^{(s)} := \frac{F_{\alpha_i}^s}{[s]_{q_i}!},$$

for any integer $s \geq 0$. Then Lusztig's integral form U^{res} is defined to be the R-subalgebra of U_q generated by K_λ ($\lambda \in P$) and all $E_{\alpha_i}^{(r)}$ and $F_{\alpha_i}^{(r)}$ for $r \geq 0$ and $1 \leq i \leq n$. Recall that for $1 \leq i \leq n$, $c, t \in \mathbb{Z}$ with $t \geq 0$ we define

$$\begin{bmatrix} K_{\alpha_i}; c \\ t \end{bmatrix} = \prod_{j=1}^{t} \frac{K_{\alpha_i} q_i^{c-j+1} - K_{\alpha_i}^{-1} q_i^{-(c-j+1)}}{q_i^j - q_i^{-j}}.$$

Then by [26, 11.1, p.238] we have that all such $\left[\begin{smallmatrix} K_{\alpha_i}; c \\ t \end{smallmatrix}\right]$ lie in U^{res}. Also note that by [34, Theorem 6.7] U^{res} has a triangular decomposition and a PBW type basis, so that U^{res} is free over R.

There is an R-subalgebra $(U^{\text{res}})^0$ generated by all K_λ and all $\left[\begin{smallmatrix} K_{\alpha_i}; c \\ t \end{smallmatrix}\right]$. We let $U^{\text{res}}(\mathfrak{b})$ denote the R-subalgebra of U^{res} generated by $(U^{\text{res}})^0$ and all $E_{\alpha_i}^{(r)}$ for $r \geq 0$ and $1 \leq i \leq n$. By [1, Lemma 1.1], for each $\lambda \in P$ there is a unique character $\psi_\lambda : (U^{\text{res}})^0 \to R$ defined by

$$\psi_\lambda(K_\mu) = q^{\langle \lambda, \mu \rangle} \quad \text{and} \quad \psi_\lambda\left(\begin{bmatrix} K_{\alpha_i}; c \\ t \end{bmatrix}\right) = \begin{bmatrix} \langle \lambda, \alpha_i^\vee \rangle + c \\ t \end{bmatrix}_{q_i}. \tag{2.1}$$

We will say these characters are of type **1**.

Given a U^{res}-module M and a character ψ as above of $(U^{\text{res}})^0$, we write M_ψ for the elements $m \in M$ such that $um = \psi(u)m$ for all $u \in (U^{\text{res}})^0$. We now recall the notion of integrable module from [1, 1.6]:

Definition 2.3. *A U^{res}-module M is said to be* integrable of type **1** *if it is a sum of weight spaces which all correspond to a character of type* **1** *as described above and if in addition, for every $m \in M$, there is $r >> 0$ such that m is killed by $E^{(r)}$ and $F^{(r)}$. Similarly we define a $U^{res}(\mathfrak{b})$-module to be integrable of type* **1** *if it is the sum of its weight spaces corresponding to type* **1** *characters and for every $m \in M$, $E^{(r)}m = 0$ for $r >> 0$.*

Since all our characters will always be of type **1** we will often just say 'integrable' to mean 'integrable of type **1**'.

The second integral form we will need is the De Concini-Kac integral form U. This is defined to be the R-subalgebra of U_q generated by $E_{\alpha_i}, F_{\alpha_i} (1 \le i \le n), K_\lambda (\lambda \in P)$. This algebra has a similar presentation to U_q. If we write $[K_{\alpha_i}; m] := \left[{K_{\alpha_i};m \atop 1}\right]$ for $m \in \mathbb{Z}$ and $1 \le i \le n$, then U is generated as an R-algebra by $E_{\alpha_i}, F_{\alpha_i}, [K_{\alpha_i}; 0] (1 \le i \le n), K_\lambda (\lambda \in P)$ with the same relations as U_q except that the commutator relation between E_{α_i} and F_{α_j} is replaced by the two relations

$$[E_{\alpha_i}, F_{\alpha_j}] = \delta_{ij}[K_{\alpha_i}; 0],$$
$$(q_i - q_i^{-1})[K_{\alpha_i}; 0] = K_{\alpha_i} - K_{\alpha_i}^{-1}.$$

Note that U is a Hopf R-algebra. For example we have the identity

$$\Delta([K_{\alpha_i}; 0]) = [K_{\alpha_i}; 0] \otimes K_{\alpha_i} + K_{\alpha_i}^{-1} \otimes [K_{\alpha_i}; 0].$$

Note that we also have the equality

$$[K_{\alpha_i}; m] = [K_{\alpha_i}; 0]q_i^{-m} + K_{\alpha_i}[m]_{q_i}$$

for all $m \in \mathbb{Z}$, and so U contains all $[K_{\alpha_i}; m]$.

We showed in [18, Section 4] that U has a triangular decomposition $U \cong U^- \otimes_R U^0 \otimes_R U^+$ where $U^\pm$ is the R-subalgebra generated by the E_{α_i}'s, respectively F_{α_i}'s, and U^0 is the R-subalgebra generated by $[K_{\alpha_i}; 0] (1 \le i \le n), K_\lambda (\lambda \in P)$. Moreover $[K_{\alpha_i}; m] \in U^0$ for all $m \in \mathbb{Z}$ by the above. We also showed that $U^\pm$ has a PBW basis, more specifically that the PBW monomials which form an L-basis of $U_q^\pm$ are also an R-basis of $U^\pm$.

Note that both of these integral forms are π-adically separated since $U \subset U^{\text{res}}$ and U^{res} is free over R. We finish by describing the relationship between the reduction modulo π of U and U^{res} and classical objects. We write $U_k := U/\pi U$ and $U_k^{\text{res}} = U^{\text{res}}/\pi U^{\text{res}}$.

Proposition 2.4. *1.* ([16, Proposition 9.2.3]) *The quotient k-algebra $U_k/(K_{\varpi_1} - 1, \dots, K_{\varpi_n} - 1)$ is isomorphic to $U(\mathfrak{g}_k)$.*

2. ([34, 8.15]) *The quotient $U_k^{res}/(K_{\varpi_1} - 1, \dots, K_{\varpi_n} - 1)$ is isomorphic to the hyperalgebra of the group G_k.*

2.3. Quantized Coordinate Rings and Their Integral Forms

We now recall the construction of the quantized coordinate algebra $\mathcal{O}_q$. For any module M over an L-Hopf algebra H, and for any $f \in H^*$ and $v \in M$, the matrix coefficient $c_{f,v}^M \in H^*$ is defined by

$$c_{f,v}^M(x) := f(xv) \qquad \text{for } x \in H.$$

Also recall from [26, Theorem 5.10] that for each $\lambda \in P$ there is a unique irreducible representation of type **1**, $V(\lambda)$, of U_q and that these form a complete list of such representations. The quantized coordinate ring $\mathcal{O}_q$ is then defined to be the L-subalgebra of the Hopf dual U_q° generated by the matrix coefficients of the modules $V(\lambda)$ for $\lambda \in P^+$. In fact, from [15, I.7-I.8], it is a finitely generated, Noetherian L-algebra, and it is a sub-Hopf algebra of U_q°. There is also a quantized coordinate algebra of the Borel $\mathcal{O}_q(B)$. Since $U_q^{\geq 0}$ is a Hopf-subalgebra of U_q, the restriction maps yields a Hopf algebra homomorphism $\mathcal{O}_q \to (U_q^{\geq 0})^\circ$ and we let $\mathcal{O}_q(B)$ denote its image.

We now recall how the integral forms of $\mathcal{O}_q$ and $\mathcal{O}_q(B)$ are defined. Let U^{res} be Lusztig's integral form defined in above. Let $\mathcal{J}$ denote the set of ideals I in U^{res} such that U^{res}/I is a finite free R-module. We now consider the set $\mathscr{I}$ consisting of ideals $I \in \mathcal{J}$ such that $I \cap (U^{\text{res}})^0$ contains a finite intersection of ideals $\ker(\psi_\lambda)$. Note that for any R-module M, we may view $\text{Hom}_R(U^{\text{res}}, M)$ as a U^{res}-module via $(x \cdot f)(y) = f(yx)$ for all $x, y \in U^{\text{res}}$. In [1, Definition 1.10], a so-called induction functor from the trivial subalgebra was defined. It takes any R-module M to the subrepresentation $H(M)$ of $\text{Hom}_R(U^{\text{res}}, M)$ given by all elements in the sum the weight spaces in $\text{Hom}_R(U^{\text{res}}, M)$ which are killed by all $E_{\alpha_i}^{(r)}$ and $F_{\alpha_i}^{(r)}$ for $r >> 0$. In other words $H(M)$ is the largest integrable subrepresentation of $\text{Hom}_R(U^{\text{res}}, M)$. We then define the integral form of the quantized coordinate algebra to be $\mathcal{A}_q := H(R)$. By [1, Corollary 1.30], we have $f \in H(M)$ if and only if f kills an ideal $I \in \mathscr{I}$. In particular,

$$\mathcal{A}_q = \{f \in (U^{\text{res}})^* : f|_I = 0 \text{ for some } I \in \mathscr{I}\}.$$

So $\mathcal{A}_q$ is a sub-Hopf algebra of $(U^{\text{res}})^\circ$ (see Definition A.1) and it may be viewed as the algebra of matrix coefficients of finite free U^{res}-modules of type **1**. In particular the comultiplication on it makes it into a $(U^{\text{res}})^\circ$-comodule and hence we may view it as a U^{res}-module by Proposition A.4 (and that agrees with the definition of the U^{res}-action on $H(R)$). Moreover by [1, Theorem 1.33], $\mathcal{A}_q$ is free over R.

Next, we look at the Borel subalgebra $U^{\text{res}}(\mathfrak{b})$ of U^{res}. Let $\mathcal{I}$ be the set of $f \in \mathcal{A}_q$ such that $f|_{U^{\text{res}}(\mathfrak{b})} = 0$. The Hopf algebra homomorphism $\mathcal{A}_q \to U^{\text{res}}(\mathfrak{b})^\circ$ given by restriction has kernel precisely $\mathcal{I}$ and so we see that $\mathcal{I}$ is a Hopf ideal and that $\mathcal{B}_q := \mathcal{A}_q/\mathcal{I} \subseteq U^{\text{res}}(\mathfrak{b})^\circ$ is a Hopf algebra. Similarly to the above, [1] defined an induction functor from the trivial subalgebra to $U^{\text{res}}(\mathfrak{b})$ in a completely analogous way: if M is an R-module, we define $\mathcal{H}(M)$ to be the largest integrable submodule of $\text{Hom}_R(U^{\text{res}}(\mathfrak{b}), M)$. By [1, Proposition 2.7(ii) and (iii)] we have that $\mathcal{B}_q = \mathcal{H}(R)$ and so it is integrable, and it is free as an R-module.

2.4. The Categories of Comodules

We now recall how the category of $\mathcal{A}_q$-comodules (respectively $\mathcal{B}_q$-comodules) can be identified with integrable U^{res}-modules (respectively $U^{\text{res}}(\mathfrak{b})$-modules). We expect this to be well-known but we did not find a suitable reference for it, so we provide proofs. To that end, we use general results about R-Hopf algebras which we've written in the appendix.

Since $\mathcal{A}_q = H(R)$, it is integrable with the U^{res}-module structure described above. Note that for any R-module M there is a natural map $M \otimes_R \mathcal{A}_q \to H(M) \subseteq \text{Hom}_R(U^{\text{res}}, M)$ which is the composite of the map $M \otimes_R \mathcal{A}_q \to M \otimes_R (U^{\text{res}})^*$, coming from the inclusion $\mathcal{A}_q \subseteq (U^{\text{res}})^*$, and the map θ_M from Corollary A.6. By abuse of notation we also denote this map by θ_M. For the Borel, we have again a map $\theta_M : M \otimes_R \mathcal{B}_q \to \mathcal{H}(M)$ for any R-module M. Moreover we have again that $f \in \mathcal{H}(M)$ if and only if f kills an ideal I of $U^{\text{res}}(\mathfrak{b})$ such that $U^{\text{res}}(\mathfrak{b})/I$ is finitely generated and $I \cap (U^{\text{res}})^0$ contains a finite intersection of ideals $\ker(\psi_\lambda)$.

The next result immediately follows from the above:

Lemma 2.5. *If M is torsion-free as an R-module then $Hom_R(U^{res}, M)/H(M)$ and $Hom_R(U^{res}(\mathfrak{b}), M)/\mathcal{H}(M)$ are torsion-free. In particular $(U^{res})^*/\mathcal{A}_q$ and $U^{res}(\mathfrak{b})^*/\mathcal{B}_q$ are torsion free.*

Proof. If $\pi^n f \in \text{Hom}_R(U^{\text{res}}, M)$ kills an ideal in $\mathscr{I}$, then so does f as M is torsion-free. An analogous argument applies to $\mathcal{H}(M)$. The last part follows by putting $M = R$. □

Since $\mathcal{A}_q$ and $\mathcal{B}_q$ are sub Hopf algebras of $(U^{\text{res}})^\circ$ and $U^{\text{res}}(\mathfrak{b})^\circ$ respectively, it follows that any comodule over $\mathcal{A}_q$ (respectively $\mathcal{B}_q$) is a comodule over $(U^{\text{res}})^\circ$ (respectively $U^{\text{res}}(\mathfrak{b})^\circ$). Thus we may view comodules over $\mathcal{A}_q$ and $\mathcal{B}_q$ as locally finite modules over U^{res} and $U^{\text{res}}(\mathfrak{b})$ respectively. This defines functors from the categories of $\mathcal{A}_q$-comodules and $\mathcal{B}_q$-comodules to the categories of locally finite U^{res}-modules and $U^{\text{res}}(\mathfrak{b})$-modules respectively.

Remark 2.6. *The following observations will be useful in the next proof and also at several points later on. Suppose that M is a $\mathcal{B}_q$-comodule, with coaction $\rho_M : M \to M \otimes_R \mathcal{B}_q$. Note that by the axioms of comodules, the composite*

$$(1 \otimes \varepsilon) \circ \rho_M = 1_M$$

so that the map ρ splits and M is a direct summand of $M \otimes_R \mathcal{B}_q$ as an R-module. Moreover, the diagram

$$\begin{array}{ccc} M & \xrightarrow{\rho_M} & M \otimes_R \mathcal{B}_q \\ {\scriptstyle \rho_M}\downarrow & & \downarrow{\scriptstyle id\otimes\Delta} \\ M \otimes_R \mathcal{B}_q & \xrightarrow{\rho_M \otimes id} & M \otimes_R \mathcal{B}_q \otimes_R \mathcal{B}_q \end{array}$$

commutes. But note that the map $1 \otimes \Delta$ makes $M \otimes_R \mathcal{B}_q$ into a $\mathcal{B}_q$-comodule, so that the above diagram and the splitting says that M identifies via ρ_M with a subcomodule of $M \otimes_R \mathcal{B}_q$ where the latter is given the comodule structure $1 \otimes \Delta$. Of course all of the above applies more generally to a comodule over an arbitrary coalgebra.

Theorem 2.7. *The category of $\mathcal{A}_q$-comodules, respectively $\mathcal{B}_q$-comodules, is isomorphic to the category of integrable U^{res}-modules, respectively $U^{res}(\mathfrak{b})$-modules.*

Proof. We first show that the above functors are fully faithful. This is the exact same argument as in Proposition A.7, using Lemma A.5 with $A = U^{\text{res}}$, $B = R$ and $C = \mathcal{A}_q$ for $\mathcal{A}_q$-comodules and with $A = U^{\text{res}}(\mathfrak{b})$, $B = R$ and $C = \mathcal{B}_q$ for $\mathcal{B}_q$-comodules. For these to apply we need to show that $(U^{\text{res}})^*/\mathcal{A}_q$ and $U^{\text{res}}(\mathfrak{b})^*/\mathcal{B}_q$ are torsion-free, but this is just the previous Lemma.

Next, the key fact we use is [1, Theorem 1.31(iii)]: for any R-module M the natural map $\theta_M : M \otimes_R \mathcal{A}_q \to \text{Hom}_R(U^{\text{res}}, M)$ is an isomorphism onto $H(M)$. Now suppose that M is an integrable U^{res}-module. Then for all $m \in M$, the action map $\varphi_M(m) : x \mapsto x \cdot m$ belongs to $H(M)$. So by the above facts the maps $\varphi_M(m)$ all belong to the image of θ_M. By Lemma A.9 with $C = \mathcal{A}_q$ we conclude that M must be an $\mathcal{A}_q$-comodule. An analogous argument shows that integrable $U^{\text{res}}(\mathfrak{b})$-modules are $\mathcal{B}_q$-comodules using [1, Proposition 2.7(iv)], which states that the natural map $\theta_M : M \otimes_R \mathcal{B}_q \to \text{Hom}_R(U^{\text{res}}(\mathfrak{b}), M)$ is an isomorphism onto $\mathcal{H}(M)$.

Thus since the functors are fully faithful we are now reduced to showing that any $\mathcal{A}_q$-comodule (respectively $\mathcal{B}_q$-comodule) is integrable when viewed as a U^{res}-module (respectively $U^{\text{res}}(\mathfrak{b})$-module). We prove it for $\mathcal{B}_q$, the proof for $\mathcal{A}_q$ being entirely analogous. Suppose M is a $\mathcal{B}_q$-comodule. Then by the above remark the map $\rho : M \to M \otimes_R \mathcal{B}_q$ is an injective comodule homomorphism where the right hand side is given the coaction map $1 \otimes \Delta$. In other words, in the language of $U^{\text{res}}(\mathfrak{b})$-modules, this is saying the action on $M \otimes_R \mathcal{B}_q$ is the tensor product of the trivial action on M with the usual action on $\mathcal{B}_q$, i.e for $u \in U^{\text{res}}(\mathfrak{b})$ we have $u(m \otimes f) = m \otimes uf$ for all $m \in M$ and $f \in \mathcal{B}_q$. Thus, since $\mathcal{B}_q$ is integrable, so is $M \otimes_R \mathcal{B}_q$ with that structure. But now the result follows since integrable modules are closed under taking submodules by [1, Note added in proof p.59]. ☐

2.5. Some Noetherianity Conditions

We record here some conditions under which we can lift the Noetherian property from the reduction mod π of a ring to the ring itself. These will be useful later in the paper.

Proposition 2.8. *1. Suppose that A is an R-algebra such that $A/\pi A$ is Noetherian. Then the π-adic completion $\widehat{A}$ is also Noetherian.*

2. Let $n \geq 1$ and suppose that we have $\mathbb{Z}^n$-graded R-algebra $\mathcal{R} = \oplus_{\mathbf{m}\in\mathbb{Z}^n}\mathcal{R}_{\mathbf{m}}$ such that each graded piece $\mathcal{R}_{\mathbf{m}}$ is finitely generated over R. If $\mathcal{R}/\pi\mathcal{R}$ is Noetherian, then $\mathcal{R}$ is graded Noetherian.

Proof. (i) is just [11, Lemma 3.2.2]. For (ii) we use the same argument as in [33, Proposition II.2.3]. Specifically, consider the π-adic filtration on $\mathcal{R}$. The associated graded ring is a quotient of the polynomial algebra $(\mathcal{R}/\pi\mathcal{R})[t]$ (where t corresponds to the symbol of π), and so is Noetherian. We will consider several graded R-submodules of $\mathcal{R}$, equipped with the subspace filtration of the π-adic filtration.

Suppose we are given two graded ideals $I \subset J$ with $I \neq J$. Then we have $\operatorname{gr} I \subset \operatorname{gr} J$ and it will suffice to show that $\operatorname{gr} I \neq \operatorname{gr} J$. Pick $\mathbf{m} \in \mathbb{Z}^n$ such that $I_{\mathbf{m}} \neq J_{\mathbf{m}}$, and assume that $\operatorname{gr} I_{\mathbf{m}} = \operatorname{gr} J_{\mathbf{m}}$. Since $I_{\mathbf{m}}$ and $J_{\mathbf{m}}$ are finitely generated over R, we will get a contradiction by Nakayama if we show that $J_{\mathbf{m}} = I_{\mathbf{m}} + \pi J_{\mathbf{m}}$.

By the Artin-Rees Lemma ([7, Theorem 10.11]) applied to $J_{\mathbf{m}}$ viewed as a submodule of $\mathcal{R}_{\mathbf{m}}$, the subspace filtration of the π-adic filtration on $\mathcal{R}_{\mathbf{m}}$ and the π-adic filtration on $J_{\mathbf{m}}$ have finite difference. So there exists a $d \in \mathbb{Z}_{<0}$ such that for all $j \in J_{\mathbf{m}}$ with degree $d(j) < d$ in the subspace filtration, $j \in \pi J_{\mathbf{m}}$. Now let $j \in J_{\mathbf{m}}$ be arbitrary. We show by induction on $d(j)$ that $j \in I_{\mathbf{m}} + \pi J_{\mathbf{m}}$, the cases $d(j) < d$ being already dealt with. Since $\operatorname{gr} I_{\mathbf{m}} = \operatorname{gr} J_{\mathbf{m}}$, there exists $i \in I_{\mathbf{m}}$ such that $d(i-j) < d(j)$. But by induction hypothesis this implies $i - j \in I_{\mathbf{m}} + \pi J_{\mathbf{m}}$, and hence we get $j = i - (i-j) \in I_{\mathbf{m}} + \pi J_{\mathbf{m}}$ as required. □

Corollary 2.9. *The ring $\widehat{\mathcal{A}_q}$ is Noetherian.*

Proof. Since $q \equiv 1 \pmod{\pi}$, the ring $\mathcal{A}_q/\pi\mathcal{A}_q$ coincides with the ring of regular functions on the group G_k and hence is Noetherian. Therefore the result follows from part (i) of the Proposition. □

3. THE QUANTUM FLAG VARIETY AND ITS INTEGRAL FORM

In this Section we review definitions and results from [8] and then adapt them to integral forms.

3.1. The Category $\mathcal{M}_{B_q}(G_q)$

We first begin by recalling the definition of the quantum flag variety.

Definition 3.1. *([8, Definition 3.1]) A B_q-*equivariant sheaf on *G_q is a triple (F, α, β) where F is an L-vector space, $\alpha : \mathcal{O}_q \otimes F \to F$ is a left $\mathcal{O}_q$-module action and $\beta : F \to F \otimes \mathcal{O}_q(B)$ is a right $\mathcal{O}_q(B)$-comodule action, such that α is an $\mathcal{O}_q(B)$-comodule homomorphism where $\mathcal{O}_q \otimes F$ is given the tensor $\mathcal{O}_q(B)$-comodule structure. We denote by $\mathcal{M}_{B_q}(G_q)$ the category of B_q-equivariant sheaves on G_q.*

Remark 3.2. *In the classical case $q = 1$, this category is equivalent to the category of B-equivariant sheaves of $\mathcal{O}_G$-modules, which in turn is equivalent to the category of quasi-coherent sheaves of $\mathcal{O}_{G/B}$-modules. So the category $\mathcal{M}_{B_q}(G_q)$ can be thought of as the quantum analogue of the flag variety.*

Obviously $\mathcal{O}_q$ is an object of this category. More generally we have a notion of line bundles. Any element $\lambda \in T_P$ may be thought of as a character of the group algebra $LP \cong L[K_\mu : \mu \in P]$, and we may extend it to a character of $U_q^{\geq 0}$ by setting it to kill the E's. This defines a one dimensional $U_q^{\geq 0}$-module L_λ. The ones among these which are integrable, and so $\mathcal{O}_q(B)$-comodules, correspond to $\lambda \in P$, and the coaction is $1 \mapsto 1 \otimes \lambda$.

Definition 3.3. *([8, Definition 3.3]) We define a line bundle in $\mathcal{M}_{B_q}(G_q)$ to be an object of the form $\mathcal{O}_q(\lambda) := \mathcal{O}_q \otimes_L L_{-\lambda}$ for $\lambda \in P$, where the $\mathcal{O}_q$-action is on the left factor and the $\mathcal{O}_q(B)$-coaction is the tensor one (or in the modules language this means we give it the tensor product $U_q^{\geq 0}$-module structure). More generally for a finite dimensional $\mathcal{O}_q(B)$-comodule V we get that $\mathcal{O}_q \otimes_L V$, with an analogous structure as above, is an element of $\mathcal{M}_{B_q}(G_q)$ and we may think of it as a vector bundle.*

Now that we have a flag variety, we turn to the notion of taking global sections.

Definition 3.4. *([8, Definition 3.4]) The* global section functor $\Gamma : \mathcal{M}_{B_q}(G_q) \to L$*-mod is defined to be*

$$\Gamma(M) := Hom_{\mathcal{M}_{B_q}(G_q)}(\mathcal{O}_q, M) = \{m \in M : \beta(m) = m \otimes 1\} =: M^{B_q},$$

*which we call the B_q-*invariants *of M.*

By [8, Lemma 3.8], the category $\mathcal{M}_{B_q}(G_q)$ has enough injectives, and so we can right derive the global section functor. It was shown in [8, Section 3] that the category $\mathcal{M}_{B_q}(G_q)$ is equivalent to a Proj category in the sense of Artin-Zhang [6]. That includes [8, Proposition 3.5] which states that the line bundles are very ample in the sense that for any coherent module M, the twist $M(\lambda)$ is Γ-acyclic and generated by its global sections for $\lambda >> 0$.

3.2. Quantum D-Modules

Let H be a Hopf algebra over a commutative ring S, and let A be an S-algebra equipped with a left H-module structure. We say that A is an *H-module algebra* if for all $u \in H$ and all $a, b \in A$, $u(ab) = \sum u_1(a)u_2(b)$. In that case, we may form the *smash product algebra $A\#H$*. As an S-module, this is just $A \otimes_S H$, but with multiplication given by

$$(a \otimes u) \cdot (b \otimes v) = \sum au_1(b) \otimes u_2 v.$$

From now on we drop the tensor signs, and write au for $a \otimes u$. Note that the action $u(a)$ of $u \in H$ on $a \in A$ coincides with the adjoint action $\sum u_1 a S(u_2)$ in $A\#H$.

Now, recall that there is a left U_q-module algebra structure on $\mathcal{O}_q$ given by

$$u(a) = \sum a_2(u) \cdot a_1,$$

for $u \in U_q$ and $a \in \mathcal{O}_q$. By viewing $\mathcal{O}_q \subseteq U_q^*$, this action amounts to the action $u(a)(x) = a(xu)$ for $a \in \mathcal{O}_q$ and $u, x \in U_q$. Following [8, Definition 4.1], we define the ring of quantum differential operators on G_q to be the smash product algebra $\mathcal{D}_q = \mathcal{O}_q\#U_q$. We will need the following result which was not proved in [8]:

Proposition 3.5. *The ring $\mathcal{D}_q$ is Noetherian.*

Proof. Since $\mathcal{D}_q = \mathcal{O}_q \otimes_L U_q$ as a vector space, and since $x \cdot (yu) = (xy)u$ for all $x, y \in \mathcal{O}_q$ and all $u \in U_q$, it follows that $\mathcal{D}_q$ is generated as an $\mathcal{O}_q$-module by U_q. Recall our PBW filtration on U_q. We now define an analogous filtration on $\mathcal{D}_q$ given by

$$F_i\mathcal{D}_q = \mathcal{O}_q \cdot F_i U_q.$$

We claim this defines an algebra filtration. Indeed, suppose that for some $i, j \geq 0$, we are given $u \in F_i U_q$ and $v \in F_j U_q$, and take $x, y \in \mathcal{O}_q$. By definition of the Hopf algebra structure on U_q, we have that $\Delta(u) \in F_i(U_q \otimes_L U_q) \subset F_i U_q \otimes_L F_i U_q$ where we give $U_q \otimes_L U_q$ the tensor filtration. Therefore, it follows that

$$(xu)(yv) = \sum \big(xu_1(y)\big)\big(u_2 v\big) \in F_{i+j}\mathcal{D}_q$$

since the filtration on U_q is an algebra filtration. Hence $\mathcal{D}_q$ is a positively filtered L-algebra.

It will therefore be enough to show that $\operatorname{gr}\mathcal{D}_q$ is Noetherian. First, observe that $F_0\mathcal{D}_q$ is generated over $\mathcal{O}_q$ by the K_μ for $\mu \in P$, which all commute. Moreover, for each generator x_i of $\mathcal{O}_q$, we have that

$$K_\mu x_i K_{-\mu} = K_\mu(x_i) \in q^{\frac{1}{d}\mathbb{Z}} x_i$$

by definition of the U_q-action on $\mathcal{O}_q$ and since the x_i's are matrix coefficients with respect to weight bases. Thus we see that the generators of F_0U_q normalise $\mathcal{O}_q$. Hence it follows from Lemma 2.2 that $F_0\mathcal{D}_q$ is Noetherian since $\mathcal{O}_q$ is Noetherian.

Next, we claim that the symbols $\overline{E_{\alpha_i}}$ and $\overline{F_{\alpha_j}}$ normalise $F_0\mathcal{D}_q$ in $\operatorname{gr}\mathcal{D}_q$ for all i, j. Indeed, we have that they q-commute with the K's and for $x \in \mathcal{O}_q$, we have

$$E_{\alpha_i}x - (K_{\alpha_i}xK_{-\alpha_i})E_{\alpha_i} = E_{\alpha_i}(x) \in \mathcal{O}_q \subseteq F_0\mathcal{D}_q,$$

where $K_{\alpha_i}xK_{-\alpha_i} \in \mathcal{O}_q$ by the above. Thus in $\operatorname{gr}\mathcal{D}_q$ we have

$$\overline{E_{\alpha_i}}x = (K_{\alpha_i}xK_{-\alpha_i})\overline{E_{\alpha_i}} \in F_0\mathcal{D}_q \cdot \overline{E_{\alpha_i}}$$

Similarly for the F's.

Finally we give to $\operatorname{gr}\mathcal{D}_q$ an analogue of the $\mathbb{Z}^{2N}_{\geq 0}$-filtration on $\operatorname{gr}U_q$ from section 2.1. More precisely, we make $\operatorname{gr}\mathcal{D}_q$ into a $\mathbb{Z}^{2N}_{\geq 0}$-filtered $F_0\mathcal{D}_q$-algebra. First we impose the reverse lexicographic total ordering on $\mathbb{Z}^{2N}_{\geq 0}$, and give a $\mathbb{Z}^{2N}_{\geq 0}$-filtration on $\operatorname{gr}\mathcal{D}_q$ by stating that a monomial

$$F_{\beta_1}^{r_1}\cdots F_{\beta_N}^{r_N}K_\lambda E_{\beta_1}^{s_1}\cdots E_{\beta_N}^{s_N}$$

has degree $(r_1, \ldots, r_N, s_1, \ldots, s_N)$. Then it follows that the corresponding associated multigraded algebra is a q-commutative $F_0\mathcal{D}_q$-algebra. Hence the associated graded algebra of $\operatorname{gr}\mathcal{D}_q$ is Noetherian by Lemma 2.2, and so it must be that $\operatorname{gr}\mathcal{D}_q$ is Noetherian. □

Note that $\mathcal{D}_q$ is a U_q-module algebra via the adjoint action in $\mathcal{D}_q$, or alternatively by tensoring the above action on $\mathcal{O}_q$ with the adjoint action on U_q. Explicitly,

$$u \cdot (a \otimes v) = \sum u_1.a \otimes u_2 v S(u_3). \tag{3.1}$$

We now are ready to define D-modules on the quantum flag variety:

Definition 3.6. *([8, Definition 4.2]) Let $\lambda \in T_P$. A (B_q, λ)-*equivariant $\mathcal{D}_q$-module *is a triple (M, α, β) where M is an L-vector space, $\alpha : \mathcal{D}_q \otimes M \to M$ is a left $\mathcal{D}_q$-module action and $\beta : M \to M \otimes \mathcal{O}_q(B)$ is a right $\mathcal{O}_q(B)$-comodule action. The map β induces a left $U_q^{\geq 0}$-action on M which we also denote by β. These actions must satisfy:*

(i) The $U_q^{\geq 0}$-actions on $M \otimes L_\lambda$ given by $\beta \otimes \lambda$ and $\alpha|_{U_q^{\geq 0}} \otimes 1$ are equal.

(ii) The map α is $U_q^{\geq 0}$-linear with respect to the β-action on M and the action (3.1) on $\mathcal{D}_q$.

In other words M is an object of $\mathcal{M}_{B_q}(G_q)$ equipped with a $U_q^{\geq 0}$-equivariant $\mathcal{D}_q$-action with in addition the condition (i).

We denote by $\mathcal{D}^\lambda_{B_q}(G_q)$ the category of such $\mathcal{D}_q$-modules. We have a forgetful functor $\mathcal{D}^\lambda_{B_q}(G_q) \to \mathcal{M}_{B_q}(G_q)$, which allows us to define a global section functor on $\mathcal{D}^\lambda_{B_q}(G_q)$ given by $\Gamma \circ$ forget. We also denote this functor by Γ.

Note that condition (i) above can be rephrased into saying that for $M \in \mathcal{D}^\lambda_{B_q}(G_q)$ and $m \in M$, we have $E_\alpha m = \beta(E_\alpha)m$ and $K_\mu m = \lambda(\mu)\beta(K_\mu)m$ for all simple roots α and $\mu \in P$. In particular if m is a global section then by B_q-invariance we must have $E_\alpha m = 0$ and $K_\mu m = \lambda(\mu)m$. In other words global sections consist of the highest weight vectors of weight λ. So we see that the $\mathcal{D}_q$-module homomorphisms $\mathcal{D}_q \to M$ corresponding to global sections factor through the ideal $\mathcal{D}_q I$ where $I = \{E_{\alpha_i}, K_\mu - \lambda(K_\mu) : 1 \leq i \leq n, \mu \in P\}$.

Based on the above, we define $\mathcal{D}_q^\lambda$ to be the quotient

$$\mathcal{D}_q^\lambda = \mathcal{D}_q/\mathcal{D}_q I$$

where I is as above. We can see that $\mathcal{D}_q^\lambda = \mathcal{O}_q \otimes_L M_\lambda$ where M_λ is the Verma module of highest weight λ. We saw that there is a surjection $U_q^{\text{fin}} \to M_\lambda$. Using this, we can view M_λ as an $\mathcal{O}_q(B)$-comodule, or an integrable $U_q^{\geq 0}$-module, via the quotient of the adjoint action. This action is just the usual action twisted by $-\lambda$ and so with this $U_q^{\geq 0}$-module structure it is isomorphic to $M_\lambda \otimes L_{-\lambda}$ and has trivial highest weight. Then, as an object of $\mathcal{M}_{B_q}(G_q)$, $\mathcal{D}_q^\lambda = \mathcal{O}_q \otimes_L M_\lambda$ with the tensor $\mathcal{O}_q(B)$-coaction and with the action of $\mathcal{O}_q$ on the left factor, where we view M_λ is an $\mathcal{O}_q(B)$-comodule just as now. It's moreover in $\mathcal{D}^\lambda_{B_q}(G_q)$: (i) follows from our discussion above of the fact that M_λ has trivial highest weight as an $\mathcal{O}_q(B)$-comodule, and (ii) simply follows from the fact that $\mathcal{D}_q$ is a $U_q^{\geq 0}$-module algebra.

Then $\mathcal{D}_q^\lambda$ represents the global section functor, i.e. $\Gamma(M) = \operatorname{Hom}_{\mathcal{D}^\lambda_{B_q}(G_q)}(\mathcal{D}_q^\lambda, M)$ by the above. In particular, $\Gamma(\mathcal{D}_q^\lambda)$ is a ring. Also one can easily check that $\mathcal{D}_q^\lambda$ is the maximal quotient of $\mathcal{D}_q$ that lies in $\mathcal{D}^\lambda_{B_q}(G_q)$, where we take the quotient $\mathcal{D}_q$-action and the quotient of the $U_q^{\geq 0}$-action (3.1) on $\mathcal{D}_q$.

Definition 3.7. *Let M_λ be the Verma module with highest weight λ. Let $J_\lambda = Ann_{U_q^{fin}}(M_\lambda)$. We write $U_q^\lambda = U_q^{fin}/J_\lambda$.*

We finally recall the notion of regular and dominant weights in this context. By [26, Lemma 6.3] the centre Z of U_q acts on any Verma module M_λ by a character χ_λ. Following [8, 2.1] we say that $\lambda \in T_P$ is *dominant* if $\chi_\lambda \neq \chi_{\lambda+\mu}$ for any $0 \neq \mu \in Q^+$, and that λ is regular dominant if for all $\mu \in P^+$ and all weight $\gamma \neq \mu$ of $V(\mu)$, where $V(\mu)$ denotes the simple U_q-module of highest weight μ, then we have $\chi_{\lambda+\mu} \neq \chi_{\lambda+\gamma}$. When $\lambda \in P$ this is equivalent to saying that it's dominant, respectively regular dominant in the classical sense.

Theorem 3.8 ([8, Theorem 4.12]). *Suppose that $\lambda \in T_P$ is regular and dominant. Then there is an equivalence of categories*

$$\Gamma : \mathcal{D}^\lambda_{B_q}(G_q) \to \Gamma(\mathcal{D}_q^\lambda)\text{-}mod.$$

whose quasi-inverse is given by the localisation functor $\operatorname{Loc}(M) = \mathcal{D}_q^\lambda \otimes_{\Gamma(\mathcal{D}_q^\lambda)} M$.

The proof of this Theorem uses an analogue of the Beilinson-Bernstein 'key lemma' [8, Lemma 4.14] and the fact that there is a natural map $U_q^\lambda \to \Gamma(\mathcal{D}_q^\lambda)$ (see the proof of [8, Proposition 4.8]).

3.3. An R-Form of $\mathcal{M}_{B_q}(G_q)$

We now return to our integral forms $\mathcal{A}_q$ and $\mathcal{B}_q$ and make completely analogous definitions to the previous section. Many of our constructions are similar to those of [8, Section 3]

Definition 3.9. *The integral quantum flag variety is the category $\mathscr{C}_R$ whose objects consist of $\mathcal{A}_q$-modules M which are equipped with a right $\mathcal{B}_q$-comodule action $M \to M \otimes_R \mathcal{B}_q$ such that the $\mathcal{A}_q$-action map $\mathcal{A}_q \otimes_R M \to M$ is a comodule homomorphism where we give $\mathcal{A}_q \otimes_R M$ the tensor comodule structure. The morphisms are just the $\mathcal{A}_q$-linear maps which are also comodule homomorphisms.*

There is an obvious functor

$$\mathscr{C}_R \longrightarrow \mathcal{M}_{B_q}(G_q)$$
$$M \longmapsto M_L := M \otimes_R L$$

to our quantum flag variety. Given $M \in \mathscr{C}_R$, we will write ρ_M (respectively ρ_{M_L}) to denote the comodule map on M (respectively M_L).

Next, there are several adjunctions we need to describe. Namely we have

$$\begin{array}{ccc} \mathcal{A}_q\text{-mod} & \xrightarrow{q} & R\text{-mod} \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\phi} \\ \mathscr{C}_R & \xrightarrow{p} & \mathcal{B}_q\text{-comod} \end{array}$$

where each arrow denotes a pair of functors. We write (θ^*, θ_*), (p^*, p_*), (q^*, q_*) and (ϕ^*, ϕ_*) where each time the 'lower star' functors are the right adjoints and go in the direction of the arrows. The functor $\theta_* : \mathcal{A}_q\text{-mod} \to \mathscr{C}_R$ is given by $N \mapsto N \otimes_R \mathcal{B}_q$ where $\mathcal{A}_q$ acts on $\theta_*(N)$ via the tensor action and the $\mathcal{B}_q$-coaction comes from the second factor, while $\theta^* : \mathscr{C}_R \to \mathcal{A}_q\text{-mod}$ is just the forgetful functor. The bijection making this an adjunction is as follows: let $M \in \mathscr{C}_R$ and $N \in \mathcal{A}_q$-mod, and let $\rho : M \to M \otimes_R \mathcal{B}_q$ and $\varepsilon : \mathcal{B}_q \to R$ be the comodule map and the counit of $\mathcal{B}_q$ respectively; given a module homomorphism $f : M \to N$, we construct a morphism $g : M \to N \otimes_R \mathcal{B}_q$ in $\mathscr{C}_R$ by taking the composite $(f \otimes \mathrm{id}) \circ \rho$. Conversely, given a morphism $g : M \to N \otimes_R \mathcal{B}_q$ in $\mathscr{C}_R$, we construct a module homomorphism $f : M \to N$ by taking the composite $(\mathrm{id} \otimes \varepsilon) \circ g$.

Moreover the adjunction between $\mathscr{C}_R$ and $\mathcal{B}_q$-comod is given by $p_* =$ forgetful one way and the functor $p^* : M \mapsto \mathcal{A}_q \otimes_R M$ the other way, where $\mathcal{A}_q$ acts on the first factor and the $\mathcal{B}_q$-coaction is the tensor coaction. The bijection is as follows: given a map $f : \mathcal{A}_q \otimes_R M \to N$ in $\mathscr{C}_R$ we get a comodule map $M \to N$ by taking $m \mapsto f(1 \otimes m)$, and conversely given a comodule map $g : M \to N$ we get a map $\mathcal{A}_q \otimes_R M \to N$ by post-composing $1 \otimes g : \mathcal{A}_q \otimes_R M \to \mathcal{A}_q \otimes_R N$ with the action map $\mathcal{A}_q \otimes_R N \to N$.

Similarly $q_* =$ forgetful, $q^* : M \mapsto \mathcal{A}_q \otimes_R M$, $\phi^* =$ forgetful and $\phi_* : M \to M \otimes_R \mathcal{B}_q$ where the coaction is on the second factor, all with similar bijections as in the above.

In particular, the maps $M \to \theta_*\theta^*(M)$ and $M \to \phi_*\phi^*(M)$ are both just the comodule map and so are injective, since the comodule map has left inverse $1 \otimes \varepsilon$. Also note that since $\mathcal{A}_q$ and $\mathcal{B}_q$ are torsion-free and so flat over R, all the functors are exact and so θ_*, p_*, q_* and ϕ_* all map injective objects to injective objects.

Lemma 3.10. *The categories $\mathscr{C}_R$ and $\mathcal{B}_q$-comod have enough injectives.*

Proof. Let $M \in \mathscr{C}_R$ and let I be an injective $\mathcal{A}_q$-module such that there is an $\mathcal{A}_q$-linear injection $M \to I$. By the above, the adjunction map $M \to \theta_*\theta^*(M)$ is injective, and so there is an injection

$$M \to \theta_*\theta^*(M) \to \theta_*(I).$$

But since θ_* is the right adjoint of an exact functor we see that $\theta_*(I) = I \otimes_R \mathcal{B}_q$ is injective and we're done for $\mathscr{C}_R$. The proof for $\mathcal{B}_q$-comod is entirely analogous working with ϕ instead of θ. □

Now we can define the global sections functor $\Gamma : \mathscr{C}_R \to R$-mod to be

$$\Gamma(M) := \mathrm{Hom}_{\mathscr{C}_R}(\mathcal{A}_q, M) = \{m \in M : \rho(m) = m \otimes 1\} := M^{\mathcal{B}_q}.$$

So in particular the above lemma shows that we can right derive this functor.

3.4. Proj Categories

Our first main aim is to show that our category $\mathscr{C}_R$ is a noncommutative projective scheme in the sense of Artin–Zhang [6, 2.3-2.4]. We quickly recall the definitions. Given a $\mathbb{Z}^n$-graded ring $\mathcal{R} = \oplus_{\mathbf{m} \in \mathbb{Z}^n} \mathcal{R}_{\mathbf{m}}$, we say that a graded (left or right) $\mathcal{R}$-module M is *torsion* if, for every $m \in M$, there exists some k such that m is killed by $\mathcal{R}_{\geq k} := \oplus_{m_1, \ldots, m_n \geq k} \mathcal{R}_{\mathbf{m}}$. Write $\mathcal{R}$-mod to denote the category of *graded* (left or right) $\mathcal{R}$-modules. The full subcategory $\mathcal{T}(\mathcal{R})$ of torsion modules is a Serre subcategory of $\mathcal{R}$-mod, and we let $\mathrm{Proj}(\mathcal{R}) := \mathcal{R}\text{-mod}/\mathcal{T}(\mathcal{R})$ denote the quotient category. Similarly we denote by $\mathrm{proj}(\mathcal{R})$

the quotient category of the category of finitely generated graded modules by the full subcategory of finitely generated torsion modules.

Now suppose that we are equipped with a tuple $(\mathcal{C}, \mathcal{O}, s_1, \ldots, s_n)$ where $\mathcal{C}$ is an abelian category, $\mathcal{O}$ is an object of $\mathcal{C}$ and $s_1, \ldots, s_n$ are pairwise commuting autoequivalences of $\mathcal{C}$. For $\mathbf{m} \in \mathbb{Z}^n$ and an object M of $\mathcal{C}$, we define twisting functors on $\mathcal{C}$ by

$$M(\mathbf{m}) = s_1^{m_1} \cdots s_n^{m_n}(M).$$

We let Γ denote the functor $\mathrm{Hom}_{\mathcal{C}}(\mathcal{O}, -)$ and we set $\underline{\Gamma}(M) = \oplus_{\mathbf{m} \in \mathbb{N}^n} \Gamma(M(\mathbf{m}))$. Note that $\underline{\Gamma}(\mathcal{O})$ is a graded ring where the multiplication is defined as follows: for $a \in \Gamma(\mathcal{O}(\mathbf{m}))$ and $b \in \Gamma(\mathcal{O}(\mathbf{m'}))$, we set

$$a \cdot b := s_1^{m'_1} \cdots s_n^{m'_n}(a) \circ b.$$

Similarly for each M in $\mathcal{C}$, $\underline{\Gamma}(M)$ is a graded right $\underline{\Gamma}(\mathcal{O})$-module. Finally, let $\mathcal{C}^0$ denote the full subcategory of Noetherian objects in $\mathcal{C}$. Then we have the following multigraded version of a result of Artin and Zhang (see also [8, Proposition 2.1]):

Proposition 3.11. ([6, Theorem 4.5], [8, Remark 2.2]) *Let $(\mathcal{C}, \mathcal{O}, s_1, \ldots, s_n)$ be a tuple as above, such that the following hold:*

1. *$\mathcal{O}$ belongs to $\mathcal{C}^0$;*
2. *$\Gamma(\mathcal{O})$ is a right Noetherian ring and $\Gamma(M)$ is a finitely generated $\Gamma(\mathcal{O})$-module for each object M of $\mathcal{C}^0$;*
3. *for each $M \in \mathcal{C}^0$ there is an epimorphism $\oplus_{i=1}^{l} \mathcal{O}(-\mathbf{m_i}) \to M$ for some $l \geq 1$ and $\mathbf{m_1}, \ldots, \mathbf{m_l} \in \mathbb{N}^n$; and*
4. *given $M, N \in \mathcal{C}^0$ and an epimorphism $M \to N$ in $\mathcal{C}$, the associated map $\Gamma(M(\mathbf{m})) \to \Gamma(N(\mathbf{m}))$ is surjective for $\mathbf{m} >> 0$.*

Then $\underline{\Gamma}(\mathcal{O})$ is right Noetherian and $\mathcal{C}^0$ is equivalent to $\mathrm{proj}(\underline{\Gamma}(\mathcal{O}))$ *(working with graded right modules). If, moreover, we assume that every object of $\mathcal{C}$ is a direct limit of objects in $\mathcal{C}^0$, then $\mathcal{C}$ is equivalent to* $\mathrm{Proj}(\underline{\Gamma}(\mathcal{O}))$.

Note that in general the assignement $M \mapsto \underline{\Gamma}(M)$ defines a left exact functor from $\mathcal{C}$ to the category of graded $\underline{\Gamma}(\mathcal{O})$-modules. Now we return to the setting of the quantum R-flag variety.

Definition 3.12. *We define the* representation ring *to be $R_q := \oplus_{\lambda \in P^+} \Gamma(\mathcal{A}_q(\lambda))$ with the induced ring structure from the multiplication in $\mathcal{A}_q$.*

Remark 3.13. *We will apply the above setup to the category $\mathscr{C}_R$. Specifically we will set the autoequivalences to be $s_i(M) := M(\varpi_i)$. The above mentioned ring structure for $\underline{\Gamma}(\mathcal{A}_q)$ is then just the ring structure of R_q^{op}. So we will apply the above results, working with R_q, by replacing every instance of the word 'right' by 'left'.*

Theorem 3.14. *The category $\mathscr{C}_R$ is equivalent to* $\mathrm{Proj}(R_q)$ *(this time working with left modules).*

We now start preparing for the proof of this theorem.

3.5. Line Bundles

We begin by proving results in $\mathscr{C}_R$ analogous to standard facts about line bundles on the flag variety. We will mostly just adapt arguments from [8, Section 3.4]. They apply essentially identically but we repeat them nevertheless.

Note that we have a functor of taking $\mathcal{B}_q$-invariants in $\mathcal{B}_q$-comod, which we denote by $\tilde{\Gamma}$. This functor is also left exact. Let Ind be the functor $\Gamma \circ p^* : M \mapsto (\mathcal{A}_q \otimes_R M)^{\mathcal{B}_q}$. Since $\mathcal{B}_q$-comod is isomorphic to the category of integrable $U^{\text{res}}(\mathfrak{b})$-modules of type $\mathbf{1}$ by Theorem 2.7, Ind is just the induction functor $M \mapsto (\mathcal{A}_q \otimes_R M)^{U^{\text{res}}(\mathfrak{b})}$ studied in [1]. This will be useful in the next result.

Proposition 3.15. *1. If $I \in \mathcal{B}_q$-comod is injective, then $p^*(I)$ is Γ-acyclic.*

2. For any $M \in \mathcal{B}_q$-comod and any $i \geq 0$, $R^i\text{Ind}(M) = R^i\Gamma(p^*(M))$.

3. For any $M \in \mathscr{C}_R$ and any $i \geq 0$, $R^i\tilde{\Gamma}(p_*(M)) \cong R^i\Gamma(M)$.

4. The functor Γ has cohomological dimension at most $N = \dim G/B$.

Proof. 1. The adjunction map $I \to \phi_*\phi^*(I) = I \otimes_R \mathcal{B}_q$ is injective. So as I is injective, this embedding splits. Therefore, as p^* is additive and since the derived functors $R^i\Gamma$ commute with finite direct sums, it suffices to show that $p^*(I \otimes_R \mathcal{B}_q)$ is acyclic. To simplify notation a bit, we write $J = \phi^*(I)$. We claim that we have an isomorphism $p^*(\phi_*(J)) \overset{\cong}{\to} \theta_*(q^*(J))$. Indeed, as R-modules they both equal $\mathcal{A}_q \otimes_R I \otimes_R \mathcal{B}_q$ and the isomorphism is given by $a \otimes i \otimes b \mapsto \sum a_1 \otimes i \otimes a_2 b$, with inverse $a \otimes i \otimes b \mapsto \sum a_1 \otimes i \otimes S(a_2)b$. These maps are easily checked to be both module and comodule homomorphisms. Hence we have that

$$\begin{aligned} R^i\Gamma(p^*(I \otimes_R \mathcal{B}_q)) &\cong R^i\Gamma(\theta_*(q^*(J))) \\ &= \text{Ext}^i_{\mathscr{C}_R}(\mathcal{A}_q, \theta_*(q^*(J))) \\ &\cong \text{Ext}^i_{\mathcal{A}_q}(\theta^*(\mathcal{A}_q), q^*(J)) \\ &= \text{Ext}^i_{\mathcal{A}_q}(\mathcal{A}_q, q^*(J)) = 0 \end{aligned}$$

for $i > 0$, as $\mathcal{A}_q$ is projective as an $\mathcal{A}_q$-module. Here we used the fact that θ_* is exact and preserves injectives in the second isomorphism.

2. Pick an injective resolution $M \to I^\bullet$. Then, by (i), $p^*(M) \to p^*(I^\bullet)$ is a Γ-acyclic resolution of $p^*(M)$, hence it computes the cohomology of Γ. The result now follows.

3. Pick an injective resolution $M \to I^\bullet$ in $\mathscr{C}_R$. Since p_* preserves injectives, it follows that $M \to I^\bullet$ is also an injective resolution in $\mathcal{B}_q$-comod. The result follows.

4. Let $M \in \mathscr{C}_R$. Since p_* maps injectives to injectives, any injective resolution of M in $\mathscr{C}_R$ is also an injective resolution of M in the category of $\mathcal{B}_q$-comodules. Thus we see that $R^i\Gamma(M) \cong R^i\tilde{\Gamma}(p_*(M))$ for all $i \geq 0$ and it suffices to show that the right hand side vanishes for $i \geq N$. To simplify notation we will drop the p_* when referring to an element of $\mathscr{C}_R$ viewed only as a comodule.

Now, note that there is a $\mathcal{B}_q$-comodule map $M \to p^*(M) = \mathcal{A}_q \otimes_R M$ given by $m \mapsto 1 \otimes m$. This map has a splitting given by the $\mathcal{A}_q$-action map, which is a comodule homomorphism by definition of $\mathscr{C}_R$. So, as $\mathcal{B}_q$-comodules, M is a direct summand of $\mathcal{A}_q \otimes_R M$. This in turn implies that $R^i\tilde{\Gamma}(M)$ is a direct summand of $R^i\tilde{\Gamma}(p^*(M))$. By 2 the latter equals $R^i\text{Ind}(M)$. But it was proved in [1, Theorem 5.8] that this induction functor has cohomological dimension at most N. So the result follows. ☐

Definition 3.16. *We let $T_P^R = \{\lambda \in T_P : \lambda((U^{res})^0) \subseteq R^\times\}$, which is a subgroup of T_P. Note that for $\lambda \in P$, the associated element of T_P belongs in T_P^R. For each $\lambda \in T_P^R$ we have a rank 1 $U^{res}(\mathfrak{b})$-module R_λ.*

When $\lambda \in P$ we may view it as a comodule with coaction $1 \mapsto 1 \otimes \lambda$. In that case, we let $\mathcal{A}_q(\lambda) := p^*(R_{-\lambda})$, which we call a line bundle. More generally, for $M \in \mathscr{C}_R$, we will write $M(\lambda)$ for $M \otimes_R R_{-\lambda}$. By letting $\mathcal{A}_q$ act on the left factor and giving it the tensor $\mathcal{B}_q$-coaction, this is also an element of $\mathscr{C}_R$.

Corollary 3.17. *For all $\lambda \in P$ and all $i \geq 0$, $R^i\Gamma(\mathcal{A}_q(\lambda))$ is finitely generated as an R-module. Moreover if $\lambda \in P^+$ then $R^i\Gamma(\mathcal{A}_q(\lambda)) = 0$ for all $i > 0$.*

Proof. By Proposition 3.15 (2), we have that $R^i\Gamma(\mathcal{A}_q(\lambda)) = R^i \operatorname{Ind}(R_{-\lambda})$. But it was proved in [1, Theorem 5.8] that $R^i \operatorname{Ind}$ sends finitely generated R-modules to finitely generated R-modules, and in [1, Corollary 5.7] that $R^i \operatorname{Ind}(R_{-\lambda}) = 0$ when $\lambda \in P^+$ and $i > 0$. □

3.6. Generators for $\mathscr{C}_R$

We now show results analogous to [8, Lemmas 3.13 & 3.16, Proposition 3.5]. The proofs are essentially identical with the exception of part 2 of the Lemma below where a few small adjustments are necessary to deal with torsion.

Suppose that M is a $\mathcal{B}_q$-comodule or in other words an integrable $U^{\mathrm{res}}(\mathfrak{b})$-module. We will write V to denote the underlying R-module of M equipped with the trivial $\mathcal{B}_q$-coaction.

Lemma 3.18. *Let M be as above.*

1. *If M is in fact a $\mathcal{A}_q$-comodule, viewed as a $\mathcal{B}_q$-comodule via restriction, then $p^*(M) \cong p^*(V)$ in $\mathscr{C}_R$.*
2. *Suppose now that M is finitely generated over R, and moreover suppose that all the weight spaces of M have weight of the form $-\lambda$ where $\lambda \in P^+$. Then*
   - *(a) M is acyclic with respect to the induction functor;*
   - *(b) there is an $\mathcal{A}_q$-comodule which surjects onto M as a $\mathcal{B}_q$-comodule.*

Proof. 1. We have $p^*(M) = \mathcal{A}_q \otimes_R M$ and $p^*(V) = \mathcal{A}_q \otimes_R V$, which are the same as R-modules. The isomorphism is given by the map $a \otimes m \mapsto \sum am_2 \otimes m_1$ where $m \mapsto \sum m_1 \otimes m_2$ denotes the $\mathcal{A}_q$-coaction. It quite evidently is an $\mathcal{A}_q$-module map, and it is straightforward to check that it is also a $\mathcal{B}_q$-comodule map. Thus this is a morphism in $\mathscr{C}_R$. Quite similarly we have a map going the other way given by $a \otimes m \mapsto \sum aS(m_2) \otimes m_1$, which is also a morphism in $\mathscr{C}_R$ by the Hopf algebra axioms. It also follows from the Hopf algebra axioms that these two maps are inverse to each other, and so we have an isomorphism.

2. Write $M = \oplus_\lambda M_{-\lambda}$ for the weight space decomposition of M, where $\lambda \in P^+$ ranges through the weights of M. Since M is finitely generated there are only finitely many weights, and we may list them as $-\lambda_1, -\lambda_2, \ldots, -\lambda_r$ so that $-\lambda_r$ is maximal among them. Hence $N := M_{-\lambda_r}$ is a $U^{\mathrm{res}}(\mathfrak{b})$-submodule. We prove (a) by induction on r. Simply note that N is acyclic by [1, Corollary 5.7(ii)], and by taking the long exact sequence associated to the short exact sequence

$$0 \to N \to M \to M/N \to 0$$

we see that M is also acyclic by induction hypothesis.

For (b), note that $\operatorname{Ind}(M)$ is finitely generated over R by [1, Proposition 3.2]. Hence the result will follow if we show that the map $\operatorname{Res}\operatorname{Ind}(M) \to M$ coming from Frobenius reciprocity (see [1, Proposition 2.12]) is surjective. We prove this by induction on r. Suppose that $r = 1$ so that M is isomorphic to a finite direct sum of modules all of the form $R_{-\lambda}$ or $R_{-\lambda}/\pi^n R_{-\lambda}$ for some $n \geq 1$. Then, it suffices to prove the claim for these summands. But it is true for $R_{-\lambda}$ by [1, Proposition 3.3] and so it follows that it also true for any $R_{-\lambda}/\pi^n R_{-\lambda}$ since we have a commutative diagram

$$\begin{array}{ccc} R_{-\lambda} & \twoheadrightarrow & R_{-\lambda}/\pi^n R_{-\lambda} \\ \uparrow\!\!\uparrow & & \uparrow \\ \operatorname{Res}\operatorname{Ind}(R_{-\lambda}) & \longrightarrow & \operatorname{Res}\operatorname{Ind}(R_{-\lambda}/\pi^n R_{-\lambda}) \end{array}$$

Now for $r > 1$ we consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & M/N & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \operatorname{Res}\operatorname{Ind}(N) & \longrightarrow & \operatorname{Res}\operatorname{Ind}(M) & \longrightarrow & \operatorname{Res}\operatorname{Ind}(M/N) & \longrightarrow & 0 \end{array}$$

in which both rows are exact by (a), and we conclude that $\operatorname{Res}\operatorname{Ind}(M) \to M$ is surjective by the induction hypothesis and the Five Lemma. $\square$

Let $\mathrm{coh}(\mathscr{C}_R)$ denote the full subcategory of $\mathscr{C}_R$ consisting of objects M which are finitely generated as $\mathcal{A}_q$-modules. We call elements of $\mathrm{coh}(\mathscr{C}_R)$ *coherent modules*.

Proposition 3.19. *Let $M \in \mathrm{coh}(\mathscr{C}_R)$. Then there exists $\lambda \in P^+$ such that for all $\mu \in \lambda + P^+$, $M(\mu)$ is generated by finitely many global sections. In particular there is finite direct sum of $\mathcal{A}_q(-\lambda)$ surjecting onto M in $\mathscr{C}_R$.*

Proof. Suppose $m_1, \dots, m_n$ generate M over $\mathcal{A}_q$. Since M is a $\mathcal{B}_q$-comodule i.e an integrable $U^{\mathrm{res}}(\mathfrak{b})$-module it is in particular locally finite. So if we let W denote the $U^{\mathrm{res}}(\mathfrak{b})$-submodule they generate, then we have that W is finitely generated over R. Moreover we have a surjection $p^*(W) \to M$ in $\mathscr{C}_R$. We may pick $\lambda \in P$ such that $W(\lambda) = W \otimes R_{-\lambda}$ satisfies the conditions of Lemma 3.18 (2) and let N be an R-finite $\mathcal{A}_q$-comodule surjecting onto $W(\lambda)$. Then $p^*(N)$ surjects onto $p^*(V(\lambda))$ and hence onto $M(\lambda)$. By Lemma 3.18 (1) and since N is finite over R, we have that $p^*(N)$ is generated as an $\mathcal{A}_q$-module by finitely many global sections, and these define a surjection $\mathcal{A}_q^r \to p^*(N)$. Thus we have a surjection $\mathcal{A}_q^r \to M(\lambda)$ and twisting by $-\lambda$ we get a surjection $\oplus_{i=1}^r \mathcal{A}_q(-\lambda) \to M$ as claimed. Of course the same argument shows that $M(\mu)$ is generated by its global sections for any $\mu \in \lambda + P^+$. $\square$

In the next section we repeatedly use a general construction, which we record here:

Lemma 3.20. *Let $M \in \mathscr{C}_R$ and let $m_1, \dots, m_i \in M$ for some $i \geq 1$. Then there is a unique minimal coherent submodule P of M such that $m_1, \dots, m_i \in P$.*

Proof. Let N be the $U^{\mathrm{res}}(\mathfrak{b})$-submodule of M generated by $m_1, \dots, m_i$. Then N is R-finite and we let P be the $\mathcal{A}_q$-submodule of M generated by N. Since the $\mathcal{A}_q$-action on M is a comodule homomorphism it follows that P is a subcomodule of M and it is in $\mathrm{coh}(\mathscr{C}_R)$ as N is finite over R. Moreover, any coherent submodule of M which contains $m_1, \dots, m_i$ must also contain N, and so must contain P. $\square$

3.7. Coherent Modules

Since we do not know whether $\mathcal{A}_q$ is Noetherian or not, it is not clear yet that $\mathrm{coh}(\mathscr{C}_R)$ is a well-behaved category. This is what we turn to next. We first need to establish:

Lemma 3.21. *The ring R_q is graded Noetherian.*

Proof. Since $q^{\frac{1}{d}} \equiv 1 \pmod{\pi}$, the $U^{\mathrm{res}} \otimes_R k$-representation $\Gamma(\mathcal{A}_q(\lambda)) \otimes_R k$ is just the global sections of the usual line bundle $\mathcal{L}_\lambda$ on the flag variety G_k/B_k over k for any $\lambda \in P^+$ by [1, 3.11], noting that $\mathcal{L}_\lambda$ has no higher cohomology by the classical Kempf vanishing theorem (see e.g. [27, Proposition II.4.5]). Hence we see that the ring $R_q/\pi R_q$ is isomorphic to the ring of regular functions on the basic affine space G_k/N_k, and so is Noetherian. Moreover the graded pieces $\Gamma(\mathcal{A}_q(\lambda))$ are all finitely generated over R by Corollary 3.17. Thus the result follows from Proposition 2.8 (2). $\square$

Theorem 3.22. *The modules in $\mathscr{C}_R$ which are finitely generated as $\mathcal{A}_q$-modules coincide exactly with the Noetherian objects.*

Proof. We prove this result in several steps. First we claim that $\mathcal{A}_q$ satisfies ACC in the category $\mathrm{coh}(\mathscr{C}_R)$. Indeed, assume we have a chain

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$$

of coherent submodules of $\mathcal{A}_q$. Recall the functor $\underline{\Gamma}$ from 3.4. By Noetherianity of $R_q = \underline{\Gamma}(\mathcal{A}_q)$ and by left exactness of $\underline{\Gamma}$, we get that there is some $m \geq 1$ such that for all $n \geq m$, $\underline{\Gamma}(M_n) = \underline{\Gamma}(M_m)$. In particular we get that $\Gamma(M_n(\lambda)) = \Gamma(M_m(\lambda))$ for all $\lambda \in P^+$. Fix any $n \geq m$. Then by Proposition 3.19, we may pick $\lambda >> 0$ such that both $M_n(\lambda)$ and $M_m(\lambda)$ are generated by their global sections. But then the above equality of global sections implies that $M_n(\lambda) = M_m(\lambda)$ and hence after untwisting that $M_n = M_m$.

Next, we claim that $\mathcal{A}_q$ satisfies ACC in $\mathscr{C}_R$. Indeed, suppose we have a chain

$$M_1 \subset M_2 \subset M_3 \subset \cdots$$

of subobjects of $\mathcal{A}_q$ with $M_i \neq M_{i+1}$ for every $i \geq 1$. Then we may pick $m_1 \in M_1$ and $m_i \in M_i \setminus M_{i-1}$ for every $i \geq 2$. By Lemma 3.20, for each $i \geq 1$ we may consider the smallest coherent submodule P_i of M_i which contains $m_1, \ldots, m_i$. Note that $P_i \subset P_{i+1}$ by the proof of Lemma 3.20. But $m_i \in P_i$ for every i, so that we get a strict ascending chain

$$P_1 \subset P_2 \subset P_3 \subset \cdots$$

of coherent submodules of $\mathcal{A}_q$, which is a contradiction by our first step.

Thus we have proved that $\mathcal{A}_q$ is a Noetherian object. It is then immediate that every line bundle $\mathcal{A}_q(\lambda)$ is also a Noetherian object. But by Proposition 3.19, this implies that every coherent module is a Noetherian object. Finally, for the converse, the above argument that $\mathcal{A}_q$ satisfies ACC in $\mathscr{C}_R$ also shows that Noetherian objects are finitely generated over $\mathcal{A}_q$. Indeed, if M is not finitely generated, pick $m_1 \in M$ and let P_1 be the smallest coherent submodule of M containing m_1, given by Lemma 3.20. Since M is not coherent we have that $M \neq P_1$. So we can pick $m_2 \in M$ such that $m_2 \notin P_1$. Then we may apply Lemma 3.20 again and set P_2 to be the smallest coherent submodule of M containing m_1, m_2. By construction, $P_1 \subset P_2$ is a strict inclusion. As M is not coherent, we may pick $m_3 \in M \setminus P_2$. Carrying on, we get a strict ascending chain

$$P_1 \subset P_2 \subset P_3 \subset \cdots$$

so that M is not a Noetherian object. □

This in particular shows that $\mathrm{coh}(\mathscr{C}_R)$ is an abelian category. This has a few consequences.

Proposition 3.23. *Let $M \in \mathrm{coh}(\mathscr{C}_R)$. Then:*

1. *there exists $\lambda \in P^+$ such that for all $\mu \in \lambda + P^+$, $M(\mu)$ is acyclic; and*
2. (Serre finiteness) *for all $i \geq 0$, $R^i\Gamma(M)$ is finitely generated as an R-module.*

Proof. (1) By Proposition 3.19 and the above Theorem, we may find a resolution of M of the form

$$F_\bullet : F_N \xrightarrow{f_N} \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} M \to 0$$

where the F_i are finite direct sums of line bundles. Pick $\lambda \in P$ sufficiently large such that all the line bundles in $F_\bullet(\lambda)$ are of the form $\mathcal{A}_q(\mu)$ for $\mu \in P^+$. Then by Corollary 3.17, all the $F_i(\lambda)$ are Γ-acyclic. Let $K_0 = M(\lambda)$ and $K_j = \ker f_j(\lambda)$ for $1 \leq j \leq N$. Then we have a short exact sequence

$$0 \to K_j \to F_j(\lambda) \xrightarrow{f_j(\lambda)} K_{j-1} \to 0$$

for every $1 \leq j \leq N$, and the long exact sequence yields isomorphisms $R^i\Gamma(K_{j-1}) \cong R^{i+1}\Gamma(K_j)$ for all $i \geq 1$. Thus, by using Proposition 3.15 (4), we obtain

$$R^i\Gamma(M(\lambda)) \cong R^{i+1}\Gamma(K_1) \cong \cdots \cong R^{i+N}\Gamma(K_N) = 0$$

for all $i \geq 1$ as required. Again the same argument works by replacing λ by any $\mu \in \lambda + P^+$.

(2) The proof we give is completely analogous to the proof in [24, Theorem III.5.2]. First note that by Proposition 3.15 (4) we have $R^i\Gamma(M) = 0$ for all $i > N$ and so we may assume that $i \leq N$. We will prove the result by downwards induction on i, the cases $i > N$ being already covered.

By Proposition 3.19 there is a surjection $f : \bigoplus_{j=1}^{n} \mathcal{A}_q(-\lambda_j) \to M$ in $\mathscr{C}_R$, where each $\lambda_j \in P^+$. This gives a short exact sequence

$$0 \to K \to \bigoplus_{j=1}^{n} \mathcal{A}_q(-\lambda_j) \to M \to 0$$

Applying the long exact sequence, we obtain

$$\cdots \to \bigoplus_{j=1}^{n} R^i\Gamma(\mathcal{A}_q(-\lambda_j)) \to R^i\Gamma(M) \to R^{i+1}\Gamma(K) \to \cdots$$

By the induction hypothesis applied to K (which we may apply by the above Theorem), we get that $R^{i+1}\Gamma(K)$ is finitely generated. Now by Corollary 3.17 and since R is Noetherian, we see that $R^i\Gamma(M)$ is finitely generated over R as well. □

One of our main aims will be to establish a D-modules version of part 1. of the Proposition. Before we get to that, we can now finally fulfill our promise:

Proof of Theorem 3.14. Note that every object of $\mathscr{C}_R$ is a direct limit of object of $\mathrm{coh}(\mathscr{C}_R)$. Indeed, it suffices to show that every element of any $M \in \mathscr{C}_R$ is contained in a coherent submodule. But this is given by Lemma 3.20.

So we just have to check all conditions 1-4 from Proposition 3.11. Condition 1 is just Theorem 3.22, 2 follows from the fact that $\Gamma(\mathcal{A}_q) = R$ and from Proposition 3.23 (2), and 3 follows from Proposition 3.19. Finally, condition 4 is easily deduced from Theorem 3.22 and Proposition 3.23 (1). Indeed, suppose $M \to N$ is a surjection between coherent modules in $\mathscr{C}_R$ and let K denote its kernel. For $\lambda >> 0$, we know that $K(\lambda)$ is Γ-acyclic, and so the corresponding map $\Gamma(M(\lambda)) \to \Gamma(N(\lambda))$ is surjective. □

3.8. Weyl Group Translates of the Big Cell

We now introduce certain localisations of $\mathcal{A}_q$ from Joseph (see [28, 3.1-3.3] and [29, 9.1.10]). For each fundamental weight ϖ_i, consider the highest weight representation $V(\varpi_i)$ of U_q. It contains a free R-lattice $M := \mathrm{Ind}(R_{\varpi_i})^*$ that is a U^{res}-module. In fact M is a cyclic module generated by a highest weight vector $v \in V(\varpi_i)$ (see [1, Proposition 3.3]). Let $f \in M^*$ be the corresponding dual vector. Let $c_{\varpi_i} := c^M_{f,v} \in \mathcal{A}_q$ be the corresponding matrix coefficient. Joseph showed in *loc. cit.* that these commute and we may define for any $\mu = \sum_i n_i \varpi_i \in P^+$ the element $c_\mu = \prod_i c_{\varpi_i}^{n_i} \in \mathcal{A}_q$. Moreover, for any $\mu \in P^+$, $c_\mu = c^{V(\mu)}_{f_\mu, v_\mu}$ is the matrix coefficient of the highest weight representation $V(\mu)$ of U_q. In fact it is the matrix coefficient of a U^{res}-lattice inside $V(\mu)$, namely $\mathrm{Ind}(R_{-\mu})^*$.

Recall that $\mathcal{A}_q$ is a U^{res}-module algebra via the action $u \cdot f = \sum f_2(u) f_1$. If we identify $\mathcal{A}_q$ with a submodule of $\mathrm{Hom}_R(U^{\mathrm{res}}, R)$, this action is given by

$$(u \cdot f)(x) = f(xu)$$

for all $u, x \in U^{\mathrm{res}}$ and all $f \in \mathcal{A}_q$. Therefore, identifying c_μ with the matrix coefficient corresponding to a highest weight vector as above, we see that $u \cdot c_\mu = \mu(u) c_\mu$ for any $u \in (U^{\mathrm{res}})^0$ and $E^{(r)}_{\alpha_i} \cdot c_\mu = 0$ for any i and any $r \geq 1$. Thus in the $\mathcal{B}_q$-comodule language, we have $\Delta(c_\mu) = c_\mu \otimes \mu \in \mathcal{A}_q \otimes \mathcal{B}_q$. So we see that $c_\mu \in \Gamma(\mathcal{A}_q(\mu))$.

Recall now that $\Gamma(\mathcal{A}_q(\mu)) = \mathrm{Ind}\, R_{-\mu}$ is an integrable U^{res}-module. The elements of it can all be identified as certain functions in $\mathrm{Hom}_R(U^{\mathrm{res}}, R)$, and the module structure is given by

$$(u \cdot f)(x) = f(S(u)x)$$

for all $u, x \in U^{\text{res}}$ and all $f \in \Gamma(\mathcal{A}_q(\mu))$. With respect to this action, the element c_μ has weight $-\mu$ and so is a lowest weight vector, since the module $\Gamma(\mathcal{A}_q(\mu))$ is a free R-lattice inside $V(-w_0\mu)$ and satisfies the Weyl character formula by [1, Corollary 3.3]. In particular we see that $\Gamma(\mathcal{A}_q(\mu))$ has a unique (up to scalars) extreme w-weight vector $c_{w\mu}$ of weight $-w\mu$ for any Weyl group element $w \in W$, which we may choose to equal

$$c_{w\mu} = E_{\alpha_{i_1}}^{(r_1)} \cdots E_{\alpha_{i_s}}^{(r_s)} \cdot c_\mu$$

where $w = s_{i_1} \cdots s_{i_s}$ and where the exponents r_j are defined by $r_s = \langle \mu, \alpha_{i_s}^\vee \rangle$ and $r_j = \langle s_{i_{j+1}} \cdots s_{i_s}\mu, \alpha_{i_j}^\vee \rangle$ for $j \leq s-1$. Then Joseph [29, 9.1.10] showed that $c_{w\lambda}c_{w\mu} = c_{w(\lambda+\mu)}$ for every $w \in W$ and every $\lambda, \mu \in P^+$. Therefore, for every $w \in W$, the set

$$S_w := \{c_{w\mu} : \mu \in P^+\}$$

is multiplicatively closed in $\mathcal{A}_q$. Moreover we still have $c_{w\mu} \in \Gamma(\mathcal{A}_q(\mu))$, so that we may view S_w as a multiplicatively closed subset of R_q. Joseph showed in *loc. cit.* that S_w is an Ore set in both $\mathcal{O}_q$ and its representation ring, but in fact his proof works equally well with $\mathcal{A}_q$ and in R_q (see also [33, III.2]). Hence we have:

Lemma 3.24. *For every $w \in W$, S_w is an Ore set in $\mathcal{A}_q$ and in R_q.*

So we may define localisations $\mathcal{A}_{q,w} := S_w^{-1}\mathcal{A}_q$ for each Weyl group element. By viewing $\mathcal{A}_q \otimes \mathcal{B}_q$ as a left $\mathcal{A}_q$-module via the comultiplication Δ, the comodule map $\mathcal{A}_q \to \mathcal{A}_q \otimes \mathcal{B}_q$, which by abuse of notation we also denote by Δ, is an $\mathcal{A}_q$-module map, and its localisation gives a map

$$\Delta_w : \mathcal{A}_{q,w} \to \mathcal{A}_{q,w} \otimes_R \mathcal{B}_q$$

which defines a $\mathcal{B}_q$-comodule structure: for $f \in \mathcal{A}_q$ and $s \in S_w$ such that $\Delta(s) = s \otimes \lambda$, Δ_w sends $s^{-1}f$ to $(s^{-1} \otimes -\lambda) \cdot \Delta(f)$. Moreover the $\mathcal{A}_{q,w}$-module structure on $\mathcal{A}_{q,w} \otimes_R \mathcal{B}_q$ is defined by Δ_w.

More generally, if $M \in \mathscr{C}_R$ with comodule map $\rho : M \to M \otimes_R \mathcal{B}_q$ then, by the axioms for $\mathscr{C}_R$, ρ is an $\mathcal{A}_q$-module map where we view $M \otimes_R \mathcal{B}_q$ as an $\mathcal{A}_q$-module via Δ, and its localisation gives rise to a map

$$\rho_w : S_w^{-1}M \to S_w^{-1}M \otimes_R \mathcal{B}_q$$

which will be $\mathcal{A}_{q,w}$-linear where $\mathcal{A}_{q,w}$ acts on $S_w^{-1}M \otimes_R \mathcal{B}_q$ via the map Δ_w.

Definition 3.25. *We define $\mathscr{C}_R^w$ to be the category of B-equivariant $\mathcal{A}_{q,w}$-modules. Specifically, the objects consist of $\mathcal{A}_{q,w}$-modules M which are equipped with a right $\mathcal{B}_q$-comodule action $M \to M \otimes_R \mathcal{B}_q$ such that the $\mathcal{A}_{q,w}$-action map $\mathcal{A}_{q,w} \otimes_R M \to M$ is a comodule homomorphism where we give $\mathcal{A}_{q,w} \otimes_R M$ the tensor comodule structure. The morphisms are just the $\mathcal{A}_{q,w}$-linear maps which are also comodule homomorphisms.*

The above discussion shows that there is a localisation functor $f_w^* : \mathscr{C}_R \to \mathscr{C}_R^w$ which sends a module M to its localisation $S_w^{-1}M$ as an $\mathcal{A}_q$-module, and it has a right adjoint $(f_w)_*$ given by the forgetful functor. Both of these are exact and they make $\mathscr{C}_R^w$ into a localisation of $\mathscr{C}_R$ in the sense of Gabriel i.e a quotient of $\mathscr{C}_R$ by a localising subcategory (see [21, Chapter III.2]).

3.9. Čech Complexes

We saw that $\mathscr{C}_R$ is equivalent to Proj(R_q) and that we may equally localise any graded R_q-module at the set S_w for any $w \in W$. Since the set S_w contains elements of arbitrarily large degree in R_q, we see that the localisation functor R_q-mod $\to S_w^{-1}R_q$-mod factors through Proj(R_q) and makes $S_w^{-1}R_q$-mod into a localisation of Proj(R_q).

We have a global section functor on $\mathscr{C}_R^w$ which corresponds to taking $\mathcal{B}_q$-invariants. This is of course the same as the composite $\Gamma \circ (f_w)_*$. Now via the proj construction we see that global sections on $\mathscr{C}_R$ correspond to projection onto the degree 0 in Proj(R_q). So we see that the global section functor on $S_w^{-1}R_q$-mod is the functor of taking the degree 0 part of the graded module, which is exact! We then get:

Lemma 3.26. *The categories $S_w^{-1}R_q$-mod and $\mathscr{C}_R^w$ have enough injectives, and they are naturally equivalent to each other as localisations of $\mathscr{C}_R$. Hence the global section functor on $\mathscr{C}_R^w$ is exact and objects of $\mathscr{C}_R^w$ are acyclic when viewed in $\mathscr{C}_R$.*

Proof. By [21, Corollary III.3.2] the first part follows from Lemma 3.10 and the fact that both categories are localisations of $\mathscr{C}_R$. By the above discussion, if the two categories are equivalent then global sections is exact. To prove that $S_w^{-1}R_q$-mod and $\mathscr{C}_R^w$ are equivalent, we just need to show that $M \in \mathscr{C}_R$ has localisation zero if and only if $\underline{\Gamma}(M)$ has localisation zero.

Clearly if $M \in \mathscr{C}_R$ has localisation zero, then so does $\underline{\Gamma}(M)$. Conversely if $\underline{\Gamma}(M)$ has localisation zero, we show that $\underline{\Gamma}(S_w^{-1}M) = 0$, which implies that $S_w^{-1}M = 0$. Indeed suppose $s \in S_w$, $m \in M$ such that $\Delta(s) = s \otimes \mu$ and $s^{-1}m \in \Gamma(S_w^{-1}M(\lambda))$ for some λ. Then

$$\rho(m) = \rho(s(s^{-1}m)) = (s \otimes \mu)\rho(s^{-1}m) = (s \otimes \mu)(s^{-1}m \otimes \lambda) = m \otimes (\lambda + \mu)$$

so that $m \in \Gamma(M(\lambda + \mu))$. By assumption there exists $t \in S_w$ such that $tm = 0$. But then that means that the image of m in $S_w^{-1}M$ is zero and so $s^{-1}m = 0$. Thus we see that $\underline{\Gamma}(S_w^{-1}M) = 0$ as required.

Finally, let $M \in \mathscr{C}_R^w$ and $M \to I^\bullet$ be an injective resolution of M in $\mathscr{C}_R^w$. Note that since $(f_w)_*$ preserves injectives as it is the right adjoint to an exact functor, we have that $(f_w)_*(I^\bullet)$ is an injective resolution of $(f_w)_*(M)$ in $\mathscr{C}_R$, and applying global sections and taking cohomology we obtain $R^i\Gamma((f_w)_*(M)) = 0$ for all $i > 0$ since $\Gamma \circ (f_w)_*$ is exact. □

We think of $\mathscr{C}_R^w$ as being an analogue of the w-translate of the big cell on the flag variety, and the above lemma tells us that it is in some sense affine. Now to such a situation Rosenberg [40, Sections 1 & 2] (see also [33, section III.3]) explained how to write down an analogue of the Čech complex which allows us to compute the cohomology of the functor Γ. Write $W = \{w_1, \ldots, w_m\}$, let $J = \{1, \ldots, m\}$ and for each $i \in J$ let $\sigma_i := (f_{w_i})_* \circ f_{w_i}^*$. Moreover for any $\mathbf{i} = (i_1, \ldots, i_n) \in J^n$, let $\sigma_{\mathbf{i}} = \sigma_{i_1} \circ \cdots \circ \sigma_{i_n}$. Then by [40, 1.2 & 1.3] we may write down a complex

$$C^{\text{aug}} : \text{id}_{\mathscr{C}_R} \to \bigoplus_{i \in J} \sigma_i \to \bigoplus_{\mathbf{i} \in J^2} \sigma_{\mathbf{i}} \to \bigoplus_{\mathbf{i} \in J^3} \sigma_{\mathbf{i}} \to \cdots$$

where the maps are given as follows. Denote the adjunction morphism $\text{id}_{\mathscr{C}_R} \to \sigma_i$ by η_i. Then for any $\mathbf{i} \in J^n$ and any $1 \le j \le n$, there is a natural transformation

$$\xi_n^j : \sigma_{i_1} \circ \cdots \circ \sigma_{i_n} \to \oplus_{i \in J} \sigma_{i_1} \circ \cdots \circ \sigma_{i_{j-1}} \circ \sigma_i \circ \sigma_{i_j} \circ \cdots \circ \sigma_{i_n}$$

given by $\xi_n^j = \oplus_{i \in J} \sigma_{i_1} \cdots \sigma_{i_{j-1}} \eta_i \sigma_{i_j} \cdots \sigma_{i_n}$. The differential in the complex is then given by taking the alternating sum (over all j) of these ξ_n^j.

We may post-compose C^{aug} with the functor of taking global sections to obtain a complex $\check{C}^{\text{aug}}$ called the *augmented standard complex* of Γ. We may also consider the complex

$$C : \bigoplus_{i \in J} \sigma_i \to \bigoplus_{\mathbf{i} \in J^2} \sigma_{\mathbf{i}} \to \bigoplus_{\mathbf{i} \in J^3} \sigma_{\mathbf{i}} \to \cdots$$

and $\check{C} = \Gamma \circ C$, which we call the *standard complex*. We then have:

Proposition 3.27. *For any $M \in \mathscr{C}_R$, the complex $C^{aug}(M)$ is exact. Moreover, for $i \ge 0$, the i-th cohomology of the complex $\check{C}(M)$ is isomorphic to $R^i\Gamma(M)$.*

Proof. By [40, Proposition 1.4 & Theorem 2.2] and by the Lemma, it will follow if we prove that the categories $\mathscr{C}_R^{w_i}$ cover the category $\mathscr{C}_R$, meaning that a morphism g in $\mathscr{C}_R$ is an isomorphism if and only if $f_{w_i}^*(g)$ is an isomorphism for all $i \in J$. This is equivalent to saying that $M \in \mathscr{C}_R$ is zero if and only if all its localisations are zero. Working with proj categories instead, suppose M is a graded R_q-module such that $S_w^{-1}M = 0$ for all w. Pick $m \in M$. Then for all $i \in J$, there exists $\mu_i \in P^+$ such that $c_{w_i\mu_i}m = 0$. Let $\mu = \sum_i \mu_i$. Then for all $w \in W$, $c_{w\mu}m = 0$. But then it follows from the Lemma below that $\Gamma(\mathcal{A}_q(\lambda + \mu))m = 0$ for all $\lambda >> 0$. Since m was arbitrary this implies that M is torsion and so zero in Proj(R_q). □

Lemma 3.28. *Let $\mu \in P^+$. Then for $\lambda >> 0$ we have*

$$\sum_{w \in W} \Gamma(\mathcal{A}_q(\lambda)) c_{w\mu} = \Gamma(\mathcal{A}_q(\lambda + \mu)).$$

Proof. This is proved in [33, Lemma III.3.3] but we reproduce it here. Clearly the left hand side is included in the right hand side, and both sides are finitely generated as R-modules, so by Nakayama it's enough to show that the equality holds modulo π, i.e. that

$$\sum_{w \in W} H^0(\lambda) \overline{c_{w\mu}} = H^0(\lambda + \mu)$$

where $H^0(\lambda)$ denotes the global sections of the line bundle $\mathcal{L}(\lambda)$ on the flag variety G_k/B_k. We are then in a classical situation, and the equality will follow from the classical fact that the Weyl group translates of the big cell cover the flag variety of G_k. The equality was proved over $\mathbb{C}$ in [30, Lemma 11]. The argument is the same here in positive characteristic, but for completeness we sketch it.

Firstly, since both sides are finite dimensional over k, to show equality is to show that the dimensions are equal, and so it will suffice to prove that the equality holds after passing to the algebraic closure of k. So without loss of generality, we may assume that $k = \bar{k}$. Moreover, for any λ' and μ', the natural map $H^0(\lambda') \otimes H^0(\mu') \to H^0(\lambda' + \mu')$ is surjective (see [27, Proposition 14.20]). Thus we may assume that $\lambda = n\mu$ for $n >> 0$.

Now, consider the Weyl module $V = V(-w_0\mu) = H^0(\mu)^*$ and let $v \in V$ have weight $-w_0\mu$. Then the flag variety G_k/B_k maps onto the G_k-orbit of the line kv in the projective space $\mathbb{P}(V)$. If we take the homogeneous cone above this, its algebra of regular functions is a quotient of $S(V^*)$, and in fact is the commutative graded ring $A = \oplus_{n \geq 0} H^0(n\mu)$ (see [27, Proposition 14.22]). The fact that the Weyl group translates of the big cell cover the flag variety now implies that the radical of the ideal of A generated by the elements $\overline{c_{w\mu}}$ is in fact the irrelevant ideal $A_{>0}$. This says that the ideal they generate contains all $H^0(m\mu)$ for m large enough, as required. □

3.10. Base Change

As an immediate application of the Čech complex, we show how the cohomology of Γ behaves under base change to the field L.

Proposition 3.29. *For any $M \in \mathscr{C}_R$ and any $i \geq 0$, we have $R^i\Gamma(M_L) = R^i\Gamma(M) \otimes_R L$.*

Proof. Let $M \in \mathscr{C}_R$ and first assume that $i = 0$. By the universal property of tensor products, we have a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\ g\ } & M \otimes_R L \\ \uparrow & & \uparrow f \\ M^{\mathcal{B}_q} & \longrightarrow & M^{\mathcal{B}_q} \otimes_R L \end{array}$$

of R-modules with injective vertical arrows, and we have to show that $\mathrm{Im}(f) = (M_L)^{B_q}$. It will be enough show that $(M_L)^{B_q} \subseteq \mathrm{Im}(f)$, the other inclusion being clear.

Pick $m \in (M_L)^{B_q}$. Then there is some $a \geq 0$ such that $\pi^a m \in \mathrm{Im}(g)$, i.e $\pi^a m = m' \otimes 1$ for some $m' \in M$. Now, given that $\rho_{M_L}(m) = m \otimes 1$, and since $\rho_{M_L} = \rho_M \otimes_R L$, we see that $\rho_M(m') - m' \otimes 1 \in M \otimes_R \mathcal{B}_q$ is π-torsion. Hence, there is some $b \geq 0$ such that $\rho_M(\pi^b m') = \pi^b m' \otimes 1$, and thus we get that $\pi^{a+b} m \in \mathrm{Im}(f)$. The result now follows since $\mathrm{Im}(f)$ is an L-vector space.

For $i > 0$, using Proposition 3.26, the case $i = 0$ and the fact that $- \otimes_R L$ is exact, we see that $R^i\Gamma(M) \otimes_R L$ is the i-th Čech cohomology group of $M \otimes_R L$. By [8, Proposition 4.5] this group is equal to $R^i\Gamma(M \otimes_R L)$. □

3.11. The Ring $\mathcal{D}$

Recall the notation and the definitions from 3.2. We now define an R-form of the ring of quantum differential operators. For $u \in U$, $a \in \mathcal{A}_q$, $i \geq 0$, we have $u(a) = \sum a_2(u) \cdot a_1 \in \mathcal{A}_q$ since $U \subset U^{\text{res}}$. From this, we can immediately see that $\mathcal{A}_q$ is a left U-module algebra. Hence we may form the smash product $\mathcal{D} = \mathcal{A}_q \# U$. Note that $\mathcal{D}$ is π-torsion free as it is equal to $\mathcal{A}_q \otimes_R U$ as an R-module, thus it follows that it is a lattice in $\mathcal{D}_q$.

Proposition 3.30. *The algebra $\mathcal{D}/\pi\mathcal{D}$ is Noetherian. Hence so is $\widehat{\mathcal{D}}$.*

Proof. By the above remarks we see that $\mathcal{D}_k := \mathcal{D}/\pi\mathcal{D}$ is the smash product algebra of $\mathcal{A}_q/\pi\mathcal{A}_q \cong \mathcal{O}(G_k)$ and U_k. We will see in [19] that $U_k \cong U(\mathfrak{g}_k) \otimes_k k(P/2Q)$, and so $\mathcal{D}_k$ is a finite module over the smash product $\mathcal{O}(G_k)\#U(\mathfrak{g}_k)$. The latter is isomorphic to the ring $\mathcal{D}(G_k)$ of crystalline differential operators on the affine variety G_k and hence is Noetherian. Thus $\mathcal{D}_k$ is Noetherian as required. The last part follows from Proposition 2.8. $\square$

3.12. $\mathcal{D}$-Modules

We now turn to an R-version of the category $\mathcal{D}^{\lambda}_{B_q}(G_q)$. We first introduce the following notation: we let $U^{\geq 0} = U \cap U_q^{\geq 0}$. It is the R-subalgebra of U generated by all E_{α_i}, all K_μ ($\mu \in P$) and all $[K_{\alpha_i}; 0]_{q_i}$. Note that $U^{\geq 0}$ is a subalgebra of $U^{\text{res}}(\mathfrak{b})$. Moreover, note that the action (3.1) restricts to an action of U^{res} on $\mathcal{D}$ making it into a U^{res}-module algebra. This is because the adjoint action of U^{res} preserves U.

Definition 3.31. *Let $\lambda \in T_P^R$. We let $\mathscr{D}^\lambda$ be the category whose objects are triples (M, α, β) where M is an R-module, $\alpha : \mathcal{D} \otimes_R M \to M$ is a left $\mathcal{D}$-module action and $\beta : M \to M \otimes_R \mathcal{B}_q$ is a right $\mathcal{B}_q$-comodule action. The map β induces a left $U^{res}(\mathfrak{b})$-action on M which we also denote by β. These actions must satisfy:*

(i) The $U^{\geq 0}$-actions on $M \otimes_R R_\lambda$ given by $\beta \otimes \lambda$ and $\alpha|_{U^{\geq 0}} \otimes 1$ are equal.

(ii) The map α is $U^{res}(\mathfrak{b})$-linear with respect to the β-action on M and the action (3.1) on $\mathcal{D}$.

We will write coh($\mathscr{D}^\lambda$) to denote the full subcategory of $\mathscr{D}^\lambda$ consisting of finitely generated $\mathcal{D}$-modules.

There is of course a forgetful functor forget : $\mathscr{D}^\lambda \to \mathscr{C}_R$, and given an object $M \in \mathscr{D}^\lambda$ we let its global sections equal $\Gamma(M)$ where we view M as an object of $\mathscr{C}_R$. By abuse of notation we also denote this global section functor by Γ. Also the functor $M \mapsto M_L$ described earlier restricts to a functor $\mathscr{D}^\lambda \to \mathcal{D}^\lambda_{B_q}(G_q)$.

Note again that condition (i) above can be rephrased into saying that for $M \in \mathscr{D}^\lambda$ and $m \in M$, we have $E_\alpha m = \beta(E_\alpha)m$, $K_\mu m = \lambda(K_\mu)\beta(K_\mu)m$, and

$$[K_\alpha; 0]m = (\lambda([K_\alpha; 0])\beta(K_\alpha) + \lambda(K_\alpha^{-1})\beta([K_\alpha; 0]))m$$

for all simple roots α and $\mu \in P$. In particular if m is a global section then by B_q-invariance we must have $E_\alpha m = 0$, $[K_\alpha; 0]m = \lambda([K_\alpha; 0])m$ and $K_\mu m = \lambda(K_\mu)m$. In other words global sections consist of the highest weight vectors of weight λ. So we see that the $\mathcal{D}$-module homomorphisms $\mathcal{D} \to M$ corresponding to global sections factor through the quotient $\mathcal{D}^\lambda = \mathcal{D}/I$ where I is the left ideal generated by

$$\{E_{\alpha_i}, K_\mu - \lambda(K_\mu), [K_{\alpha_i}; 0] - \lambda([K_{\alpha_i}; 0]) : 1 \leq i \leq n, \mu \in P\}.$$

Our aim now is to show that $\mathcal{D}^\lambda \in \mathscr{D}^\lambda$.

Recall the notation from 2.2. Note that we may define a Verma module $\mathcal{M}_\lambda$ for U, namely it is the cyclic U-module with generator v_λ and relations $E_{\alpha_i} v_\lambda = 0$, $K_\mu v_\lambda = \lambda(\mu) v_\lambda$ and $[K_{\alpha_i}; 0]v_\lambda = \lambda([K_{\alpha_i}; 0])v_\lambda$.

By the triangular decomposition for U and the PBW basis for U^- (see [18, Sections 4.5-4.6]), we see that $\mathcal{M}_\lambda$ is a free R-module with basis given by the monomials

$$F_{\beta_1}^{r_1} \cdots F_{\beta_N}^{r_N} v_\lambda$$

and so we also see that it is a lattice in the Verma module M_λ for U_q. In fact it is the image of U under the canonical surjection $U_q \to M_\lambda$. Recall that the quotient of the adjoint action of $U_q^{\geq 0}$ gave rise to an integrable module structure on M_λ. Since the adjoint action of $U^{\mathrm{res}}(\mathfrak{b})$ preserves U (see [50, Lemma 1.2]), we immediately get:

Lemma 3.32. *The above adjoint $U^{res}(\mathfrak{b})$-action on M_λ preserves $\mathcal{M}_\lambda$, making it into a $\mathcal{B}_q$-comodule.*

Now since $\mathcal{D}^\lambda = \mathcal{A}_q \otimes_R \mathcal{M}_\lambda$ as an R-module, we identify it with $p^*(\mathcal{M}_\lambda) \in \mathscr{C}_R$. Just as for $\mathcal{D}_q^\lambda$, we then have that $\mathcal{D}^\lambda$ is in fact an object of $\mathscr{D}^\lambda$, and our previous discussion shows that it represents the global section functor on $\mathscr{D}^\lambda$, i.e

$$\Gamma(M) = \mathrm{Hom}_{\mathscr{D}^\lambda}(\mathcal{D}^\lambda, M)$$

for all $M \in \mathscr{D}^\lambda$.

3.13. Cohomology of the Induction Functor mod π

We will need to investigate the cohomology of $M_k := M/\pi M$ for $M \in \mathcal{B}_q$-comod for the induction functor $\mathrm{Ind} : \mathcal{B}_q\text{-comod} \to \mathcal{A}_q\text{-comod}$ defined in [1]. Note that M_k is in fact a $\mathcal{B}_q/\pi\mathcal{B}_q \cong \mathcal{O}(B_k)$-comodule, and the global section functor applied to $p^*(M_k)$ coincides with the functor of taking $\mathcal{O}(B_k)$-coinvariants in $\mathcal{O}(G_k) \otimes_k M_k$, i.e. with the classical induction functor $\mathrm{Ind}_{B_k}^{G_k} M_k$ (c.f. [1, Proposition 3.7]). We will compare the cohomology groups $R^i\,\mathrm{Ind}(M_k)$ and $R^i\,\mathrm{Ind}_{B_k}^{G_k}(M_k)$.

By [1, 2.17-2.19], if M is a $\mathcal{B}_q$-comodule that is free as an R-module then it has a resolution

$$0 \to M \to Q_0 \to Q_1 \to \cdots$$

in the category of $\mathcal{B}_q$-comodules, which is R-split and such that each Q_i is R-free and acyclic. This is called the *standard resolution* of M. This construction is completely canonical, so that M_k also has similarly such a resolution in the category of $\mathcal{B}_q/\pi\mathcal{B}_q$-comodules, which we also call the standard resolution of M_k.

Lemma 3.33. *Suppose $M \in \mathscr{C}_R$ is free as an R-module. Then there is a canonical isomorphism $R^i\mathrm{Ind}(M_k) \cong R^i\mathrm{Ind}_{B_k}^{G_k}(M_k)$ for all $i \geq 0$.*

Proof. Since each Q_i is free, we have a short exact sequence

$$0 \longrightarrow Q_i \xrightarrow{\cdot\pi} Q_i \longrightarrow Q_i \otimes_R k \longrightarrow 0.$$

Applying the long exact sequence and using the fact that Q_i is acyclic, we immediately obtain that $Q_i \otimes_R k$ is also acyclic. Now since the standard resolution is split exact, it follows from the above that

$$0 \to M_k \to Q_0 \otimes_R k \to Q_1 \otimes_R k \to \cdots$$

is an acyclic resolution of M_k whose cohomology therefore computes $R\,\mathrm{Ind}(M_k)$. On the other hand this resolution coincides with the standard resolution of M_k by [1, page 24, after equation (6)], so computes $R\,\mathrm{Ind}_{B_k}^{G_k}(M_k)$ by [1, Proposition 3.7]. ☐

This will be useful because $R\mathrm{Ind}(M_k) \cong R\Gamma(p^*(M_k))$ by Proposition 3.15 (2). We now apply the above to Verma modules, viewed as $\mathcal{B}_q$-comodules via the adjoint action of $U^{\mathrm{res}}(\mathfrak{b})$. First we recall some well-known generalities.

Recall from [26, I.5.8 & Proposition I.5.12] that if N is a representation of B_k, then there is a corresponding G_k-equivariant sheaf $\mathcal{L}(N)$ on the flag variety $X_k := G_k/B_k$ such that $R^i\,\mathrm{Ind}_{B_k}^{G_k} N$ is canonically isomorphic to the sheaf cohomology $H^i(X_k, \mathcal{L}(N))$ for each $i \geq 0$. Moreover, the sheaf $\mathcal{D}^\lambda_{X_k}$ of crystalline λ-twisted differential operators on X_k, for $\lambda \in \mathfrak{h}_k^*$, is the G_k-equivariant sheaf corresponding to the Verma module $\mathcal{M}(\lambda) = U(\mathfrak{g}_k)/(x - \lambda(x) : x \in \mathfrak{b}_k)$ viewed as a B_k-representation via the adjoint action (c.f. [37, pages 12& 20] in characteristic 0, see also [12, 3.1.3] in positive characteristic). Thus we see that the sheaf cohomology of $\mathcal{D}^\lambda_{X_k}$ coincides with $R\,\mathrm{Ind}_{B_k}^{G_k}(\mathcal{M}(\lambda))$.

Definition 3.34. *We set*

$$T_P^k := \{\gamma \in T_P^R : \gamma(K_\mu) \equiv 1 \pmod{\pi} \text{ for all } \mu \in P\}.$$

This is a subgroup of T_P^R containing P. Note that any $\lambda \in T_P^k$ induces an element of $\mathfrak{h}_k^*$ which we also denote by λ. Hence the corresponding Verma module $\mathcal{M}_\lambda$ satisfies $(\mathcal{M}_\lambda)_k \cong \mathcal{M}(\lambda)$.

Finally recall that the object $\mathcal{D}^\lambda \in \mathscr{C}_R$ is given by $\mathcal{D}^\lambda = p^*(\mathcal{M}_\lambda)$ and so similarly $\mathcal{D}^\lambda_k = p^*((\mathcal{M}_\lambda)_k)$. Thus we have $R\Gamma(\mathcal{D}^\lambda_k) \cong R\,\mathrm{Ind}((\mathcal{M}_\lambda)_k)$. Putting everything together, we get by the Lemma:

Proposition 3.35. *Let $\lambda \in T_P^k$. Then the cohomology of $\mathcal{D}^\lambda_k$ with respect to Γ coincides with the classical sheaf cohomology of twisted differential operators on the flag variety, i.e. there is a canonical isomorphism $R^i\Gamma(\mathcal{D}^\lambda_k) \cong H^i(X_k, \mathcal{D}^\lambda_{X_k})$ for every $i \geq 0$.*

The above will allow us to compute the global sections of $\mathcal{D}^\lambda_k$. But first we need to mention some restrictions on the prime p.

Definition 3.36. *Recall that the prime p is said to be* bad *for an irreducible root system Φ if*

- *$p = 2$ when $\Phi = B_l, C_l$ or D_l;*
- *$p = 2$ or 3 when $\Phi = E_6, E_7, F_4$ or G_2; and*
- *$p = 2, 3$ or 5 when $\Phi = E_8$.*

We say that p is bad *for U_q if it is bad for some irreducible component of the associated root system, and we say that p is* good *if it is not bad. Finally, we say that p is* very good *for U_q if it is a good and no irredicuible component of the root system is of type A_{mp-1} for some integer $m \geq 1$.*

Corollary 3.37. *Let $\lambda \in T_P^k$ and assume that p is a very good prime. Then $\mathcal{D}^\lambda_k$ is Γ-acyclic and $\Gamma(\mathcal{D}^\lambda_k) \cong U(\mathfrak{g}_k)_{\chi_\lambda}$, where χ_λ is the corresponding character of the Harish-Chandra centre of $U(\mathfrak{g}_k)$.*

Proof. This follows from the Proposition and [12, Proposition 3.4.1] (this is where the restrictions on p are required). ☐

3.14. Twists of Coherent $\mathcal{D}$-Modules

Observe that for $\mu \in T_P^R$ and $M \in \mathscr{D}^\lambda$, the left $\mathcal{D}$-action on $M(\mu)$ makes $M(\mu)$ into an element of $\mathscr{D}^{\lambda+\mu}$. We investigate those twists.

Proposition 3.38. *Let $\mu \in P^+$ and $\lambda \in T_P^k$. Assume that p is a good prime. Then*

$$R^i\Gamma(\mathcal{D}^\lambda(\mu)_k) = 0$$

for all $i > 0$.

Proof. We have $\mathcal{D}^\lambda(\mu)_k = p^*((\mathcal{M}_\lambda)_k \otimes_k k_{-\mu})$. Using [27, II.4.1.(2)], we see that the corresponding G_k-equivariant sheaf on the flag variety X_k is $\mathcal{D}^\lambda_{X_k} \otimes_{\mathcal{O}_{X_k}} \mathcal{L}(\mu)$. Thus we are reduced to showing that

$$H^i\left(X_k, \mathcal{D}^\lambda_{X_k} \otimes_{\mathcal{O}_{X_k}} \mathcal{L}(\mu)\right) = 0 \tag{3.2}$$

for all $i > 0$.

Now consider the filtration on $\mathcal{D}^\lambda_{X_k}$ by degree of differential operators, which naturally induces a filtration on $\mathcal{D}^\lambda_{X_k} \otimes_{\mathcal{O}_{X_k}} \mathcal{L}(\mu)$ by

$$F_i\left(\mathcal{D}^\lambda_{X_k} \otimes_{\mathcal{O}_{X_k}} \mathcal{L}(\mu)\right) = F_i(\mathcal{D}^\lambda_{X_k}) \otimes_{\mathcal{O}_{X_k}} \mathcal{L}(\mu).$$

Let $\tau : T^*X_k \to X_k$ be the cotangent bundle. Since $\mathcal{L}(\mu)$ is locally free, the corresponding associated graded is

$$\mathrm{gr}\left(\mathcal{D}^\lambda_{X_k} \otimes_{\mathcal{O}_{X_k}} \mathcal{L}(\mu)\right) \cong \mathrm{gr}(\mathcal{D}^\lambda_{X_k}) \otimes_{\mathcal{O}_{X_k}} \mathcal{L}(\mu) \cong \tau_*\mathcal{O}_{T^*X_k} \otimes_{\mathcal{O}_{X_k}} \mathcal{L}(\mu).$$

Next, by [22, 0.5.4.10],

$$\tau_*\mathcal{O}_{T^*X_k} \otimes_{\mathcal{O}_{X_k}} \mathcal{L}(\mu) \cong \tau_*\left(\mathcal{O}_{T^*X_k} \otimes_{\mathcal{O}_{T^*X_k}} \tau^*\mathcal{L}(\mu)\right) \cong \tau_*\tau^*\mathcal{L}(\mu).$$

Moreover, because τ is an affine morphism, we get from [23, Cor. I.3.3] that

$$H^i\left(X_k, \tau_*\tau^*\mathcal{L}(\mu)\right) \cong H^i\left(T^*X_k, \tau^*\mathcal{L}(\mu)\right)$$

for all $i \geq 0$. Finally, under the assumption that p is a good prime, it was shown in [32, Theorem 2] that $H^i\left(T^*X_k, \tau^*\mathcal{L}(\mu)\right) = 0$ for all $i > 0$.

Putting everything together, we have obtained that $\mathrm{gr}\left(\mathcal{D}^\lambda_{X_k} \otimes_{\mathcal{O}_{X_k}} \mathcal{L}(\mu)\right)$ is Γ-acyclic. Now, since X_k is Noetherian, cohomology commutes with direct limits by [24, Proposition III.2.9], and so each homogeneous component $\mathrm{gr}_i\left(\mathcal{D}^\lambda_{X_k} \otimes_{\mathcal{O}_{X_k}} \mathcal{L}(\mu)\right)$ is Γ-acyclic, and hence each filtered piece $F_i\left(\mathcal{D}^\lambda_{X_k} \otimes_{\mathcal{O}_{X_k}} \mathcal{L}(\mu)\right)$ is Γ-acyclic as well. Therefore (3.2) holds as required since $\mathcal{D}^\lambda_{X_k} \otimes_{\mathcal{O}_{X_k}} \mathcal{L}(\mu)$ is the direct limit of the $F_i\left(\mathcal{D}^\lambda_{X_k} \otimes_{\mathcal{O}_{X_k}} \mathcal{L}(\mu)\right)$. ☐

Corollary 3.39. *Assume p is a good prime. Then for $\mu \in P^+$ and for any $n \geq 1$, we have that $\mathcal{D}^\lambda(\mu)/\pi^n\mathcal{D}^\lambda(\mu)$ is Γ-acyclic.*

Proof. We proceed by induction on n. The case $n = 1$ is just the previous Proposition. Now for $n \geq 1$, we have a short exact sequence

$$0 \to \mathcal{D}^\lambda(\mu)/\pi\mathcal{D}^\lambda(\mu) \to \mathcal{D}^\lambda(\mu)/\pi^{n+1}\mathcal{D}^\lambda(\mu) \to \mathcal{D}^\lambda(\mu)/\pi^n\mathcal{D}^\lambda(\mu) \to 0$$

where by the Proposition and by induction hypothesis, the two side terms are acyclic. Hence by the long exact sequence the middle term is acyclic. ☐

As a consequence of this we can obtain a D-modules version of Proposition 3.19 which will be useful to us later. We first need a lemma:

Lemma 3.40. *Let $M \in \text{coh}(\mathscr{D}^\lambda)$. Then there is an $\mathcal{A}_q$-submodule N of M such that $N \in \text{coh}(\mathscr{C}_R)$ and N generates M as a $\mathcal{D}$-module.*

Proof. Let $m_1, \ldots, m_n$ be a generating set for M as a $\mathcal{D}$-module. Viewing M as an object of $\mathscr{C}_R$, we simply let N be the smallest coherent submodule of M containing $m_1, \ldots, m_n$, as given by Lemma 3.20. □

Theorem 3.41. *Let $M \in \text{coh}(\mathscr{D}^\lambda)$. Then $M(\mu)$ is generated by finitely many global sections for $\mu >> 0$. Moreover, if $\pi M = 0$ and if p is a good prime, then $M(\mu)$ is also Γ-acyclic for $\mu >> 0$.*

Proof. Let $N \in \text{coh}(\mathscr{C}_R)$ be as in the previous lemma. Note that $M(\mu) \in \text{coh}(\mathscr{D}_n^{\lambda+\mu})$ for any μ. By Proposition 3.19 we see that $N(\mu)$ is generated by finitely many global sections for $\mu >> 0$. Since M is generated by N as a $\mathcal{D}$-module, the first claim follows.

Now assume $\pi M = 0$. Fix any μ_1 such that $M(\mu_1)$ is generated by its global sections. Then we have a surjection $(\mathcal{D}^{\lambda+\mu_1})^a \to M(\mu_1)$ which in fact factors through a surjection $f_1 : (\mathcal{D}_k^{\lambda+\mu_1})^a \to M(\mu_1)$. Let $K = \ker f_1$. Note that $K \in \text{coh}(\mathscr{D}^{\lambda+\mu_1})$ by Proposition 3.30 and that $\pi K = 0$. So by the above argument applied to K, we can find $\mu_2 >> 0$ and a surjection $f_2 : (\mathcal{D}_k^{\lambda+\mu_1+\mu_2})^b \to K(\mu_2)$. Carrying on we obtain $\mu_1, \ldots, \mu_N \in P^+$ and a resolution in $\text{coh}(\mathscr{D}^{\lambda+\mu})$

$$F_N \xrightarrow{f_N} \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} M(\mu) \to 0$$

where $\mu = \sum_{j=1}^N \mu_i$ and for $1 \leq i \leq N$, F_i is a direct sum of finitely many copies of modules of the form $\mathcal{D}_k^{\lambda+\mu_1+\ldots+\mu_i}(\mu_{i+1} + \ldots + \mu_N)$. Note that all the F_i are Γ-acyclic by Proposition 3.38. Write $K_0 = M(\mu)$ and $K_i = \ker f_i$ for $1 \leq i \leq N$. Then for each $1 \leq i \leq N$ we have a short exact sequence

$$0 \to K_i \to F_i \to K_{i-1} \to 0$$

Since F_i is acyclic the long exact sequence implies that $R^j\Gamma(K_{i-1}) \cong R^{j+1}\Gamma(K_i)$ for all $j \geq 1$. Thus we obtain

$$R^j\Gamma(M(\mu)) \cong R^{j+1}\Gamma(K_1) \cong R^{j+2}\Gamma(K_2) \cong \ldots \cong R^{j+N}\Gamma(K_N) = 0$$

for any $j \geq 1$ as required. □

Remark 3.42. *We expect the above result to hold for all modules, not just for those killed by π.*

4. BANACH $\widehat{\mathcal{O}_q(B)}$-COMODULES

In this Section, we define various categories of comodules over certain π-adically complete or Banach coalgebras. In doing so, we will often use techniques to do with topologies on tensor products, and so we begin by establishing the necessary facts on this topic.

4.1. Completed Tensor Products and Banach Hopf Algebras

Recall from [42, Section 17B] that given two seminorms p and p' on the vector spaces V and W respectively, the *tensor product seminorm* $p \otimes p'$ on $V \otimes_L W$ is defined in the following way: for $x \in V \otimes_L W$, we have

$$p \otimes p'(x) := \inf \Big\{ \max_{1 \leq i \leq r} p(v_i) \cdot p'(w_i) : x = \sum_{i=1}^r v_i \otimes w_i, v_i \in V, w_i \in W \Big\}.$$

If V and W are locally convex spaces, then we will always only consider the projective tensor topology on $V \otimes_L W$, i.e the topology obtained via these tensor product seminorms. One can then construct the Hausdorff completion $V \widehat{\otimes}_L W$ of this space, which we call the completed tensor product of V and W. Note that this construction is functorial, so that two continuous linear maps $f : V \to W$ and $g : X \to Y$ induce a continuous linear map $f \widehat{\otimes} g : V \widehat{\otimes}_L X \to W \widehat{\otimes}_L Y$. In general, if V and W are Hausdorff, so is

$V \otimes_L W$. When V and W are Banach spaces, so is $V \widehat{\otimes}_L W$, and $\widehat{\otimes}_L$ is a monoidal structure on the category of L-Banach spaces.

Given an L-vector space V and an R-lattice $V^\circ \subset V$, we may define a norm on V called the *gauge norm*, given by

$$||v||_{\text{gauge}} = \inf_{\substack{a \in L \\ v \in aV^\circ}} |a| .$$

This infimum simply equals $|\pi^n|$ where $n \in \mathbb{Z}$ is the largest integer such that $v \in \pi^n V^\circ$, hence the topology induced by the gauge norm is the topology induced by the π-adic filtration on V°. Recall then that if V is a normed L-vector space, then its norm is equivalent to the gauge norm associated to the unit ball V°, see [42, Lemma 2.2]. Hence, without loss of generality, we will always assume that our normed vector spaces are equipped with the π-adic norm induced from their unit balls. Moreover, recall that given two normed L-vector spaces V and W with unit balls V° and W°, the unit ball of $V \otimes_L W$ equipped with the tensor product norm as above is $V^\circ \otimes_R W^\circ$, see [13, Lemma 2.2]. This is a fact we will often use without further mention.

Recall that a bounded linear map $f : X \to Y$ between two L-locally convex spaces is called *strict* if it induces a topological isomorphism

$$\hat{f} : X/\ker f \to \operatorname{Im} f.$$

The following result says that strict maps behave well under tensor products:

Lemma 4.1. *Suppose that V is a vector subspace of a locally convex space W equipped with the subspace topology and let U be any other locally convex space. Then*

1. *the canonical maps $V \otimes_L U \to W \otimes_L U$ and $U \otimes_L V \to U \otimes_L W$ are strict embeddings where we give the left hand side the tensor product topology;*
2. *the canonical maps $V \widehat{\otimes}_L U \to W \widehat{\otimes}_L U$ and $U \widehat{\otimes}_L V \to U \widehat{\otimes}_L W$ are strict embeddings;*
3. *the functor of taking tensor product with U, in the category of locally convex spaces, preserves strict surjections.*

Proof. (i) The map $V \otimes_L U \to W \otimes_L U$ is clearly injective and by [42, Proposition 17.4.iii] we see that it is isometric, hence an isomorphism onto its image.

(ii) This follows from (i) and [14, 1.1.9 Cor 6].

(iii) This follows immediately from the proof of [10, Appendix A, Lemma A.34]. ☐

Remark 4.2. *From now on, given any U, V, W as in the Lemma, we shall not distinguish between $V \widehat{\otimes}_L U$ and the subspace of $W \widehat{\otimes}_L U$ isomorphic to it.*

Next we turn to Banach coalgebras and Hopf algebras.

Definition 4.3. *An L-*Banach coalgebra *is a coalgebra object in the monoidal category of L-Banach spaces. In other words it is a Banach space C equipped with continuous linear maps $\Delta : C \to C \widehat{\otimes}_L C$ and $\varepsilon : C \to L$ which satisfy the usual axioms:*

$$(\Delta \widehat{\otimes}\, id_C) \circ \Delta = (id_C \widehat{\otimes} \Delta) \circ \Delta, \quad (id_C \widehat{\otimes} \varepsilon) \circ \Delta = (\varepsilon \widehat{\otimes}\, id_C) \circ \Delta = id_C .$$

A morphism of coalgebras $f : C \to D$ is a continuous linear map such that $\varepsilon_D \circ f = \varepsilon_C$ and $(f \widehat{\otimes} f) \circ \Delta_C = \Delta_D \circ f$.

Given a Banach coalgebra C as above, a Banach C-comodule *is a Banach space $\mathcal{M}$ equipped with a continuous linear map $\rho_{\mathcal{M}} : M \to \mathcal{M} \widehat{\otimes}_L C$, which satisfies:*

$$(id_{\mathcal{M}} \widehat{\otimes} \Delta) \circ \rho_{\mathcal{M}} = (\rho_{\mathcal{M}} \widehat{\otimes}\, id_C) \circ \rho_{\mathcal{M}}, \quad (id_{\mathcal{M}} \widehat{\otimes} \varepsilon) \circ \rho_{\mathcal{M}} = id_{\mathcal{M}} .$$

A morphism of comodules $f : \mathcal{M} \to \mathcal{N}$ is then a continous linear map such that $\rho_{\mathcal{N}} \circ f = \rho_{\mathcal{M}} \circ (f \widehat{\otimes}\, id_C)$. We denote by $\mathbf{Comod}(C)$ the category of Banach C-comodules.

An L-Banach Hopf algebra *is an* L*-Banach algebra* H *which is also a coalgebra such that* Δ *and* ε *are algebra homomorphisms, and furthemore* H *is equipped with a continuous linear map* $S : H \to H$*, which satisfies*

$$m \circ (S \widehat{\otimes}\, id_H) \circ \Delta = \iota \circ \varepsilon = m \circ (id_H \widehat{\otimes} S) \circ \Delta$$

where $m : H \widehat{\otimes}_L H \to H$ *and* $\iota : L \to H$ *denote the multiplication map and the unit in* H *respectively. A morphism of Hopf algebras* $f : H \to S$ *is then a continuous algebra homomorphism which is also a morphism of coalgebras, such that* $S_D \circ f = f \circ S_H$*.*

We showed in [18, Proposition 3.3] that for any two R-modules M and N, there is a canonical isomorphism of Banach spaces $\widehat{M_L} \widehat{\otimes}_L \widehat{N}_L \cong (\widehat{M \otimes_R N})_L$. In particular this implies that if H is an R-Hopf algebra, then $\widehat{H}_L$ is a Banach Hopf algebra. Indeed, the maps ε and S extend to the completion, and the comultiplication gives rise to a map $\widehat{\Delta} : \widehat{H}_L \to (\widehat{H \otimes_R H})_L \cong \widehat{H}_L \widehat{\otimes}_L \widehat{H}_L$. These satisfy the Hopf algebra axioms since they do on the dense subset H_L. So in particular we see that $\widehat{\mathcal{O}_q} := \widehat{\mathcal{A}_q} \otimes_R L$ and $\widehat{\mathcal{O}_q(B)} := \widehat{\mathcal{B}_q} \otimes_R L$ are Banach Hopf algebras.

4.2. $\widehat{\mathcal{B}_q}$-Comodules

We now define a suitable version of comodules over $\widehat{\mathcal{B}_q}$.

Notation 1. *Given two* R*-modules* M *and* N*, we write* $M \widehat{\otimes}_R N$ *to denote the* π*-adic completion* $\widehat{M \otimes_R N}$ *of* $M \otimes_R N$*. This construction satisfies the usual associativity and additivity properties of tensor products, and is functorial.*

Definition 4.4. *A* $\widehat{\mathcal{B}_q}$*-comodule is a* π*-adically complete* R*-module* $\mathcal{M}$ *equipped with a map* $\rho : \mathcal{M} \to \mathcal{M} \widehat{\otimes}_R \mathcal{B}_q$ *such that*

$$(\rho \widehat{\otimes} 1) \circ \rho = (1 \widehat{\otimes} \Delta) \circ \rho, \quad \text{and} \quad (1 \widehat{\otimes} \varepsilon) \circ \rho = 1_{\mathcal{M}}.$$

A morphism of $\widehat{\mathcal{B}_q}$*-comodules is an* R*-module map* $f : \mathcal{M} \to \mathcal{N}$ *such that* $(f \widehat{\otimes} 1) \circ \rho_{\mathcal{M}} = \rho_{\mathcal{N}} \circ f$*. We denote the set of comodule morphisms* $\mathcal{M} \to \mathcal{N}$ *by* $Hom_{\widehat{\mathcal{B}_q}}(\mathcal{M}, \mathcal{N})$*.*

Lemma 4.5. *Suppose that* $\mathcal{M}$ *is a* $\widehat{\mathcal{B}_q}$*-comodule. Then* $\mathcal{M}/\pi^n\mathcal{M}$ *is a* $\mathcal{B}_q$*-comodule for every* $n \geq 1$*. Hence* $\mathcal{M}$ *is a* $\widehat{U^{res}(\mathfrak{b})}$*-module and, moreover, if* ρ_n *denotes the* $\mathcal{B}_q$*-comodule map on* $\mathcal{M}/\pi^n\mathcal{M}$ *and* ρ *denotes the* $\widehat{\mathcal{B}_q}$*-comodule map on* $\mathcal{M}$*, then* $\rho = \varprojlim \rho_n$*.*

Proof. There are isomorphisms

$$(\mathcal{M} \widehat{\otimes}_R \mathcal{B}_q)/\pi^n(\mathcal{M} \widehat{\otimes}_R \mathcal{B}_q) \cong (\mathcal{M} \otimes_R \mathcal{B}_q)/\pi^n(\mathcal{M} \otimes_R \mathcal{B}_q) \cong (\mathcal{M}/\pi^n\mathcal{M}) \otimes_R \mathcal{B}_q,$$

and hence $\rho : \mathcal{M} \to \mathcal{M} \widehat{\otimes}_R \mathcal{B}_q$ induces a map

$$\rho_n : \mathcal{M}/\pi^n\mathcal{M} \to (\mathcal{M}/\pi^n\mathcal{M}) \otimes_R \mathcal{B}_q$$

for every $n \geq 1$. The comodule axioms are satisfied since they are obtained by reducing the equalities

$$(\rho \widehat{\otimes} 1) \circ \rho = (1 \widehat{\otimes} \Delta) \circ \rho, \quad \text{and} \quad (1 \widehat{\otimes} \varepsilon) \circ \rho = 1_{\mathcal{M}}$$

modulo π^n. Hence $\mathcal{M}/\pi^n\mathcal{M}$ is a $U^{\mathrm{res}}(\mathfrak{b})$-module and even a $U^{\mathrm{res}}(\mathfrak{b})/\pi^n U^{\mathrm{res}}(\mathfrak{b})$-module, and the structures are compatible with the maps $\mathcal{M}/\pi^{n+1}\mathcal{M} \to \mathcal{M}/\pi^n\mathcal{M}$. Taking inverse limits we see that $\mathcal{M}$ is a $\widehat{U^{\mathrm{res}}(\mathfrak{b})}$-module. The last part is immediate since $\varprojlim \rho_n = \widehat{\rho} = \rho$ as $\mathcal{M}$ is π-adically complete. □

Corollary 4.6. *For any two* $\widehat{\mathcal{B}_q}$*-comodules* $\mathcal{M}$ *and* $\mathcal{N}$*, there is a canonical isomorphism* $\mathrm{Hom}_{\widehat{\mathcal{B}_q}}(\mathcal{M}, \mathcal{N}) \cong \varprojlim \mathrm{Hom}_{\mathcal{B}_q}(\mathcal{M}/\pi^n\mathcal{M}, \mathcal{N}/\pi^n\mathcal{N})$*. Moreover every* $\widehat{\mathcal{B}_q}$*-comodule homomorphism is* $\widehat{U^{res}(\mathfrak{b})}$*-linear.*

Proof. Given a $\widehat{\mathcal{B}_q}$-comodule homomorphism $f : \mathcal{M} \to \mathcal{N}$, the induced map $f_n : \mathcal{M}/\pi^n\mathcal{M} \to \mathcal{N}/\pi^n\mathcal{N}$ is a $\mathcal{B}_q$-comodule map for every $n \geq 1$: since f is a comodule homomorphism we have that $\rho_{\mathcal{N}} \circ f = \rho_{\mathcal{M}} \circ (f\widehat{\otimes}1)$, which gives that f_n is a comodule homomorphism by reducing modulo π^n. Moreover the maps f_n uniquely determine f since $f = \varprojlim f_n$. Hence this implies that f is $\widehat{U^{\mathrm{res}}(\mathfrak{b})}$-linear since the maps f_n are all $U^{\mathrm{res}}(\mathfrak{b})$-linear. All this defines an injective map

$$\mathrm{Hom}_{\widehat{\mathcal{B}_q}}(\mathcal{M},\mathcal{N}) \to \varprojlim \mathrm{Hom}_{\mathcal{B}_q}(\mathcal{M}/\pi^n\mathcal{M},\mathcal{N}/\pi^n\mathcal{N})$$

and we need to check that it is surjective. But given an inverse system of maps $f_n : \mathcal{M}/\pi^n\mathcal{M} \to \mathcal{N}/\pi^n\mathcal{N}$, passing to the inverse limit gives rise to a map $f : \mathcal{M} \to \mathcal{N}$ which is a comodule homomorphism since the axioms are satisfied modulo π^n for every $n \geq 1$. □

4.3. Topologically Integrable $\widehat{U^{res}(\mathfrak{b})}$-Module

We now start preparing for an equivalent notion to the notion of $\widehat{\mathcal{B}_q}$-comodules. Note that by Lemma 4.5, if $\mathcal{M}$ is a $\widehat{\mathcal{B}_q}$-comodule, then $\mathcal{M}/\pi^n\mathcal{M}$ is an integrable $U^{\mathrm{res}}(\mathfrak{b})$-module. We want an analogous notion of integrable modules at this π-adically complete level.

Definition 4.7. *Let $\mathcal{M}$ be a π-adically complete $\widehat{U^{res}(\mathfrak{b})}$-module. Given $\lambda \in P$, we define the λ-weight space $\mathcal{M}_\lambda$ to be the corresponding weight space of $\mathcal{M}$ viewing it as a $U^{res}(\mathfrak{b})$-module. We say that $\mathcal{M}$ is* topologically integrable *as a $\widehat{U^{res}(\mathfrak{b})}$-module if:*

1. $\mathcal{M}$ *is* topologically $(U^{\mathrm{res}})^0$-semisimple, *i.e for every $m \in M$ there exists a family $(m_\lambda)_{\lambda\in P}$ such that $m_\lambda \in \mathcal{M}_\lambda$ and $\sum_{\lambda\in P} m_\lambda$ converges to m in M; and*
2. *for every i the action of E_{α_i} on M is* locally topologically nilpotent, *i.e for every $m \in \mathcal{M}_\lambda$ the sequence $E_{\alpha_i}^{(r)} \cdot m \to 0$ as $r \to \infty$.*

Proposition 4.8. *Let M be a $U^{res}(\mathfrak{b})$-module and let $\mathcal{M}$ be a $\widehat{U^{res}(\mathfrak{b})}$-module. Then:*

1. *if $\mathcal{M}$ is topologically integrable, then it has a canonical $\widehat{\mathcal{B}_q}$-comodule structure; and*
2. *if M is integrable, then $\widehat{M}$ is a topologically integrable $\widehat{U^{res}(\mathfrak{b})}$-module.*

Proof. For 1., note that it follows immediately from the definition of topologically integrable $\widehat{U^{\mathrm{res}}(\mathfrak{b})}$-module that $\mathcal{M}/\pi^n\mathcal{M}$ is integrable as a $U^{\mathrm{res}}(\mathfrak{b})$-module for every $n \geq 1$. So there are comodule maps

$$\rho_n : \mathcal{M}/\pi^n\mathcal{M} \to \mathcal{M}/\pi^n\mathcal{M} \otimes_R \mathcal{B}_q \cong (\mathcal{M} \otimes_R \mathcal{B}_q)/\pi^n(\mathcal{M} \otimes_R \mathcal{B}_q)$$

for every $n \geq 1$, which are compatible with the maps $\mathcal{M}/\pi^{a+1}\mathcal{M} \to \mathcal{M}/\pi^a\mathcal{M}$. Taking inverse limits gives a map

$$\rho : \mathcal{M} \to \mathcal{M}\widehat{\otimes}_R\mathcal{B}_q$$

which gives a comodule structure to $\mathcal{M}$: the comodule axioms hold modulo π^n for every $n \geq 1$ so hold for ρ. The module structure arising from ρ agrees by definition with the initial module structure on $\mathcal{M}$.

For 2., let $m \in \widehat{M}$. Then there exists $m_0, m_1, \ldots$ in M such that

$$m = \sum_{i=0}^{\infty} \pi^i m_i.$$

Now, as M is integrable, we can find ascending chain of finite subsets $S_j \subseteq P$ such that $m_j = \sum_{\lambda\in S_j} m_{j,\lambda}$ for some $m_{j,\lambda} \in M_\lambda$. Let $S = \bigcup_{j\geq 0} S_j$. For each $\lambda \in S$, let

$$n(\lambda) = \inf\{j : \lambda \in S_j\}.$$

Then set

$$m_\lambda = \sum_{j \geq n(\lambda)} \pi^j m_{j,\lambda} \in \pi^{n(\lambda)} \widehat{M}_\lambda.$$

Since each set S_j is finite, each set $\{\lambda : n(\lambda) < j\}$ is also finite and so $\sum_{\lambda \in S} m_\lambda$ converges to m.

Finally, write $m = \sum_{j\geq 0} \pi^j m_j$ again, and pick $N \in \mathbb{N}$. Since M is integrable, for every $0 \leq j < N$, $E_{\alpha_i}^{(r)} m_j = 0$ for $r >> 0$. So there exists $R > 0$ such that for all $r > R$ and for any $0 \leq j < N$, $E_{\alpha_i}^{(r)} m_j = 0$. Then we have

$$E_{\alpha_i}^{(r)} m = \sum_{j \geq N} \pi^j E_{\alpha_i}^{(r)} m_j \in \pi^N \widehat{M}$$

for $r > R$. So $E_{\alpha_i}^{(r)} m \to 0$ as $r \to \infty$ as required. □

We aim to show a converse to Proposition 4.8 (1). Similarly to the uncompleted situation, it boils down to showing that closed submodules of topologically integrable modules are topologically integrable. We are only able to do this for torsion-free modules, but this is sufficient for our needs.

Definition 4.9. *A Banach $\widehat{U^{res}(\mathfrak{b})}_L$-module $\mathcal{M}$ is called* topologically integrable *if its unit ball $\mathcal{M}^\circ$ is a topologically integrable $\widehat{U^{res}(\mathfrak{b})}$-module.*

Note that topologically integrable $\widehat{U^{\mathrm{res}}(\mathfrak{b})}_L$-modules are automatically topologically semisimple as Banach $(\widehat{U^{\mathrm{res}}})^0_L$-modules, in the sense of [18, Section 5.1]. Thus we have:

Theorem 4.10 ([18, Theorem 5.1])**.** *Suppose that $\mathcal{M}$ is a topologically integrable $\widehat{U^{res}(\mathfrak{b})}_L$-module. Then for each $m \in \mathcal{M}$, there exists a unique family $(m_\lambda)_{\lambda \in P}$ with $m_\lambda \in \mathcal{M}_\lambda$ such that $\sum_{\lambda \in P} m_\lambda$ converges to m. Moreover, if $m \in \mathcal{N}$ where $\mathcal{N}$ is a closed U_q^0-invariant subspace, then each $m_\lambda \in \mathcal{N}$.*

4.4. An Equivalence of Categories

We now use above results to obtain a description of the category of Banach $\widehat{\mathcal{O}_q(B)}$-comodules.

Proposition 4.11. *Let $\mathcal{M}$ be a π-torsion free $\widehat{\mathcal{B}_q}$-comodule. Then $\mathcal{M}$ is a topologically integrable $\widehat{U^{res}(\mathfrak{b})}$-module.*

Proof. We have the comodule map $\mathcal{M} \to \mathcal{M}\widehat{\otimes}_R \mathcal{B}_q$ which is a split injection. As it is split, we must have

$$\pi^n(\mathcal{M}\widehat{\otimes}_R \mathcal{B}_q) \cap \rho(\mathcal{M}) = \pi^n \rho(\mathcal{M}) = \rho(\pi^n \mathcal{M})$$

so that ρ is in fact an isometry with respect to the π-adic norms. Moreover, ρ is a comodule homomorphism if we give $\mathcal{M}\widehat{\otimes}_R \mathcal{B}_q$ the comodule map $1\widehat{\otimes}\Delta$, c.f. Remark 2.6. Hence this gives rise to a $\widehat{U^{\mathrm{res}}(\mathfrak{b})}_L$-linear isometry $\mathcal{M}_L \to \mathcal{M}_L \widehat{\otimes}_L \widehat{\mathcal{O}_q(B)}$ by Corollary 4.6.

Note that $\mathcal{M}\widehat{\otimes}_R \mathcal{B}_q$ is the π-adic completion of $\mathcal{M} \otimes_R \mathcal{B}_q$, which is a $\mathcal{B}_q$-comodule via $1 \otimes \Delta$. Since $\mathcal{B}_q$-comodules are integrable $U^{\mathrm{res}}(\mathfrak{b})$-modules by Theorem 2.7, it follows from Proposition 4.8 (2) that $\mathcal{M}\widehat{\otimes}_R \mathcal{B}_q$ is topologically integrable, hence so is $\mathcal{M}_L \widehat{\otimes}_L \widehat{\mathcal{O}_q(B)}$. Now we identify $\mathcal{M}$ with its image in $\mathcal{M}\widehat{\otimes}_R \mathcal{B}_q = (\mathcal{M}_L \widehat{\otimes}_L \widehat{\mathcal{O}_q(B)})^\circ$. Since the map was an isometry we also have $\mathcal{M} = \mathcal{M}_L \cap \mathcal{M}\widehat{\otimes}_R \mathcal{B}_q$. Pick $m \in \mathcal{M}$. Then inside $\mathcal{M}_L \widehat{\otimes}_L \widehat{\mathcal{O}_q(B)}$ we automatically have $m = \sum_{\lambda \in P} m_\lambda$ and $E_{\alpha_i}^{(r)} m \to 0$ as $r \to \infty$. So we just need to check that each $m_\lambda \in \mathcal{M}$. However by Theorem 4.10, the m_λ are uniquely determined by m and must belong to $\mathcal{M}_L$ since it is complete hence closed in $\mathcal{M}_L \widehat{\otimes}_L \widehat{\mathcal{O}_q(B)}$. On the other hand, since $\mathcal{M}\widehat{\otimes}_R \mathcal{B}_q$ is topologically integrable and $m \in \mathcal{M} \subset \mathcal{M}\widehat{\otimes}_R \mathcal{B}_q$ we must have $m_\lambda \in \mathcal{M}\widehat{\otimes}_R \mathcal{B}_q$ for all λ. Therefore each $m_\lambda \in (\mathcal{M}\widehat{\otimes}_R \mathcal{B}_q) \cap \mathcal{M}_L = \mathcal{M}$ as required. □

Note that by Proposition 4.8 (1) there is a canonical functor between the category of topologically integrable $\widehat{U^{\mathrm{res}}(\mathfrak{b})}$-modules and $\widehat{\mathcal{B}_q}$-comodules. Indeed, given a module map $f : \mathcal{M} \to \mathcal{N}$ its restriction modulo π^n is a module map between two integrable $U^{\mathrm{res}}(\mathfrak{b})$-modules by definition, hence is a comodule homomorphism. Passing to the inverse limit, f is a $\widehat{\mathcal{B}_q}$-comodule homomorphism.

Corollary 4.12. *The canonical functor between the category of topologically integrable $\widehat{U^{res}(\mathfrak{b})}$-modules and the category of $\widehat{\mathcal{B}_q}$-comodules restricts to an equivalence of categories between the full subcategories of π-torsion free objects.*

Proof. By Proposition 4.11, the restriction of the functor to the torsion-free modules is essentially surjective. It is evidently faithful. Moreover, it is full by Corollary 4.6. □

If $\mathcal{M}$ is a topologically integrable $\widehat{U^{\mathrm{res}}(\mathfrak{b})}_L$-module then we may apply the above functor to its unit ball and extend scalars to construct a functor to $\mathbf{Comod}(\widehat{\mathcal{O}_q(B)})$. This gives our promised Theorem A:

Proof of Theorem A. By the proof of the above Corollary, this functor is full and faithful so that we just need to show that it is essentially surjective. Now suppose that $\mathcal{N}$ is a Banach $\widehat{\mathcal{O}_q(B)}$-comodule. Then there is a split injection $\rho : \mathcal{N} \to \mathcal{N}\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)}$ which is therefore strict by the Lemma below. Moreover ρ is a comodule homomorphism where we give the right hand side the comodule map $1\widehat{\otimes}\widehat{\Delta}$. Hence $\mathcal{N}$ is topologically isomorphic to a subcomodule $\mathcal{M}$ of $\mathcal{N}\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)}$, where $\mathcal{M}$ is equipped with the subspace topology. We note that since $1\widehat{\otimes}\widehat{\Delta}$ has norm ≤ 1, so does its restriction to $\mathcal{M}$, and so it preserves unit balls. Thus we see that $\mathcal{M}^\circ$ is a $\widehat{\mathcal{B}_q}$-comodule, and therefore is a topologically integrable $\widehat{U^{\mathrm{res}}(\mathfrak{b})}$-module by Proposition 4.11. So we have that $\mathcal{M}$ is in the image of our functor. □

Lemma 4.13. *If X and Y are two L-Banach spaces and $f : X \to Y$ is a split continuous linear map, then f is strict.*

Proof. Suppose the splitting is given by $g : Y \to X$. Then we have

$$||x||_X = ||g(f(x))||_X \leq ||g||\,||f(x)||_Y$$

for all $x \in X$. This implies that f is strict by [14, Lemma 1.1.9/2]. □

Appendix

APPENDIX A. HOPF DUALS OF R-HOPF ALGEBRAS AND THEIR COMODULES

We wish to establish some duality facts to do with Hopf algebras over R which are well known when working over fields but for which we couldn't find references for Hopf algebras over more general commutative rings. Most of our proofs work using the usual arguments but one has to be a bit careful when dealing with torsion.

1. Hopf Duals Over R

For the entirety of this Section, H will denote a fixed Hopf R-algebra. For our purposes, it will be enough to work in the case where H is *torsion-free*. First we wish to define a notion of Hopf dual. Since H has no torsion, it embeds as a sub-Hopf algebra of $H_L = H \otimes_R L$. We will define the Hopf dual to be a sub-Hopf algebra of $(H_L)^\circ$. Let $\mathfrak{J}$ denote the set of ideal I in H such that H/I is a finitely generated R-module. Moreover, denote by $\mathfrak{J}'$ the set of ideals I in H such that H/I is free of finite rank. Finally, write H^* for $\mathrm{Hom}_R(H, R)$. Note that H^* is always torsion-free since R is a domain: if $\pi f = 0$ then $\pi f(u) = 0$ for all $u \in H$ and so $f(u) = 0$ for all u.

Definition A.1. *We define the Hopf dual of H to be*

$$H^\circ := \{f \in H^* : f|_I = 0 \text{ for some } I \in \mathfrak{J}\}.$$

By the above H° is torsion-free.

If $n \geq 0$ and $x \in H$ we have for any $f \in H^*$ that $f(x) = 0$ if and only if $f(\pi^n x) = 0$. Thus if $0 \neq f \in H^\circ$ then $f|_I = 0$ for some $I \in \mathfrak{J}$ where H/I is not torsion. Moreover we then have $f|_{I_L \cap H} = 0$ and so by replacing I with $I_L \cap H$ we may in addition assume that H/I is torsion-free. Since R is a PID this shows that

$$H^\circ = \{f \in H^* : f|_I = 0 \text{ for some } I \in \mathfrak{J}'\}.$$

Moreover by extending scalars we may identify H° with an R-submodule of H_L°. From this it follows by the standard arguments that H° is the algebra of matrix coefficients of H-modules which are free of finite rank over R. Since this collection of H-modules is closed under taking tensor products, direct sums and duals, and we can take dual bases, we have proved

Lemma A.2. *H° is an sub-Hopf R-algebra of H_L°. In particular the algebra maps on H° are just the dual maps of the coalgebra maps on H and vice-versa.*

Remark A.3. *Some of the above arguments were implicit in Lusztig's work, see [34, 7.1].*

2. H°-Comodules as H-Modules

We now wish to establish some correspondence between comodules over H° and certain H-modules. We call an H-module M *locally finite* if for all $m \in M$, Hm is finitely generated over R.

Proposition A.4. *Every H°-comodule has a canonical structure of a locally finite H-module with respect to which every comodule homomorphism is an H-modules homomorphism. In other words there is a canonical faithful embedding of categories between the category of H°-comodules and the category of locally finite H-modules.*

Proof. This is just the usual argument. If M is an H°-comodule with coaction $\rho : M \to M \otimes_R H^\circ$, write $\rho(m) = \sum m_1 \otimes m_2$. Then we set

$$u \cdot m = \sum m_2(u) m_1$$

for all $u \in H$. It follows from the comodule axioms that this gives a well defined module structure, i.e that $1 \cdot m = m$ and that $u \cdot (u' \cdot m) = (uu') \cdot m$ for all $u, u' \in H$ and all $m \in M$. Moreover by definition of the module structure, $H \cdot m$ is finitely generated over R for all $m \in M$. Finally it follows from the definition of the action that any comodule homomorphism is also a module homomorphism. □

Next, we want to show that the functor we just defined is full, i.e that every H-module map between two H°-comodules is a comodule homomorphism. We first need a technical result. Suppose M is a locally finite H-module. Note that we have an R-module injection $\phi_M : M \to \mathrm{Hom}_R(H, M)$ given by $\phi_M(m)(u) = um$ for all $u \in H$ and $m \in M$. Moreover we have a map

$$\theta_M : M \otimes_R H^\circ \to \mathrm{Hom}_R(H, M)$$

given by $\theta_M(m \otimes f)(u) = f(u)m$. When the H-module structure on M arises from an H°-comodule structure then we have $\phi_M = \theta_M \circ \rho$. Therefore we can use this expression for ϕ_M as an alternative definition of the module structure on M. We claim that the map θ_M is injective. More generally we have the following

Lemma A.5. *Let A and B be R-modules, $A^* = Hom_R(A, R)$ and suppose C is any R-submodule of A^* such that A^*/C has no π-torsion. Let M be any R-module and set*

$$\theta_{M,C} : Hom_R(B, M) \otimes_R C \to Hom_R(A \otimes_R B, M)$$

to be defined by $\theta_{M,C}(g \otimes f)(x \otimes y) = f(x)g(y)$. Then the map $\theta_{M,C}$ is injective.

Proof. Suppose that $0 \neq u = \sum_{i=1}^{s} g_i \otimes f_i \in \mathrm{Hom}_R(B, M) \otimes_R C$. The R-submodule N of $\mathrm{Hom}_R(B, M)$ generated by the g_i is finitely generated, so since R is a PID we can pick a generating set $n_1, \ldots, n_l, t_1, \ldots, t_m$ for N such that $n_1, \ldots, n_l$ are torsion-free while $t_1, \ldots, t_m$ are π-torsion, and

$$N = \bigoplus_{i=1}^{l} Rn_i \oplus \bigoplus_{j=1}^{m} Rt_j.$$

For each $1 \leq j \leq m$, let a_j be the positive integer such that $Rt_j \cong R/\pi^{a_j}R$.

Now, to show that $\theta_{M,C}(u) \neq 0$ it suffices to show that the restriction of $\theta_{M,C}$ to the span of the $n_i \otimes f_j$ and $t_k \otimes f_j$ is injective. So suppose we are given

$$v = \sum r_{ij} n_i \otimes f_j + \sum r'_{kj} t_k \otimes f_j \in \ker \theta_{M,C}.$$

Evaluating at $x \otimes y$ we get $\sum_{i,j} r_{ij} f_j(x) n_i(y) + \sum_{k,j} r'_{kj} f_j(x) t_k(y) = 0$ for all $x \in A$ and $y \in B$. In particular we have $\sum_{i,j} r_{ij} f_j(x) n_i + \sum_{k,j} r'_{kj} f_j(x) t_k = 0$ for any fixed $x \in A$. Since we have a direct sum decomposition of N it follows that

$$\sum_j r_{ij} f_j(x) = 0 \quad \text{and} \quad \sum_j r'_{kj} f_j(x) \in \pi^{a_k} R$$

for all $x \in A$ and all $1 \leq i \leq l$ and $1 \leq k \leq m$. In particular, for all k, $\sum_j r'_{kj} f_j = \pi^{a_k} g_k$ for some $g_k \in C$ since A^*/C has no π-torsion.

Therefore we have

$$\sum_j r_{ij} f_j = 0 \quad \text{and} \quad \sum_j r'_{kj} f_j \in \pi^{a_k} C,$$

and hence

$$n_i \otimes \sum_j r_{ij} f_j = 0 = t_k \otimes \sum_j r'_{kj} f_j$$

for all i, k, and so $v = 0$ as required. □

Corollary A.6. *Let M be an R-module.*

1. *The map $\theta_M : M \otimes_R H^\circ \to Hom_R(H, M)$ is injective.*
2. *The map $M \otimes_R H^\circ \otimes_R H^\circ \to Hom_R(H \otimes_R H, M)$ sending $m \otimes f \otimes g$ to $x \otimes y \mapsto f(x)g(y)m$ is injective.*

Proof. Let $A = H$ and $C = H^\circ$. From the definition of H° it follows that A^*/C is torsion-free. Then (i) follows immediately from the Lemma by putting $B = R$. For (ii) note that this map is simply the composite

$$M \otimes_R H^\circ \otimes_R H^\circ \xrightarrow{\theta_M \otimes 1} \mathrm{Hom}_R(H, M) \otimes_R H^\circ \xrightarrow{\varpi} \mathrm{Hom}_R(H \otimes_R H, M)$$

where $\varpi(f \otimes g)(x \otimes y) = g(y)f(x)$. The map $\theta_M \otimes 1$ is injective by (i) and because H° is flat while the map ϖ is injective by putting $B = H$ in the Lemma. □

We can now deduce the result we were aiming for.

Proposition A.7. *The functor associating any H°-comodule to the corresponding H-module is a fully faithful embedding.*

Proof. From what we have done already we just need to show that any H-module map $f : M \to N$ between two H°-comodules is a comodule homomorphism. Write ρ_M and ρ_N for the coactions on M and N respectively, and pick $m \in M$ and $u \in H$. Then we know that $um = \sum m_2(u)m_1$ and we have $uf(m) = \sum m_2(u)f(m_1)$ since f is a module homomorphism. On the other hand by definition of the action on N we have $uf(m) = \sum f(m)_2(u)f(m)_1$. To show that f is a comodule map we need to show that

$$\sum f(m_1) \otimes m_2 = \sum f(m)_1 \otimes f(m)_2$$

or in other words that $\rho_N \circ f = (f \otimes 1) \circ \rho_M$.

Write $\tilde{\rho}_1 = \rho_N \circ f$ and $\tilde{\rho}_2 = (f \otimes 1) \circ \rho_M$. Moreover recall the map $\phi : M \to \mathrm{Hom}_R(H, M)$ given by $\phi(m)(u) = um$. Then let

$$\tilde{\phi} = \phi \circ f : M \to \mathrm{Hom}_R(H, N)$$

so that $\tilde{\phi}(m)(u) = uf(m)$. Then by definition $\tilde{\phi} = \theta_N \circ \tilde{\rho}_1$. On the other hand by our above observation we see that $\tilde{\phi} = \theta_N \circ \tilde{\rho}_2$. Since θ_N is injective the result follows. □

From now on, if M is a locally finite H-module we will say that it is an H°-comodule to mean that its H-module structure arises from an H°-comodule structure.

In order for the above functor to be an isomorphism of categories we therefore just need to show that it is surjective. This may not be true in general, however we can write a very simple necessary and sufficient condition for an isomorphism of categories to hold. Suppose M is a locally finite H-module and let $\phi_M : M \to \mathrm{Hom}_R(H, M)$ be given by $\phi_M(m)(x) = x \cdot m$. We have the map $\theta_M : M \otimes_R H^\circ$ as before.

Proposition A.8. *A locally finite H-module M is an H°-comodule if and only if $\phi_M(m)$ belongs to the image of θ_M for all $m \in M$.*

Proof. If M is a comodule with coaction ρ, then by our observation preceding Lemma A.5 we have $\phi_M = \theta_M \circ \rho$ where ϕ_M comes from the induced H-module structure, and the result is clear. Conversely assume $\phi_M(m)$ belongs to the image of θ_M for all $m \in M$. Fix $m \in M$. Then there exists $m_1, \ldots, m_n \in M$ and $f_1, \ldots, f_n \in H^\circ$ such that for all $x \in H$, $x \cdot m = \sum_{i=1}^n f_i(x)m_i$ and we define

$$\rho(m) = \sum_{i=1}^n m_i \otimes f_i,$$

i.e $\rho(m)$ is the unique element of $M \otimes_R H^\circ$ such that $\theta_M(\rho(m)) = \phi_M(m)$. We now have to check that this satisfies the comodule axioms. By definition, the counit on H° is defined by $\varepsilon(f) = f(1)$ and so

$$(1 \otimes \varepsilon) \circ \rho(m) = \sum_{i=1}^n f_i(1)m_i = 1 \cdot m = m$$

as required. Finally we aim to show that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\rho} & M \otimes_R H^\circ \\ \Big\downarrow{\scriptstyle \rho} & & \Big\downarrow{\scriptstyle 1\otimes\Delta} \\ M \otimes_R H^\circ & \xrightarrow{\rho\otimes 1} & M \otimes_R H^\circ \otimes_R H^\circ \end{array}$$

By Corollary A.6 (2), the natural map $M \otimes_R H^\circ \otimes_R H^\circ \to \mathrm{Hom}_R(H \otimes_R H, M)$ is injective. Hence it suffices to show that $(1 \otimes \Delta) \circ \rho(m)$ and $(\rho \otimes 1) \circ \rho(m)$ act in the same way on $H \otimes_R H$ for all $m \in M$. But the former sends $x \otimes y$ to $(xy) \cdot m$ while the latter sends $x \otimes y$ to $x \cdot (y \cdot m)$ for any $x, y \in H$, which are clearly equal. □

Since Lemma A.5 was quite general, the same argument as in the above proof shows the following

Lemma A.9. *Suppose M is a locally finite H-module and let C be a subcoalgebra of H° such that H^*/C is torsion-free. If $\phi_M(m)$ belongs to the image of $\theta_{M,C}$ for all $m \in M$ then M is a C-comodule.*

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