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RESEARCH ARTICLES

On the Solutions of Cauchy Problem for Two Classes of Semi-Linear Pseudo-Differential Equations over p-Adic Field*

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Abstract—Throughout this paper, using the p-adic wavelet basis together with the help of separation of variables and the Adomian decomposition method (as a scheme in numerical analysis) we initially investigate the solution of Cauchy problem for two classes of the first and second order of pseudo-differential equations involving the pseudo-differential operators such as Taibleson fractional operator in the setting of p-adic field.

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1. INTRODUCTION

From the beginning of 1960s, the theory of pseudo-differential operators has played a significant role in several stimulating and profound investigations into linear PDE. Since the 1980s, this tool has also provided various remarkable results in nonlinear PDE.

Treatments of pseudo-differential operators most frequently focus on operators with smooth coefficients, but it would be also good to employ these operators with symbols of minimal smoothness, which have many applications to diverse problems in PDE, from nonlinear problems to problems in non-smooth domains (see [5, 8, 40, 41] and the references therein).

One of the major role of pseudo-differential operator is its usage in p-adic analysis and the modeling of phenomena in the setting of p-adic field $\mathbb{Q}_p$. The theory of p-adic field $\mathbb{Q}_p$ (as the completion of the field $\mathbb{Q}$ with respect to specific norm which is called p-norm) has been initiated a few decades ago for its motivations in various fields such as string theory, cluster networks and buildings, models in geophysics, dynamic systems, Brownian motions in stochastic process, etc. (see for example [8, 20] and the references therein). The pseudo-differential equations over p-adic fields have been investigated in numerous results and can be found in the literature (see also [4, 10–12, 17, 19, 21–25, 27, 33, 36, 37, 42, 43, 46, 47]).

The *Taibleson fractional operator* is one of the most considerable operators which is obtained by taking a certain *symbol* in the definition of pseudo-differential operator.

In Sections 3 and 4, motivated by the results of Albeverio et al. [6] and using the mentioned operator we first deal with the following Cauchy problems, respectively

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + a\dfrac{\partial u}{\partial t} + bD_x^\alpha u + cu|u|^{2m} = f(x,t), \quad x \in \mathbb{Q}_p,\ a \neq 0,\ t > 0, \\ u(x,0) = \phi(x),\ u_t(x,0) = g(x),\ x \in \mathbb{Q}_p,\ t = 0 \end{cases} \tag{1.1}$$

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and

$$\begin{cases} \dfrac{\partial u}{\partial t} + aD_x^\alpha u + bu|u|^{2m} = f(x,t), & x \in \mathbb{Q}_p,\ t > 0, \\ u(x,0) = \phi(x), & x \in \mathbb{Q}_p,\ t = 0\,, \end{cases} \tag{1.2}$$

where $\alpha > 0$ and D_x^α is the Taibleson fractional operator introduced on the space of distributions [38].

In Section 5, replacing the Taibleson fractional operator D_x^α considered in problems as above by an arbitrary pseudo-differential operator A with a symbol $\mathcal{A}$ satisfying the equality (4) in [3] we conclude some results inspired by the proof of theorems relevant to the problems (1.1) and (1.2) in the vectorial case of p-adic fields $\mathbb{Q}_p^r$. Finally, some computations are presented in an appendix at the end of paper.

Remark 1.1. *We remind that a class of homogeneous semi-linear pseudo-differential equation with certain coefficients for the first-order case has been previously considered (see [6], Section 6) while our comparable problem* (1.2) *is non-homogeneous with arbitrary constant coefficients.*

2. PRELIMINARIES AND AUXILIARY FACTS

Let us recall here some basic facts needed in the sequel from [44]. Here and in what follows, we will indicate by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{C}$ the sets of natural, integer, rational and complex numbers, respectively, and by p the prime numbers, $p = 2, 3, 5, \ldots$. Also, $\mathbb{Q}_p$ stands for the field of p-adic numbers which is the completion of the field $\mathbb{Q}$ with the following p-adic norm $|\cdot|_p : |0|_p = 0$; if an arbitrary rational number $x \neq 0$ is represented as $x = p^\gamma \dfrac{m}{n}$ uniquely, where $\gamma = \gamma(x) \in \mathbb{Z}$ and m, n are not divisible by p then $|x|_p = p^{-\gamma}$. This norm satisfies the following properties:

(i) $|x|_p \geq 0$ for every $x \in \mathbb{Q}_p$, and $|x|_p = 0$ if and only if $x = 0$;

(ii) $|xy|_p = |x|_p |y|_p$ for every $x, y \in \mathbb{Q}_p$;

(iii) $|x + y|_p \leq \max\{|x|_p, |y|_p\}$, for every $x, y \in \mathbb{Q}_p$, and when $|x|_p \neq |y|_p$, we have $|x + y|_p = \max\{|x|_p, |y|_p\}$.

That is, the norm $|\cdot|_p$ is non-Archimedean and the space $(\mathbb{Q}_p, |\cdot|_p)$ is an ultrametric space.

Any p-adic number $x \in \mathbb{Q}_p$, $x \neq 0$, is represented in the canonical form

$$x = \sum_{j=\gamma}^{\infty} x_j p^j\,, \tag{2.1}$$

where $\gamma = \gamma(x) \in \mathbb{Z}$, and $x_k = 0, 1, \ldots, p-1,\ x_0 \neq 0,\ k = 0, 1, \ldots$ This series converges in the p-adic norm $|\cdot|_p$ to $p^{-\gamma}$. The fractional part of a p-adic number $x \in \mathbb{Q}_p$ defined by (2.1) is given by

$$\{x\}_p = \begin{cases} 0, & \text{if } \gamma(x) \geq 0 \text{ or } x = 0, \\ p^\gamma(x_0 + x_1 p + x_2 p^2 + \cdots + x_{|\gamma|-1} p^{|\gamma|-1}), & \text{if } \gamma(x) < 0. \end{cases} \tag{2.2}$$

The additive character χ_p of the field $\mathbb{Q}_p$ is defined by

$$\chi_p(x) = e^{2\pi i \{x\}_p}, \quad x \in \mathbb{Q}_p.$$

To construct the topology induced by $|\cdot|_p$ in $\mathbb{Q}_p$ we suppose that

$$B_\gamma(a) = \{x \in \mathbb{Q}_p : |x - a|_p \leq p^\gamma\}, \quad S_\gamma(a) = \{x \in \mathbb{Q}_p : |x - a|_p = p^\gamma\}$$

are ball and sphere of radius p^γ with center at a, respectively. For simplicity, we also let $B_\gamma(0) = B_\gamma$ and $S_\gamma(0) = S_\gamma$. We remark that any point of the ball is its center, moreover, any two balls in $\mathbb{Q}_p$ are either disjoint or one is contained in the other. Furthermore, all balls and spheres are open and closed sets in $\mathbb{Q}_p$.

The topological group $(\mathbb{Q}_p, +)$ is locally compact commutative and thus there is an additive Haar measure dx, which is positive and invariant under the translation, i.e., $d(x+a) = dx, a \in \mathbb{Q}_p$. This measure is unique by normalizing dx so that

$$\int_{B_0} dx = 1, \quad d(ax+b) = |a|_p dx,\ a \in \mathbb{Q}_p^* = \mathbb{Q}_p - \{0\}.$$

Regarding with the additive normalized character $\chi_p(x)$ on $\mathbb{Q}_p$ we get

$$\int_{B_\gamma} \chi_p(\xi x) dx = p^\gamma \Omega(p^\gamma |\xi|_p),$$

where $\Omega(t)$ is the characteristic function of the segment $[0,1] \subset \mathbb{R}$, that is,

$$\Omega(t) = \begin{cases} 1, & \text{if } t \in [0,1], \\ 0, & \text{if } t > 1. \end{cases}$$

A complex-valued function in $\mathbb{Q}_p$ is said to be a *locally constant function* if for any $x \in \mathbb{Q}_p$, there exists an integer $l(x) \in \mathbb{Z}$ such that $f(x+y) = f(x)$, for every $y \in B_{l(x)}$. We signify by $\mathcal{E}(\mathbb{Q}_p)$ the linear space of such functions in $\mathbb{Q}_p$. By $D(\mathbb{Q}_p)$ we mean the subspace of $\mathcal{E}(\mathbb{Q}_p)$ consisting of locally constant functions with compact support (so-called test function). Moreover, indicate by $D'(\mathbb{Q}_p)$ the set of all linear functionals on $D(\mathbb{Q}_p)$ (see also [44, VI.3]).

The Fourier transform of test function $\varphi \in D(\mathbb{Q}_p)$ is given by

$$\hat{\varphi}(\xi) = F[\varphi](\xi) = \int_{\mathbb{Q}_p} \varphi(x) \chi_p(\xi x) dx.$$

We have $\hat{\varphi}(\xi) \in D(\mathbb{Q}_p)$ and $\varphi(x) = F^{-1}[\varphi](\xi) = \int_{\mathbb{Q}_p} \hat{\varphi}(\xi) \chi_p(-\xi x) d\xi$ as the inverse Fourier transform.

Let $L^2(\mathbb{Q}_p)$ be the set of measurable $\mathbb{C}$-valued functions f on $\mathbb{Q}_p$ such that

$$\|f\|_{L^2(\mathbb{Q}_p)} = \left(\int_{\mathbb{Q}_p} |f(x)|^2 dx \right)^{\frac{1}{2}} < \infty$$

which is obviously a Hilbert space with the inner product

$$\langle f, g \rangle = \int_{\mathbb{Q}_p} f(x) \overline{g(x)} dx, \quad f, g \in L^2(\mathbb{Q}_p),$$

and $\|f\|^2_{L^2(\mathbb{Q}_p)} = \langle f, f \rangle$.

This yields a linear isomorphism taking $D(\mathbb{Q}_p)$ onto $D(\mathbb{Q}_p)$. It can be uniquely extended to a linear isomorphism of $L^2(\mathbb{Q}_p)$. Moreover, the Plancherel equality holds

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle, \quad f, g \in L^2(\mathbb{Q}_p).$$

2.1. p-Adic Wavelet Bases

In the current section, we recall some facts concerning with the theory of p-adic wavelets which are extensively utilized in several areas of application. In 1910, Haar [15] initially introduced the wavelet basis by presenting an orthonormal basis in $L^2(\mathbb{R})$ including dyadic translations and dilations of a single function:

$$\psi^H_{jn}(x) = 2^{-\frac{j}{2}}\psi^H(2^{-j}x - n),\ x \in \mathbb{R},\ j, n \in \mathbb{Z} \tag{2.3}$$

where

$$\psi^H(x) = \chi_{[0,\frac{1}{2})}(x) - \chi_{[\frac{1}{2},1]}(x)$$

is called a *Haar wavelet* and χ_A denotes the characteristic function of a set $A \subset \mathbb{R}$. The generalization of Haar basis (2.3) has been studied in various results. In 2002, Kozyrev [18] initially introduced a basis of complex-valued wavelets with compact support in $L^2(\mathbb{Q}_p^m)$. This basis has some resemblance to the Haar basis and takes the form below

$$\psi_{k;jn}(x) = p^{-\frac{mj}{2}}\chi_p(p^{-1}k \cdot (p^j x - n))\Omega(|p^j x - n|_p),\ x \in \mathbb{Q}_p^m \tag{2.4}$$

where $k \in J_p^m := \underbrace{J_p \times J_p \times \cdots \times J_p}_{\longleftarrow m \longrightarrow}, J_p = \{1, 2, \ldots, p-1\},\ j \in \mathbb{Z}$, and n can be taken as an element of the m-direct product of factor group

$$\mathbb{Q}_p/\mathbb{Z}_p = \left\{ \sum_{i=a}^{-1} n_i p^i \,\middle|\, n_i = 0, 1, \ldots, p-1,\ a \in \mathbb{Z}^- \right\}$$

and here, χ_p and Ω are the standard additive character of $\mathbb{Q}_p$ and characteristic function of $[0,1]$, respectively, as defined before.

2.2. p-Adic Pseudo-Differential Operators and Lizorkin Spaces

Introduced by V. S. Vladimirov [44], pseudo-differential operator A (on the field of p-adic numbers) in an open set $\mathcal{O} \subset \mathbb{Q}_p$ is given by

$$(A\varphi)(x) = \int_{\mathbb{Q}_p} \mathcal{A}(\xi, x)\hat{\varphi}(\xi)\chi_p(-\xi x)d\xi, \quad x \in \mathcal{O} \tag{2.5}$$

which acts on $\mathbb{C}$-valued functions $\varphi(x)$ of p-adic arguments $x \in \mathcal{O}$. Here we assume that functions $\varphi(x)$ are extended by zero from the set $\mathcal{O}$ on whole space $\mathbb{Q}_p$, and $\hat{\varphi}(\xi)$ are their Fourier transforms recalled previously. The function $\mathcal{A}(\xi, x), \xi \in \mathbb{Q}_p, x \in \mathcal{O}$ is called *symbol* of the operator A.

Consider the subspaces of the space of test functions $D(\mathbb{Q}_p)$

$$\Psi = \Psi(\mathbb{Q}_p) = \{\psi \in D(\mathbb{Q}_p),\ \psi(0) = 0\}, \quad \Phi = \Phi(\mathbb{Q}_p) = \{\phi\ :\ \phi = F[\psi],\ \psi \in \Psi\}.$$

Clearly, $\Psi, \Phi \neq \emptyset$. Following the fact that Fourier transform is a linear isomorphism $D(\mathbb{Q}_p)$ into $D(\mathbb{Q}_p)$, we get $\Psi, \Phi \in D(\mathbb{Q}_p)$. The space Φ can be described by the following criterion: $\phi \in \Phi$ if and only if $\phi \in D(\mathbb{Q}_p)$ and

$$\int_{\mathbb{Q}_p} \phi(x)dx = 0.$$

The space Φ is called the *p-adic Lizorkin space of test functions of the first kind* which is a complete space under the topology of the space $D(\mathbb{Q}_p)$. Furthermore, The space $\Phi' = \Phi'(\mathbb{Q}_p)$ is said to be the *p-adic Lizorkin space of distributions of the first kind* which is the topological dual space of $\Phi(\mathbb{Q}_p)$ (see also [4]).

The Taibleson fractional operator $D^\alpha : \varphi \to D^\alpha\varphi$ is defined as a convolution of the following functions:

$$D^\alpha\varphi(x) = f_{-\alpha}(x) * \varphi(x) = \langle f_{-\alpha}(x), \varphi(x - \xi)\rangle, \quad \varphi \in \Phi'(\mathbb{Q}_p),\ \alpha \in \mathbb{C}\,,$$

where the distribution $f_\alpha \in \Phi'(\mathbb{Q}_p)$ is called the *Riesz kernel* given by

$$f_\alpha(x) = \begin{cases} \dfrac{|x|_p^{\alpha-1}}{\Gamma_p(\alpha)}, & \text{if } \alpha \neq 0, 1, \\ \delta(x), & \text{if } \alpha = 0, \\ \dfrac{p^{-1}-1}{\log p} \log |x|_p, & \text{if } \alpha = 1, \end{cases} \qquad x \in \mathbb{Q}_p$$

and $\Gamma_p(\alpha) = \frac{1-p^{\alpha-1}}{1-p^{-\alpha}}$ is the Γ-function (for more details see [44]).

The domain of D^α is given by

$$\mathcal{M}(D^\alpha) = \{\varphi \in L^2(\mathbb{Q}_p) \mid D^\alpha \varphi \in L^2(\mathbb{Q}_p)\}.$$

For the vectorial case we use the following notation:

$$\mathcal{M}_n(D^\alpha) = \{\varphi \in L^2(\mathbb{Q}_p^r) \mid D^\alpha \varphi \in L^2(\mathbb{Q}_p^r)\}.$$

Moreover, we remark that all the concepts mentioned untill now can be considered in the setting of $\mathbb{Q}_p^r$.

2.3. Julia Construction

Throughout this part, in order to present our main results we need to turn our attention into theory of ODEs and recall some facts and details regarding with a class of ordinary differential equations of the first order, so-called Abel's equation of the second kind. The general form of this equation is

$$[g_1(x)u + g_0(x)]u_x = f_2(x)u^2 + f_1(x)u + f_0(x) \tag{2.6}$$

where u_x means $u_x = \frac{du}{dx}$. Regarding with Abel's equations of the first and second kinds, there are a few papers in the literature devoting themselves to study this class of ODEs (see for example [35]). There are some admissible functional transformations ([16, 32]) which can simplify (2.6) to a normal form. In 1933, Julia [32, p. 27; type (b)] found an equality including the functional coefficients of Eq. (2.6) which can lead us to the exact general solution of this equation. Based on the *Julia construction*, if the coefficients of (2.6) enjoy the functional relation

$$g_0(2f_2 + (g_1)_x) = g_1(f_1 + (g_0)_x), \quad g_1 \neq 0, \tag{2.7}$$

then Eq. (2.6) has the general solution

$$\frac{g_1 u^2 + 2g_0 u}{g_1 J} = 2\int \frac{f_0}{g_1 J} dx + C, \tag{2.8}$$

such that C is an integration constant and J the integrating factor $J(x) = \exp(2\int \frac{2f_2}{g_1} dx)$.

2.4. Adomian Decomposition Method (ADM)

Regarding with the tools utilized in our paper we now focus on a numerical method which will provide a basis for further investigation. The standard Adomian decomposition method (ADM) has been introduced at the beginning of 1980s [1, 2]. Comparing the performance of the Adomian decomposition method and the Taylor series method shows that the ADM is reliable, efficient and easy to apply from a computational aspect (see [45]). Moreover, the decomposition method supplies a fast convergent series of simply computable components and removes massive computational work needed by Taylor series method.

To describe this method for ODEs let us consider the differential equation

$$\mathcal{L}u + Ru + Nu = g\,, \tag{2.9}$$

where $\mathcal{L}$ is the highest order derivative which is assumed to be easily invertible, R is a linear differential operator of order less than $\mathcal{L}$, Nu represents the nonlinear terms, and g is the source term (or non-homogeneity term). By employing the inverse operator $\mathcal{L}^{-1}$ to both sides of Eq. (2.9), and applying the presumed conditions we get

$$u = \mathcal{L}^{-1}(Ru) - \mathcal{L}^{-1}Nu + \mathcal{L}^{-1}g. \tag{2.10}$$

By ADM we let

$$u = \sum_{n=0}^{\infty} u_n \qquad \text{and} \qquad \mathcal{L}^{-1}Nu = \sum_{n=0}^{\infty} A_n\,,$$

where A_n are the *Adomian polynomials* which depend upon $u_0, u_1, \ldots, u_n$ determined by

$$A_n = \left[\frac{1}{n!}\frac{d^n}{d\lambda^n} N\Big(\sum_{i=0}^{n} \lambda^i u_i\Big)\right]_{\lambda=0}.$$

That is, Eq. (2.10) takes the form:

$$\sum_{n=0}^{\infty} u_n = \mathcal{L}^{-1}g + \mathcal{L}^{-1}R\Big(\sum_{n=0}^{\infty} u_n\Big) - \mathcal{L}^{-1}\sum_{n=0}^{\infty} A_n. \tag{2.11}$$

We set

$$\begin{aligned} u_0 &= \mathcal{L}^{-1}g,\\ u_{n+1} &= \mathcal{L}^{-1}Ru_n - \mathcal{L}^{-1}A_n, \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{2.12}$$

Hence, the solution u of Eq. (2.9) can be determined by the series of recursive sequence u_n given as (2.12).

In the following, for the convenience of computation we will write $\sum$ instead of $\sum_{k\in J_p, j\in\mathbb{Z}, n\in\mathbb{Q}_p/\mathbb{Z}_p}$.

3. CAUCHY PROBLEM FOR THE SEMI-LINEAR PSEUDO-DIFFERENTIAL EQUATION OF THE SECOND ORDER

Let us focus on the Cauchy problem for a class of semi-linear pseudo-differential equation as follows:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + a\dfrac{\partial u}{\partial t} + bD_x^{\alpha}u + cu|u|^{2m} = f(x,t), \quad x \in \mathbb{Q}_p,\ a \neq 0,\ t > 0,\\ u(x,0) = \phi(x),\ u_t(x,0) = g(x),\ x \in \mathbb{Q}_p,\ t = 0 \end{cases} \tag{3.1}$$

where $\alpha > 0$ and D_x^{α} is the Taibleson fractional operator as introduced before. We also assume that f is represented by the continuous components $f_{k;jn}$ based on wavelet functions $\psi_{k;jn}$.

Theorem 3.1. *Let us consider $\phi \in \mathcal{M}(D^{\alpha})$. Then the pseudo-differential equation as form of* (3.1) *possesses a unique solution $u \in \mathcal{U} = C([0,\infty), \mathcal{M}(D^{\alpha})) \cap C^2([0,\infty), L^2(\mathbb{Q}_p))$ of the form*

$$u(x,t) = \sum R_{k;jn}(t) \exp\left(i\left(\int_0^t \frac{e^{-aR_{k;jn}(s)+M_{k;jn}(0)}}{R_{k;jn}(s)} ds + \arg(\langle\phi, \psi_{k;jn}\rangle)\right)\right)\psi_{k;jn}(x)$$

where

$$R_{k;jn}(t) = \frac{2C_0}{a} + \sum_{r=0}^{\infty} \frac{(r+1)^{r-1}}{2^r r!}(-a)^r \overline{B}^r(-2C_0)e^{-\frac{(r+1)a}{2}(\overline{B}(-2C_0)t+\frac{t}{3}+C_1)},$$

and $\overline{B}$ is the real root of the following cubic equation

$$\overline{B}^3 + p\overline{B} + q = 0,$$

such that

$$p=-\frac{\mu^2}{3}+\nu,\quad q=2(\frac{\mu}{3})^3-\frac{\mu\nu}{3}+\eta,\quad \mu=-4,\quad \nu=3-\eta,\quad \eta=-\frac{4(H+\mathcal{F})}{z+2C_0},$$

$$\mathcal{F}=-\frac{1}{a}(-bp^{\alpha(1-j)}R_{k;jn}(t)-cp^{-mj}R_{k;jn}(t)^{2m+1}+f_{k;jn}(t)),$$

$$H=e^{\tau}\frac{\cos\tau[|\tau|sgn(\tau)Si(\tau)]-\sin\tau[Si(\tau)+\sin\tau]}{8\tau^2Si^2(\tau)}\left(\frac{\sin\tau}{2|\tau|sgn(\tau)Si(\tau)}+2\right)-2\mathcal{F}(\tau)$$

where $\tau=\ln|-aR(t)+2C_0|$, *and*

$$C_0=\frac{|\langle g,\psi_{k;jn}\rangle|\cos(\arg(\langle g,\psi_{k;jn}\rangle\cdot[\langle\phi,\psi_{k;jn}\rangle]^{-1}))}{\overline{B}(-a|\langle\phi,\psi_{k;jn}\rangle|)+\frac{1}{3}}+\frac{1}{2}a|\langle\phi,\psi_{k;jn}\rangle|,$$

$$C_1=-\frac{2}{a}\ln\left|\frac{2}{a}\frac{|\langle g,\psi_{k;jn}\rangle|\cos(\arg(\langle g,\psi_{k;jn}\rangle\cdot[\langle\phi,\psi_{k;jn}\rangle]^{-1}))}{\overline{B}(-a|\langle\phi,\psi_{k;jn}\rangle|)+\frac{1}{3}}\right|,$$

$$M_{k;jn}(0)=\ln\left|\frac{\sin(\arg(\langle g,\psi_{k;jn}\rangle\cdot[\langle\phi,\psi_{k;jn}\rangle]^{-1}))|\langle g,\psi_{k;jn}\rangle|}{|\langle\phi,\psi_{k;jn}\rangle|}\right|+\ln|\langle\phi,\psi_{k;jn}\rangle|+a|\langle\phi,\psi_{k;jn}\rangle|.$$

Proof. Let us suppose that there exists $u\in\mathcal{U}$ of the form as given below

$$u(x,t)=\sum u_{k;jn}(t)\psi_{k;jn}(x). \tag{3.2}$$

We intend to find an explicit form for u in terms of the wavelet functions $\{\psi_{k;jn}\}$. For any (k,j,n) by $u\in\mathcal{U}$, we obtain

$$\frac{\partial^2u}{\partial t^2}+\frac{\partial u}{\partial t}=\sum(u''_{k;jn}(t)+u'_{k;jn}(t))\psi_{k;jn}(x),\ D_x^{\alpha}u(x,t)=\sum p^{\alpha(1-j)}u_{k;jn}(t)\psi_{k;jn}(x). \tag{3.3}$$

Now, considering Eq. (3.2) we get

$$|u|^2=\sum|u_{k;jn}(t)|^2p^{-j}\Omega(|p^jx-n|_p)\implies u|u|^{2m}=\sum u_{k;jn}(t)|u_{k;jn}(t)|^{2m}p^{-mj}\psi_{k;jn}(x).$$

From (3.1) it follows that

$$\sum[u''_{k;jn}(t)+au'_{k;jn}(t)+bp^{\alpha(1-j)}u_{k;jn}(t)+cp^{-mj}u_{k;jn}(t)|u_{k;jn}(t)|^{2m}-f_{k;jn}(t)]\psi_{k;jn}(x)=0,$$

that is,

$$u''_{k;jn}(t)+au'_{k;jn}(t)+bp^{\alpha(1-j)}u_{k;jn}(t)+cp^{-mj}u_{k;jn}(t)|u_{k;jn}(t)|^{2m}-f_{k;jn}(t)=0\,,$$

where $f_{k;jn}$ are continuous functions and

$$f(x,t)=\sum f_{k;jn}(t)\psi_{k;jn}(x),\ f_{k;jn}(t)=\langle f(\cdot,t),\psi_{k;jn}(\cdot)\rangle,\quad k\in J_p,j\in\mathbb{Z}.$$

Now, using the polar coordinates let us denote $u_{k;jn}(t)=R_{k;jn}(t)e^{i\alpha_{k;jn}(t)}$, for $\alpha_{k;jn}(t),R_{k;jn}(t)\in C^2(\mathbb{R}^+,\mathbb{R})$, then we derive the following system of ODEs:

$$\begin{cases}R''_{k;jn}(t)+aR'_{k;jn}(t):=F(R_{k;jn}(t))=-bp^{\alpha(1-j)}R_{k;jn}(t)-cp^{-mj}R_{k;jn}(t)^{2m+1}+f_{k;jn}(t),\\ R'_{k;jn}(t)\alpha'_{k;jn}(t)+R_{k;jn}(t)\alpha''_{k;jn}(t)+aR_{k;jn}(t)\alpha'_{k;jn}(t)=0.\end{cases}$$

From now on, to simplify the notations we drop all the indexes and variable t, that is,

$$\begin{cases} R'' + aR' := F(R) = -bp^{\alpha(1-j)}R - cp^{-mj}R^{2m+1} + f, \\ R'\alpha' + R\alpha'' + aR\alpha' = 0. \end{cases}$$

First, from the second equation of the system, one can define α in terms of R. Indeed, if α is constant then proof is complete, otherwise, setting $M = \alpha' \neq 0$ we obtain

$$(R' + aR)M + RM' = 0 \implies \frac{M'}{M} = -\frac{R' + aR}{R}$$
$$\implies \alpha_{k;jn}(t) = \int_0^t \frac{e^{-as+M_{k;jn}(0)}}{R_{k;jn}(s)} ds + \alpha_{k;jn}(0)\,, \tag{3.4}$$

where $\alpha_{k;jn}(0), M_{k;jn}(0)$ are constants with respect to their indexes.

Now, we focus on the non-homogeneous equation of the form

$$R'' + aR' = F(R). \tag{3.5}$$

We remark that existence of the nonlinear term $F(R)$ makes the process of finding the solution more complicated. If $c = 0$ then we have a linear differential equation and it is easy to study. Besides, for this case, motivation of the problem (3.1) is lost. However, here we utilize a novel technique for general case. For arbitrary constant c, and $a \neq 0$ suppose that

$$W = R' \quad\implies\quad R'' = \frac{dR'}{dt} = \frac{dW}{dt} = W\frac{dW}{dR}$$

which turns (3.5) into the following

$$W\frac{dW}{dR} + aW = F(R). \tag{3.6}$$

This equation is called Abel equation of the second kind. If $a = -1$ then we say that the Abel equation is in canonical form, otherwise, after long calculations (which can be seen in Apppendix) we find that

$$W = \frac{1}{2}\Big(z + 2C_0\Big)\Big(\overline{B}(z) + \frac{1}{3}\Big)$$

where C_0 is an arbitrary constant, $z = -aR$ and $\overline{B}(z)$ is a real root of a certain cubic function (see Eqs. (6.14)-(6.19) in Subsection 6.6.1 in Appendix). Since $W = R'$ we observe that

$$-\frac{2}{a}\ln\left|R(t) - \frac{2C_0}{a}\right| = \int_0^t \overline{B}(-aR(s))ds + \frac{1}{3}t + C_1. \tag{3.7}$$

Now, let us impose the initial conditions in the obtained results to find the constants C_0, C_1.

$$u(x,0) = \sum R_{k;jn}(0)e^{i\alpha_{k;jn}(0)}\psi_{k;jn}(x) = \varphi(x), \tag{3.8}$$

$$u_t(x,0) = \sum \Big(R'_{k;jn}(0) + i\alpha'_{k;jn}(0)R_{k;jn}(0)\Big)e^{i\alpha_{k;jn}(0)}\psi_{k;jn}(x) = g(x). \tag{3.9}$$

Suppose that

$$\langle\phi, \psi_{k;jn}\rangle = |\langle\phi, \psi_{k;jn}\rangle|e^{i\beta_{k;jn}}, \quad \langle g, \psi_{k;jn}\rangle = |\langle g, \psi_{k;jn}\rangle|e^{i\gamma_{k;jn}}\,,$$

where $\gamma_{k;jn} - \beta_{k;jn} \neq \frac{k\pi}{2}, k = 0, 1, 2, 3,$ then following (3.8) we get

$$R_{k;jn}(0)e^{i\alpha_{k;jn}(0)} = \langle\phi, \psi_{k;jn}\rangle = |\langle\phi, \psi_{k;jn}\rangle|e^{i\beta_{k;jn}}$$

which shows that

$$R_{k;jn}(0) = |\langle\phi, \psi_{k;jn}\rangle|, \quad \alpha_{k;jn}(0) = \arg(\langle\phi, \psi_{k;jn}\rangle) = \beta_{k;jn}. \tag{3.10}$$

On the other hand, based on (3.9) we see that

$$(R'_{k;jn}(0) + i\alpha'_{k;jn}(0)R_{k;jn}(0))e^{i\alpha_{k;jn}(0)} = \langle g, \psi_{k;jn}\rangle = |\langle g, \psi_{k;jn}\rangle| e^{i\gamma_{k;jn}}. \tag{3.11}$$

This together with (6.24) and (3.10) yields

$$\begin{aligned} R'_{k;jn}(0) &= |\langle g, \psi_{k;jn}\rangle| \cos(\gamma_{k;jn} - \beta_{k;jn}) = W_{k;jn}(0) \\ &= \frac{1}{2}\Big(- a|\langle \phi, \psi_{k;jn}\rangle| + 2C_0\Big)\Big(\overline{B}(-a|\langle \phi, \psi_{k;jn}\rangle|) + \frac{1}{3}\Big), \end{aligned}$$

that is,

$$C_0 = \frac{|\langle g, \psi_{k;jn}\rangle| \cos(\gamma_{k;jn} - \beta_{k;jn})}{\overline{B}(-a|\langle \phi, \psi_{k;jn}\rangle|) + \frac{1}{3}} + \frac{1}{2}a|\langle \phi, \psi_{k;jn}\rangle|. \tag{3.12}$$

Moreover, for the constant C_1 we derive that

$$C_1 = -\frac{2}{a}\ln\left|R_{k;jn}(0) - \frac{2C_0}{a}\right| = -\frac{2}{a}\ln\left|\frac{2}{a}\frac{|\langle g, \psi_{k;jn}\rangle| \cos(\gamma_{k;jn} - \beta_{k;jn})}{\overline{B}(a|\langle \phi, \psi_{k;jn}\rangle|) + \frac{1}{3}}\right|. \tag{3.13}$$

Finally, for the last constant $M_{k;jn}(0)$ defined in (3.4), with the help of (3.11) one can easily see that

$$\begin{aligned} M_{k;jn}(0) &= \ln|\alpha'_{k;jn}(0)| + \ln|\langle \phi, \psi_{k;jn}\rangle| + a|\langle \phi, \psi_{k;jn}\rangle| \\ &= \ln\left|\frac{\sin(\gamma_{k;jn} - \beta_{k;jn})|\langle g, \psi_{k;jn}\rangle|}{|\langle \phi, \psi_{k;jn}\rangle|}\right| + \ln|\langle \phi, \psi_{k;jn}\rangle| + a|\langle \phi, \psi_{k;jn}\rangle|. \end{aligned} \tag{3.14}$$

Therefore, the solution to problem (3.1) is given by

$$u(x,t) = \sum R_{k;jn}(t)\exp\left(i\Big(\int_0^t \frac{e^{-aR_{k;jn}(s)+M_{k;jn}(0)}}{R_{k;jn}(s)}ds + \beta_{k;jn}\Big)\right)\psi_{k;jn}(x)$$

where $M_{k;jn}(0)$ is defined by (3.14) and the functions $R_{k;jn}$ are derived by solving the integral equation

$$-\frac{2}{a}\ln\left|R_{k;jn}(t) - \frac{2C_0}{a}\right| = \int_0^t \overline{B}(-aR_{k;jn}(s))ds + \frac{1}{3}t + C_1 \tag{3.15}$$

in which the constants C_0, C_1 are given by (3.12) and (3.13), respectively.

To complete the proof we must find a solution of Eq. (3.7) employing the ADM. Let us first rewrite the equation as below

$$R(t) = \frac{2C_0}{a} + e^{-\frac{a}{2}(\frac{t}{3}+C_1)} \cdot \exp\left(- \int_0^t \frac{a}{2}\overline{B}(-aR(s))ds\right) \tag{3.16}$$

where the constants C_0 and C_1 are given by (3.12) and (3.13), respectively.

Following the Adomian decomposition method (ADM) we find that

$$R(t) = \frac{2C_0}{a} + \sum_{k=0}^{\infty}\frac{(k+1)^{k-1}}{2^k k!}(-a)^k\overline{B}^k(-2C_0)e^{-\frac{(k+1)a}{2}(\overline{B}(-2C_0)t+\frac{t}{3}+C_1)},$$

as desired solution of (3.7) (for more details see Subsection 6.6.2 in Appendix). This completes the proof. □

Remark 3.2. *Based on Appendix A, it is worth mentioning that during proof of Theorem 3.1 since we deal with so many parameters and cases for the function $\overline{B}$ and besides f is unknown, it seems unlikely to find an explicit and compact form for $R_{k;jn}(t)$.*

4. CAUCHY PROBLEM FOR A FIRST ORDER SEMI-LINEAR PSEUDO-DIFFERENTIAL EQUATION

In this section we consider a Cauchy problem for a class of first order semi-linear pseudo-differential equation as form of below:

$$\begin{cases} \dfrac{\partial u}{\partial t} + aD_x^\alpha u + bu|u|^{2m} = f(x,t), \quad x \in \mathbb{Q}_p,\ t > 0, \\ u(x,0) = \phi(x), \qquad x \in \mathbb{Q}_p,\ t = 0\,, \end{cases} \tag{4.1}$$

where $\phi \in \mathcal{M}(D^\alpha)$ and f is represented by the continuous components $f_{k;jn}$ corresponding to the wavelet basis $\psi_{k;jn}$.

Theorem 4.1. *The pseudo-differential equation as form of* (4.1) *possesses a unique solution* $u \in \mathcal{V} = C([0,\infty), \mathcal{M}(D^\alpha)) \cap C^1([0,\infty), L^2(\mathbb{Q}_p))$ *of the form*

$$u(x,t) = \sum_{k,j,n} \left(\left[|\langle \phi, \psi_{k;jn}\rangle| + \sum_{r=0}^{\infty} (R_{k;jn}^{(r)}(t) - R_{k;jn}^{(r)}(0)) \right] e^{i \arg(\langle \phi, \psi_{k;jn}\rangle)} \right) \psi_{k;jn}(x)\,, \tag{4.2}$$

where

$$R_{k;jn}^{(0)}(t) = \mathcal{L}^{-1} f_{k;jn}(t),$$
$$R_{k;jn}^{(r+1)}(t) = -ap^{\alpha(1-j)} \mathcal{L}^{-1} R_{k;jn}^{(r)}(t) - bp^{-mj} \mathcal{L}^{-1} A_{k;jn}^{(r)}(t), \quad r = 0, 1, 2, \ldots.$$

and $A_{k;jn}^{(r)}$ *are the associate Adomian polynomials.*

Proof. Obeying the proof presented for Theorem 3.1, let us denote $u_{k;jn}(t) = R_{k;jn}(t) e^{i\alpha_{k;jn}(t)}$, for any $\alpha_{k;jn}(t)$, $R_{k;jn}(t)$ belonging to $C^1(\mathbb{R}^+, \mathbb{R})$, then we arrive at the following system of ODEs:

$$\begin{cases} R'_{k;jn}(t) + ap^{\alpha(1-j)} R_{k;jn}(t) = -bp^{-mj} R_{k;jn}(t)^{2m+1} + f_{k;jn}(t), \\ \alpha'_{k;jn}(t) = 0. \end{cases}$$

From the second equation as above we set $\alpha_{k;jn} := \alpha_{k;jn}(t)$ as an arbitrary constant. For the convenience, in the following we omit the index $k; jn$ and the variable t. By $\mathcal{L} := \frac{d}{dt}$ we rewrite the first equation as follows:

$$\mathcal{L}R = -ap^{\alpha(1-j)} R - bp^{-mj} R^{2m+1} + f \tag{4.3}$$

and then apply the integral operator $\mathcal{L}^{-1}$ to obtain

$$R = -ap^{\alpha(1-j)} \mathcal{L}^{-1} R - bp^{-mj} \mathcal{L}^{-1} R^{2m+1} + f_0 \tag{4.4}$$

where f_0 stands for the terms arising from integrating the given term f and from utilizing the given conditions, all are supposed to be specified.

Following the standard Adomian decomposition method (ADM) we give the solution R for the very recent equation by the series

$$R = \sum_{n=0}^{\infty} R_n,$$

and following (2.11), (2.12), equation (4.4) takes the form as follows

$$\sum_{n=0}^{\infty} R_n = f_0 - ap^{\alpha(1-j)} \sum_{n=0}^{\infty} \mathcal{L}^{-1} R_n - bp^{-mj} \sum_{n=0}^{\infty} \mathcal{L}^{-1} A_n\,, \tag{4.5}$$

where the components R_n are specified by the following recursive sequence

$$
\begin{aligned}
&R_0 = \mathcal{L}^{-1} f = f_0,\\
&R_1 = -ap^{\alpha(1-j)}\mathcal{L}^{-1}R_0 - bp^{-mj}\mathcal{L}^{-1}A_0 = -ap^{\alpha(1-j)}\mathcal{L}^{-1}R_0 - bp^{-mj}\mathcal{L}^{-1}R_0^{2m+1},\\
&R_2 = -ap^{\alpha(1-j)}\mathcal{L}^{-1}R_1 - bp^{-mj}\mathcal{L}^{-1}A_1 = -ap^{\alpha(1-j)}\mathcal{L}^{-1}R_1 - b(2m+1)p^{-mj}\mathcal{L}^{-1}R_1R_0^{2m}\\
&R_3 = -ap^{\alpha(1-j)}\mathcal{L}^{-1}R_2 - bp^{-mj}\mathcal{L}^{-1}A_2\\
&\quad\; = -ap^{\alpha(1-j)}\mathcal{L}^{-1}R_2 - bp^{-mj}\mathcal{L}^{-1}\left[(2m+1)R_2R_0^{2m} + (2m+1)(2m)\frac{R_1^2}{2}R_0^{2m-1}\right]\\
&\vdots\\
&R_{n+1} = -ap^{\alpha(1-j)}\mathcal{L}^{-1}R_n - bp^{-mj}\mathcal{L}^{-1}A_n, \quad n \geq 1,
\end{aligned}
\tag{4.6}
$$

and A_n are the associate Adomian polynomials.

Hence, the solution to problem (4.1) in terms of wavelet functions is given by

$$
u(x,t) = \sum_{k,j,n}\left(\sum_{r=0}^{\infty} R_r(t)e^{i\alpha_{k;jn}}\right)\psi_{k;jn}(x), \tag{4.7}
$$

where

$$
\begin{aligned}
&R_0(t) = \mathcal{L}^{-1}f(t) = f_0(t) + C, \quad C = \text{const.},\\
&R_1(t) = -ap^{\alpha(1-j)}\mathcal{L}^{-1}R_0(t) - bp^{-mj}\mathcal{L}^{-1}R_0^{2m+1}(t),\\
&R_2(t) = -ap^{\alpha(1-j)}\mathcal{L}^{-1}R_1(t) - b(2m+1)p^{-mj}\mathcal{L}^{-1}R_1(t)R_0^{2m}(t)\\
&R_3(t) = -ap^{\alpha(1-j)}\mathcal{L}^{-1}R_2(t) - bp^{-mj}\mathcal{L}^{-1}\left[(2m+1)R_2(t)R_0^{2m}(t) + m(2m+1)R_1^2(t)R_0^{2m-1}(t)\right]\\
&\vdots\\
&R_{n+1}(t) = -ap^{\alpha(1-j)}\mathcal{L}^{-1}R_n(t) - bp^{-mj}\mathcal{L}^{-1}A_n(t), \quad n \geq 1.
\end{aligned}
$$

Relying on the initial condition and considering a unique integration constant C without loss of generality one can see that

$$
\begin{aligned}
&\phi(x) = \sum_{k,j}\left(f_0(0) + C + \sum_{r=1}^{\infty} R_r(0)\right)e^{i\alpha_{k;jn}}\psi_{k;jn}(x)\\
&\qquad \Longrightarrow \quad \left(f_0(0) + C + \sum_{r=1}^{\infty} R_r(0)\right)e^{i\alpha_{k;jn}} = \langle \phi, \psi_{k;jn}\rangle := |\langle \phi, \psi_{k;jn}\rangle| e^{i\beta_{k;jn}}
\end{aligned}
$$

which easily shows that

$$
\alpha_{k;jn} = \beta_{k;jn}, \quad C = |\langle \phi, \psi_{k;jn}\rangle| - \sum_{r=0}^{\infty} R_r(0).
$$

This completes the proof. □

5. PROBLEMS (1.1)–(1.2) FOR p-ADIC PSEUDO-DIFFERENTIAL OPERATOR A IN $\mathbb{Q}_p^r$

During this section, we replace the fractional operator D_x^{α} with the p-adic pseudo-differential operator A satisfying the condition (5.1) of the following criterion, then study the solutions of new problems in vectorial p-adic field $\mathbb{Q}_p^r$ by a brief discussion and same reasoning. Note that D^{α} is such operator which fulfills (5.1).

Theorem 5.1 ([3]). *Let A be a pseudo-differential operator* (2.5) *with a symbol $\mathcal{A} \in \mathcal{E}(\mathbb{Q}_p^r \backslash \{0\})$. Then the Haar wavelet function* (2.4) *is an eigenfunction of A, i.e.,*

$$A\psi_{k;jn}(x) = \mathcal{A}(-p^{j-1}k)\psi_{k;jn}(x) \quad \Longleftrightarrow \quad \mathcal{A}(p^j(-p^{-1}k+\eta)) = \mathcal{A}(-p^{j-1}k),\ \forall \eta \in \mathbb{Z}_p^r \tag{5.1}$$

where $k \in J_p^r$, $j \in \mathbb{Z}$, and n belongs to the r-direct product of factor group $\mathbb{Q}_p/\mathbb{Z}_p$.

Let us mention the subjected problems as follows.

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + a\dfrac{\partial u}{\partial t} + bA_x u + cu|u|^{2m} = f(x,t), \quad x \in \mathbb{Q}_p^r,\ a \neq 0,\ t > 0, \\ u(x,0) = \phi(x),\ u_t(x,0) = g(x),\ x \in \mathbb{Q}_p^r,\ t = 0 \end{cases} \tag{5.2}$$

and

$$\begin{cases} \dfrac{\partial u}{\partial t} + aA_x u + bu|u|^{2m} = f(x,t), \quad x \in \mathbb{Q}_p^r,\ t > 0, \\ u(x,0) = \phi(x), \qquad x \in \mathbb{Q}_p^r,\ t = 0\,, \end{cases} \tag{5.3}$$

where $\alpha > 0$ and A_x is a p-adic pseudo-differential operator satisfying the condition (5.1). Moreover, $\phi \in \mathcal{M}_r(D^\alpha)$ and f is represented by the continuous components $f_{k;jn}$ corresponding to the wavelet basis $\psi_{k;jn}$.

In order to simplify the notation we consider $\sum$ instead of $\sum_{k\in J_p^r, j\in\mathbb{Z}, n\in(\mathbb{Q}_p/\mathbb{Z}_p)^r}$.

Theorem 5.2. *The pseudo-differential equation as form of* (5.2) *possesses a unique solution $u \in \mathcal{U}_r = C([0,\infty), \mathcal{M}_r(D^\alpha)) \cap C^2([0,\infty), L^2(\mathbb{Q}_p^r))$ of the form*

$$u(x,t) = \sum R_{k;jn}(t) \exp\left[i\biggl(\int_0^t \frac{-e^{-as}}{R_{k;jn}(s)} \biggl(\sigma \int_0^s R_{k;jn}(\theta) e^{a\theta} d\theta + \sin(\gamma_{k;jn} - \beta_{k;jn}) |\langle g, \psi_{k;jn} \rangle| \biggr) ds \right.$$
$$\left. + \arg(\langle \phi, \psi_{k;jn} \rangle) \biggr) \right] \psi_{k;jn}(x)\,,$$

where $\sigma := -b\,\mathrm{Im}\,\mathcal{A}(-p^{j-1}k)$ and

$$R_{k;jn}(t) = \frac{2C_0}{a} + \sum_{s=0}^{\infty} \frac{(s+1)^{s-1}}{2^s s!} (-a)^s \overline{B}^s (-2C_0) e^{-\frac{(s+1)a}{2}(\overline{B}(-2C_0)t + \frac{t}{3} + C_1)},$$

and $C_0, C_1, \overline{B}$ are same as given in Theorem 3.1 .

Proof. Inspired by the proof of Theorem 3.1 we suppose that there exists $u \in \mathcal{U}_r$ given as (3.2). Substituting u in terms of the wavelet functions $\{\psi_{k;jn}\}$ into (5.2) we get

$$\sum [u''_{k;jn}(t) + au'_{k;jn}(t) + b\mathcal{A}(-p^{j-1}k)u_{k;jn}(t) + cp^{-mj}u_{k;jn}(t)|u_{k;jn}(t)|^{2m} - f_{k;jn}(t)]\psi_{k;jn}(x) = 0,$$

that is,

$$u''_{k;jn}(t) + au'_{k;jn}(t) + b\mathcal{A}(-p^{j-1}k)u_{k;jn}(t) + cp^{-mj}u_{k;jn}(t)|u_{k;jn}(t)|^{2m} - f_{k;jn}(t) = 0$$

such that

$$f(x,t) = \sum f_{k;jn}(t)\psi_{k;jn}(x),\ f_{k;jn}(t) = \langle f(\cdot,t), \psi_{k;jn}(\cdot) \rangle, \quad k \in J_p^r, j \in \mathbb{Z}.$$

Now, representing $u_{k;jn}(t)$ by the polar coordinates let us denote $u_{k;jn}(t) = R_{k;jn}(t)e^{i\alpha_{k;jn}(t)}$, for $\alpha_{k;jn}(t), R_{k;jn}(t) \in C^2(\mathbb{R}^+, \mathbb{R})$, and $\mathcal{A}(-p^{j-1}k) = \mathrm{Re}\,\mathcal{A}(-p^{j-1}k) + i\,\mathrm{Im}\,\mathcal{A}(-p^{j-1}k)$, then we derive the following system of ODEs:

$$\begin{cases} R''_{k;jn}(t) + aR'_{k;jn}(t) := F(R_{k;jn}(t)) = -b\,\mathrm{Re}\,\mathcal{A}(-p^{j-1}k)R_{k;jn}(t) - cp^{-mj}R_{k;jn}(t)^{2m+1} + f_{k;jn}(t), \\ R'_{k;jn}(t)\alpha'_{k;jn}(t) + R_{k;jn}(t)\alpha''_{k;jn}(t) + aR_{k;jn}(t)\alpha'_{k;jn}(t) + b\,\mathrm{Im}\,\mathcal{A}(-p^{j-1}k)R_{k;jn}(t) = 0 \end{cases}$$

Simplifying the notations we rewrite the system as follows:

$$\begin{cases} R'' + aR' := F(R) = -b\mathcal{A}(-p^{j-1}k)R - cp^{-mj}R^{2m+1} + f, \\ R'\alpha' + R\alpha'' + aR\alpha' + b\,\mathrm{Im}\,\mathcal{A}(-p^{j-1}k)R = 0. \end{cases}$$

First, from the second equation of the system, one can define α in terms of R. Setting $M = \alpha' \neq 0$ we immediately arrive at

$$\begin{aligned} &M' + \frac{(R' + aR)}{R}M = \sigma := -b\,\mathrm{Im}\,\mathcal{A}(-p^{j-1}k) \\ &\qquad \Longrightarrow \alpha_{k;jn}(t) = \int_0^t \frac{-e^{-as}}{R_{k;jn}(s)}\left(\sigma\int_0^s R_{k;jn}(\theta)e^{a\theta}d\theta + M_{k;jn}(0)\right)ds + \alpha_{k;jn}(0) \end{aligned} \tag{5.4}$$

where $\alpha_{k;jn}(0), M_{k;jn}(0)$ are integration constants with respect to their indexes. For the equation

$$R'' + aR' = F(R)$$

one can follow the proof of Theorem 3.1 and use the Adomian decomposition method (ADM) to find that

$$R(t) = \frac{2C_0}{a} + \sum_{s=0}^{\infty}\frac{(s+1)^{s-1}}{2^s s!}(-a)^s\overline{B}^s(-2C_0)e^{-\frac{(s+1)a}{2}(\overline{B}(-2C_0)t + \frac{t}{3} + C_1)}$$

where the constants C_0, C_1 are defined by (3.12)-(3.13). Moreover, taking account on the proof of Theorem 3.1 we see

$$\begin{aligned} &R_{k;jn}(0) = |\langle\phi, \psi_{k;jn}\rangle|, \quad \alpha_{k;jn}(0) = \arg(\langle\phi, \psi_{k;jn}\rangle), \\ &M_{k;jn}(0) = R_{k;jn}(0)\alpha'_{k;jn}(0) = \sin(\gamma_{k;jn} - \beta_{k;jn})|\langle g, \psi_{k;jn}\rangle|. \end{aligned} \tag{5.5}$$

Consequently, we get

$$\begin{aligned} u(x,t) = \sum R_{k;jn}(t)\exp\Bigg[i\Bigg(\int_0^t \frac{-e^{-as}}{R_{k;jn}(s)}\left(\sigma\int_0^s R_{k;jn}(\theta)e^{a\theta}d\theta + \sin(\gamma_{k;jn} - \beta_{k;jn})|\langle g, \psi_{k;jn}\rangle|\right)ds \\ + \arg(\langle\phi, \psi_{k;jn}\rangle)\Bigg)\Bigg]\psi_{k;jn}(x) \end{aligned}$$

as desired. □

Theorem 5.3. *The pseudo-differential equation as form of* (5.3) *possesses a unique solution* $u \in \mathcal{V}_r = C([0,\infty), \mathcal{M}_r(D^\alpha)) \cap C^1([0,\infty), L^2(\mathbb{Q}_p^r))$ *of the form*

$$u(x,t) = \sum\left(\left[|\langle\phi, \psi_{k;jn}\rangle| + \sum_{s=0}^{\infty}\left(R^{(s)}_{k;jn}(t) - R^{(s)}_{k;jn}(0)\right)\right]e^{i(-\sigma t + \arg(\langle\phi, \psi_{k;jn}\rangle))}\right)\psi_{k;jn}(x), \tag{5.6}$$

where

$$\begin{aligned} &R^{(0)}_{k;jn}(t) = \mathcal{L}^{-1}f_{k;jn}(t), \\ &R^{(s+1)}_{k;jn}(t) = -a\,\mathrm{Re}\,\mathcal{A}(-p^{j-1}k)\mathcal{L}^{-1}R^{(s)}_{k;jn}(t) - bp^{-mj}\mathcal{L}^{-1}A^{(s)}_{k;jn}(t), \quad s = 0,1,2,\dots. \end{aligned}$$

and $\sigma := -a\,\mathrm{Im}\,\mathcal{A}(-p^{j-1}k)$ *and* $A^{(s)}_{k;jn}$ *are the associate Adomian polynomials.*

Proof. Let us first denote $u_{k;jn}(t) = R_{k;jn}(t)e^{i\alpha_{k;jn}(t)}$, for any $\alpha_{k;jn}(t), R_{k;jn}(t)$ belonging to $C^1(\mathbb{R}^+, \mathbb{R})$, then we get the following system of ODEs:

$$\begin{cases} R'_{k;jn}(t) + a\,\mathrm{Re}\,\mathcal{A}(-p^{j-1}k)R_{k;jn}(t) = -bp^{-mj}R_{k;jn}(t)^{2m+1} + f_{k;jn}(t), \\ \alpha'_{k;jn}(t) + a\,\mathrm{Im}\,\mathcal{A}(-p^{j-1}k) = 0. \end{cases}$$

First, from the second equation we find that $\alpha_{k;jn}(t) = -\sigma t + \alpha_{k;jn}(0)$. For the convenience of notations, in the following we cancel the index $k; jn$ and the variable t. By $\mathcal{L} := \frac{d}{dt}$ we first rewrite the first equation as follows:

$$\mathcal{L}R = -a\,\mathrm{Re}\,\mathcal{A}(-p^{j-1}k)R - bp^{-mj}R^{2m+1} + f \tag{5.7}$$

and then utilize the integral operator $\mathcal{L}^{-1}$ to derive

$$R = -a\,\mathrm{Re}\,\mathcal{A}(-p^{j-1}k)\mathcal{L}^{-1}R - bp^{-mj}\mathcal{L}^{-1}R^{2m+1} + f_0 \tag{5.8}$$

where f_0 stands for the terms arising from integrating the given term f and from utilizing the given conditions, all are supposed to be determined.

Following the standard Adomian decomposition method (ADM) we give the solution R for the very recent equation by the series

$$R = \sum_{n=0}^{\infty} R_n\,,$$

and following (2.11), (2.12), equation (5.8) takes the following form

$$\sum_{n=0}^{\infty} R_n = f_0 - a\,\mathrm{Re}\,\mathcal{A}(-p^{j-1}k)\sum_{n=0}^{\infty}\mathcal{L}^{-1}R_n - bp^{-mj}\sum_{n=0}^{\infty}\mathcal{L}^{-1}A_n \tag{5.9}$$

where the components R_n are specified by the following recursive sequence

$$\begin{aligned}
&R_0 = \mathcal{L}^{-1}f = f_0,\\
&R_1 = -a\,\mathrm{Re}\,\mathcal{A}(-p^{j-1}k)\mathcal{L}^{-1}R_0 - bp^{-mj}\mathcal{L}^{-1}R_0^{2m+1},\\
&R_2 = -a\,\mathrm{Re}\,\mathcal{A}(-p^{j-1}k)\mathcal{L}^{-1}R_1 - b(2m+1)p^{-mj}\mathcal{L}^{-1}R_1R_0^{2m}\\
&R_3 = -a\,\mathrm{Re}\,\mathcal{A}(-p^{j-1}k)\mathcal{L}^{-1}R_2 - bp^{-mj}\mathcal{L}^{-1}\left[(2m+1)R_2R_0^{2m} + (2m+1)(2m)\frac{R_1^2}{2}R_0^{2m-1}\right]\\
&\vdots\\
&R_{n+1} = -a\,\mathrm{Re}\,\mathcal{A}(-p^{j-1}k)\mathcal{L}^{-1}R_n - bp^{-mj}\mathcal{L}^{-1}A_n, \quad n \geq 1,
\end{aligned} \tag{5.10}$$

and A_n are the associate Adomian polynomials.

Hence, the solution to problem (5.3) in terms of wavelet functions is given by

$$u(x,t) = \sum_{k,j,n}\left(\sum_{s=0}^{\infty} R_s(t)e^{i\alpha_{k;jn}}\right)\psi_{k;jn}(x) \tag{5.11}$$

where

$$\begin{aligned}
&R_0(t) = \mathcal{L}^{-1}f(t) = f_0(t) + C, \quad C = \text{const.},\\
&R_1(t) = -a\,\mathrm{Re}\,\mathcal{A}(-p^{j-1}k)\mathcal{L}^{-1}R_0(t) - bp^{-mj}\mathcal{L}^{-1}R_0^{2m+1}(t),\\
&R_2(t) = -a\,\mathrm{Re}\,\mathcal{A}(-p^{j-1}k)\mathcal{L}^{-1}R_1(t) - b(2m+1)p^{-mj}\mathcal{L}^{-1}R_1(t)R_0^{2m}(t),\\
&R_3(t) = -a\,\mathrm{Re}\,\mathcal{A}(-p^{j-1}k)\mathcal{L}^{-1}R_2(t)\\
&- bp^{-mj}\mathcal{L}^{-1}\left[(2m+1)R_2(t)R_0^{2m}(t) + m(2m+1)R_1^2(t)R_0^{2m-1}(t)\right],\\
&\vdots\\
&R_{n+1}(t) = -a\,\mathrm{Re}\,\mathcal{A}(-p^{j-1}k)\mathcal{L}^{-1}R_n(t) - bp^{-mj}\mathcal{L}^{-1}A_n(t), \quad n \geq 1.
\end{aligned}$$

Viewing the initial condition, considering a unique integration constant C without loss of generality we see

$$\phi(x) = \sum_{k,j} \left(f_0(0) + C + \sum_{s=1}^{\infty} R_s(0) \right) e^{i\alpha_{k;jn}(0)} \psi_{k;jn}(x)$$
$$\Longrightarrow \quad \left(f_0(0) + C + \sum_{s=1}^{\infty} R_s(0) \right) e^{i\alpha_{k;jn}(0)} = \langle \phi, \psi_{k;jn} \rangle := |\langle \phi, \psi_{k;jn} \rangle| e^{i\beta_{k;jn}}$$

which easily shows that

$$\alpha_{k;jn}(0) = \beta_{k;jn}, \quad C = |\langle \phi, \psi_{k;jn} \rangle| - \sum_{s=0}^{\infty} R_s(0).$$

This completes the proof. □

6. APPENDIX

6.1. Solution to the Abel Equation (3.6)

Throughout the appendix, we focus the solution to the Abel equation (3.6). This equation reduces to the following canonical form:

$$z = -aR \quad \Longrightarrow \quad W \cdot W'_z - W = \mathcal{F}(z) := \frac{F(\frac{z}{-a})}{-a} \tag{6.1}$$

Now, let us look for the exact analytic solution of (6.1) as normal form of (3.6).

Define a functional transformation as follows:

$$W(z) = W_1(z) \cdot W_2(r), \ r = r(z)$$

which reduces (6.1) to the following

$$W_1 W_2 (\frac{dW_1}{dz} \cdot W_2 + W_1 \cdot \frac{dr}{dz} \frac{dW_2}{dr}) - W_1 W_2 = \mathcal{F}(z). \tag{6.2}$$

Here, the functions W_1, W_2 and r must be specified later at the right moments. In the sequel, to simplify the computations let us define an auxiliary differentiable function h and introduce an equivalent form for (6.2) by

$$(W_1^2 \frac{dr}{dz} W_2 + h) \frac{dW_2}{dr} - 2\mathcal{F} = (-W_1^2 \frac{dr}{dz} W_2 + h) \frac{dW_2}{dr} - 2W_1 \frac{dW_1}{dz} \cdot W_2^2 + 2W_1 W_2. \tag{6.3}$$

Now, indicate the recent equality by $H(z)$, then (6.2) is divided into two Abel equations of the second kind:

$$(W_1^2 \frac{dr}{dz} W_2 + h) \frac{dW_2}{dr} = 2\mathcal{F}(z) + H(z), \tag{6.4}$$

$$(-W_1^2 \frac{dr}{dz} W_2 + h) \frac{dW_2}{dr} - 2W_1 \frac{dW_1}{dz} \cdot W_2^2 + 2W_1 W_2 = H(z). \tag{6.5}$$

It is obvious that both unknown functions h, H must be determined. To do this, we utilize the well-known Julia construction mentioned in Section 2 to Abel's equations (6.4)-(6.5). Simple integrations together with considering $r(z)$ as the identity function implies that

$$h = W_1^2, \quad W_1 = \frac{z + 2C_0}{2} \quad (z \neq -2C_0), \quad C_0 = \text{const.}, \tag{6.6}$$

$$W_2^2 + 2W_2 - 8\int \frac{H(z) + 2\mathcal{F}(z)}{(z+2C_0)^2}dz = 0, \tag{6.7}$$

$$W_2^2 - 2W_2 + \frac{8}{(z+2C_0)^4}\int (z+2C_0)^2 H(z)dz = 0. \tag{6.8}$$

Therefore, it only remains to find the functions $H(z), W_2(z)$. Considering the real roots of (6.7)-(6.8) and solving them for W_2 at the same time we derive the following

$$\sqrt{\lambda^2 - A} = \lambda\sqrt{1+B} - 2\lambda \tag{6.9}$$

where

$$\lambda = (z+2C_0)^2 = (2W_1)^2, \quad A = 8\int \lambda H dz, \quad B = 8\int \frac{H}{\lambda}dz + 16\int \frac{\mathcal{F}}{\lambda}dz. \tag{6.10}$$

Now, one can square and then differentiate (6.9), respectively, to obtain that

$$6\lambda\lambda_z + A_z + 2(1+B)\lambda\lambda_z + \lambda^2 B_z - 8\sqrt{1+B}\lambda\lambda_z - 2\lambda^2\frac{B_z}{\sqrt{1+B}} = 0. \tag{6.11}$$

Mixing (6.10) together with (6.11) implies the following form

$$(1+B)^{\frac{3}{2}} - 4(1+B) + \left[3 + \frac{4(H+\mathcal{F})}{z+2C_0}\right](1+B)^{\frac{1}{2}} - 4\frac{H+2\mathcal{F}}{z+2C_0} = 0. \tag{6.12}$$

Inspiring by the Cardano's substitution

$$(1+B)^{\frac{1}{2}} = \overline{B} + \frac{4}{3} \tag{6.13}$$

we derive a depressed cubic form as follows

$$\overline{B}^3 + p\overline{B} + q = 0, \tag{6.14}$$

where

$$p = -\frac{\mu^2}{3} + \nu, \quad q = 2(\frac{\mu}{3})^3 - \frac{\mu\nu}{3} + \eta, \quad \mu = -4, \quad \nu = 3 - \eta, \quad \eta = -\frac{4(H+\mathcal{F})}{z+2C_0}. \tag{6.15}$$

From the calculus, determined by the discriminant of the cubic equation one can consider the real roots as follows:

$$\Delta = (\frac{p}{3})^3 + (\frac{q}{2})^2. \tag{6.16}$$

Case 1: $\Delta < 0$ $(p < 0)$

$$\begin{gathered} \overline{B}_1 = 2\sqrt{\frac{-p}{3}}\cos\frac{\theta}{3}, \quad \overline{B}_2 = -2\sqrt{\frac{-p}{3}}\cos\frac{\theta-\pi}{3}, \quad \overline{B}_3 = -2\sqrt{\frac{-p}{3}}\cos\frac{\theta+\pi}{3}, \\ \cos\theta = -\frac{q}{2\sqrt{-(\frac{p}{3})^3}}, \ 0 < \theta < \pi. \end{gathered} \tag{6.17}$$

Case 2: $\Delta > 0$

$$\overline{B} = \sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}}. \tag{6.18}$$

Case 3: $\Delta = 0$

$$\overline{B}_1 = 2\sqrt[3]{-\frac{q}{2}}, \quad \overline{B}_2 = \overline{B}_3 = -\sqrt[3]{-\frac{q}{2}}. \tag{6.19}$$

Following the relation (6.13) one can easily find the real roots of Eq. (6.12) by (6.15)-(6.19). Now, considering the definition of B in (6.10) along with the replacement (6.13) we observe that

$$B(z) = 8\int \frac{H}{\lambda}dz + 16\int \frac{\mathcal{F}}{\lambda}dz = (\overline{B} + \frac{4}{3})^2 - 1, \tag{6.20}$$

and linking (6.7) together with (6.10)

$$\sqrt{1+B} = \sqrt{1 + 8\int \frac{H + 2\mathcal{F}}{\lambda}dz} = 1 + W_2, \tag{6.21}$$

and following (6.20) it implies

$$W_2 = \overline{B}(z) + \frac{1}{3} = \sqrt{B(z)+1} - 1, \tag{6.22}$$

which is related to the unknown function $H(z)$. Summing up, one can list the following:

$$\begin{aligned} &W_2 = \overline{B}(z) + \frac{1}{3}, \quad B = 8\int \frac{H + 2\mathcal{F}}{(z+2C_0)^2}dz = \left(\overline{B}(z) + \frac{4}{3}\right)^2 - 1, \\ &\frac{A(z)}{\lambda^2} = \frac{8}{(z+2C_0)^4}\int (z+2C_0)^2 H(z)dz = \left[\overline{B}(z) + \frac{1}{3}\right]\left[\frac{5}{3} - \overline{B}(z)\right] \end{aligned} \tag{6.23}$$

where $\overline{B}(z)$ is given as in Eqs. (6.17)-(6.19) with respect to the unknown function $H(z)$. Now (6.5) together with (6.23) implies that

$$W = W_1 \cdot W_2 = \frac{1}{2}\Big(z + 2C_0\Big)\Big(\overline{B}(z) + \frac{1}{3}\Big) \tag{6.24}$$

where $\overline{B}(z)$ is given as in Eqs. (6.17)-(6.19) and related to unknown parameter η in (6.15). Now, since $W = R'$ and $z = -aR$ one can easily see that

$$-\frac{2}{a}\ln\left|R(t) - \frac{2C_0}{a}\right| = \int_0^t \overline{B}(-aR(s))ds + \frac{1}{3}t + C_1 \tag{6.25}$$

where $\overline{B}$ is given as in Eqs. (6.17)-(6.19).

In order to find the subsidiary function $H(z)$ we first rewrite the Abel equations (6.4) and (6.5) and then by equating the results we obtain the following Riccati equation

$$W_2' = \frac{W_1'}{W_1}W_2^2 - \frac{1}{W_1}W_2 + \frac{\mathcal{F} + H}{W_1^2}. \tag{6.26}$$

It is easily seen that Riccati equation (6.26) can be transformed into (6.14) (or (6.12)) with the help of (6.22) and the following equality which is derived by the second relation in (6.23)

$$\overline{B}'_z = \frac{4(H+2\mathcal{F})}{(z+2C_0)^2[\overline{B}(z)+\frac{4}{3}]}.$$

The Riccati equation (6.26) can be reduced to the normal form by a functional transformation as follows:

$$W_2(z)-1=\overline{B}(z)-\frac{2}{3}=G(\tau),\ \tau=\ln|z+2C_0| \quad\Longrightarrow\quad G'_\tau=G^2-[1-4(\mathcal{F}+H)e^{-\tau}]. \tag{6.27}$$

It is obvious that the following set of equations satisfies the right hand side of (6.27).

$$G(\tau)=\overline{B}(\tau)-\frac{2}{3},\quad \overline{B}'_\tau=\frac{4(H+2\mathcal{F})}{\overline{B}(\tau)+\frac{4}{3}}e^{-\tau}. \tag{6.28}$$

By [32, page 105], it is known that, if Eqs. (6.28) constitute a particular solution of (6.27) then the general solution is as form of below:

$$\begin{aligned} &G_g(\tau)=\overline{B}(\tau)-\frac{2}{3}+A(\tau);\quad A(\tau)=\frac{\omega(\tau)}{C_2-\int_0^\tau\omega(s)ds};\\ &\omega(\tau)=\exp\left\{2\int_0^\tau\left[\overline{B}(s)-\frac{2}{3}\right]ds\right\},\quad \overline{B}'_\tau=\frac{4(H+2\mathcal{F})}{\overline{B}(\tau)+\frac{4}{3}}e^{-\tau}\end{aligned} \tag{6.29}$$

in which C_2 is an integration constant. Following the cubic equation (6.12), it has been checked that the particular solution of the Riccati equation ($(W_2)_p := \overline{B}(\tau)+\frac{1}{3}$) satisfies the Abel equations (6.4) and (6.5). This implies that the Abel equation must be satisfied by the general solution of the Riccati equation, as well. Accordingly, it is interpreted as $A(\tau)$ may vanish. This term is formulated as follows:

$$\lim_{\tau\to\pm\infty}G_g(\tau)=\lim_{\tau\to\pm\infty}\left(\overline{B}(\tau)-\frac{2}{3}\right),\quad \text{i.e.}\quad \lim_{\tau\to\pm\infty}A(\tau)=0. \tag{6.30}$$

The limit equalities (6.30) are given by the following improper integral:

$$\int_{-\infty}^{+\infty}A'_\tau d\tau=0\quad \text{s.t.}\quad A'_\tau=\frac{\omega'_\tau}{C_2-\int_0^\tau\omega(s)ds}+\frac{\omega^2(\tau)}{[C_2-\int_0^\tau\omega(s)ds]^2}, \tag{6.31}$$

hence

$$\int_{-\infty}^{+\infty}\frac{\omega'_\tau}{C_2-\int_0^\tau\omega(s)ds}d\tau+\int_{-\infty}^{+\infty}\frac{\omega^2(\tau)}{[C_2-\int_0^\tau\omega(s)ds]^2}d\tau=0. \tag{6.32}$$

The recent equality holds if both integral terms in (6.32) are identically zero. Therefore, the following statements can be derived

$$\int_{-\infty}^{+\infty}\omega'_\tau d\tau=0,\quad C_2-\int_0^\tau\omega(s)ds\neq0,\quad \int_{-\infty}^{+\infty}\left|\frac{\omega(\tau)}{C_2-\int_0^\tau\omega(s)ds}\right|^2d\tau=0 \tag{6.33}$$

such that the function ω is supposed to be specified. In the following, some mathematical logical statements are provided to facilitate the evaluation of this function. In view of the first equation in (6.33) it is understood that the derivative of ω, i.e., ω'_τ can be taken as odd function. Moreover, considering $C_2=0$ as above, from the second part in (6.33) we see that $\omega\neq0$ and also should be non-singular. The third part of (6.33) shows us that it is appropriate to consider ω as constant or convergent at infinity.

In conclusion, it is more suitable to suppose both functions ω, ω'_τ as semi-constant even and odd, respectively. From the calculus [13, type 3.721.1], comparing the first part of (6.33) and the fact that $\int_{-\infty}^{+\infty} \frac{\sin\tau}{|\tau|} d\tau = 0$ together with the argument as above we suppose that

$$\omega'_\tau = \frac{\sin\tau}{|\tau|} \tag{6.34}$$

which implies that $\omega(\tau) = \mathrm{sgn}(\tau)\mathrm{Si}(\tau)$ where Si stands for the Sine integral given by

$$\mathrm{Si}(\tau) = \int_0^\tau \frac{\sin x}{x} dx = -\frac{\pi}{2} + \sum_{i=1}^{\infty} \frac{(-1)^{i+1}\tau^{2i-1}}{(2i-1)(2i-1)!}.$$

It is clear that ω, ω'_τ hold in all conditions mentioned above and hence following the third part of (6.29) we obtain that

$$\ln\omega(\tau) = 2\int_0^\tau \left[\overline{B}(s) - \frac{2}{3}\right] ds = \ln[\mathrm{sgn}(\tau)\mathrm{Si}(\tau)] \quad \Longrightarrow \quad 2(\overline{B}(\tau) - \frac{2}{3}) = \frac{\sin\tau}{|\tau|\mathrm{sgn}(\tau)\mathrm{Si}(\tau)}. \tag{6.35}$$

Now, considering (6.29) with the values obtained above we observe that

$$G_g(\tau) = \frac{\sin\tau}{2|\tau|\mathrm{sgn}(\tau)\mathrm{Si}(\tau)} + \frac{\mathrm{sgn}(\tau)\mathrm{Si}(\tau)}{C_2 - \int_0^\tau \mathrm{sgn}(s)\mathrm{Si}(s)ds}$$

which satisfies the condition (6.30), that is,

$$\lim_{\tau\to\pm\infty} A(\tau) = \lim_{\tau\to\pm\infty} \frac{\mathrm{sgn}(\tau)\mathrm{Si}(\tau)}{C_2 - \int_0^\tau \mathrm{sgn}(s)\mathrm{Si}(s)ds} = 0.$$

Differentiating from both sides in the second part of (6.35) and then equating with second part of (6.28) we conclude that

$$\frac{4(H + 2\mathfrak{F})}{\overline{B}(\tau) + \frac{4}{3}} e^{-\tau} = \frac{\cos\tau[|\tau|\mathrm{sgn}(\tau)\mathrm{Si}(\tau)] - \sin\tau[\mathrm{Si}(\tau) + \sin\tau]}{2\tau^2\mathrm{Si}^2(\tau)}.$$

This shows that H can be found by the following

$$H = e^\tau \frac{\cos\tau[|\tau|\mathrm{sgn}(\tau)\mathrm{Si}(\tau)] - \sin\tau[\mathrm{Si}(\tau) + \sin\tau]}{8\tau^2\mathrm{Si}^2(\tau)} \left(\frac{\sin\tau}{2|\tau|\mathrm{sgn}(\tau)\mathrm{Si}(\tau)} + 2\right) - 2\mathfrak{F}(\tau) \tag{6.36}$$

where $\tau = \ln|-aR + 2C_0|$.

Remark 6.1. *We note that in order to determine the function H in the process as above one can also utilize the fact that*

$$\int_{-\infty}^{+\infty} \frac{\cos\alpha x}{\beta - x} dx = \pi\sin\alpha\beta, \quad \alpha > 0,\ Im\beta \geq 0$$

[13, type 3.722.8] and consider the following replacement

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{\cos x}{x} dx = 0 \quad &\Longrightarrow \quad \omega'_\tau = \frac{\cos\tau}{\tau} \\ &\Longrightarrow \quad \omega(\tau) = -\int_\tau^{+\infty} \frac{\cos x}{x} dx = Ci(\tau) = \gamma + \ln\tau + \int_0^\tau \frac{\cos t - 1}{t} dt \\ &\Longrightarrow \quad \omega(\tau) = Ci(\tau) = \gamma + \ln\tau + \sum_{k=1}^{\infty} \frac{(-1)^k\tau^{2k}}{(2k)(2k)!}, \end{aligned}$$

where $\gamma \cong 0.57721$ is the Euler-Mascheroni constant, and proceed the process of proof to result another appropriate function H.

6.2. Adomian Decomposition Method for (3.16)

In this section let us focus the solution of Eq. (3.16). The nonlinear operator

$$N(R) = \exp\Big(- \int_0^t \frac{a}{2}\overline{B}(-aR(s))ds \Big)$$

is decomposed as

$$\exp\Big(- \int_0^t \frac{a}{2}\overline{B}(-aR(s))ds \Big) = \sum_{n=0}^{\infty} A_n$$

where A_n are so-called the Adomian polynomials given by the following formula

$$A_m = \left[\frac{1}{m!} \frac{d^m}{d\lambda^m} N(\sum_{i=0}^{m} \lambda^i R_i) \right]_{\lambda=0}.$$

That is,

$$\begin{aligned}
&A_0 = N(R_0) = e^{-\frac{a}{2}\overline{B}(-2C_0)t} \quad \text{where} \quad R_0 = \frac{2C_0}{a},\\
&A_1 = R_1 N'(R_0),\\
&A_2 = R_2 N'(R_0) + \frac{1}{2!} R_1^2 N''(R_0),\\
&A_3 = R_3 N'(R_0) + R_1 R_2 N''(R_0) + \frac{1}{3!} R_1^3 N'''(R_0),\\
&A_4 = R_4 N'(R_0) + (R_1 R_3 + \frac{R_2^2}{2}) N''(R_0) + \frac{R_1^2 R_2}{2} N'''(R_0) + \frac{R_1^4}{4!} N^{(4)}(R_0),\\
&A_5 = R_5 N'(R_0) + (R_1 R_4 + R_3 R_2) N''(R_0) + (\frac{R_2^2 R_1 + R_1^2 R_3}{2}) N'''(R_0) + \frac{R_1^3 R_2}{3!} N^{(4)}(R_0)\\
&+ \frac{R_1^5}{5!} N^{(5)}(R_0),\\
&\vdots
\end{aligned}$$

Now by the ADM we have the recursive scheme as follows

$$\begin{aligned}
&R_0 = \frac{2C_0}{a},\\
&R_1 = e^{-\frac{a}{2}(\frac{t}{3}+C_1)} \cdot A_0 = e^{-\frac{a}{2}(\overline{B}(-2C_0)t+\frac{t}{3}+C_1)},\\
&R_2 = e^{-\frac{a}{2}(\frac{t}{3}+C_1)} \cdot A_1 = -\frac{a}{2}\overline{B}(-2C_0) e^{-a(\overline{B}(-2C_0)t+\frac{t}{3}+C_1)}\\
&R_3 = e^{-\frac{a}{2}(\frac{t}{3}+C_1)} \cdot A_2 = \frac{3a^2}{8}\overline{B}^2(-2C_0) e^{-\frac{3a}{2}(\overline{B}(-2C_0)t+\frac{t}{3}+C_1)}\\
&R_4 = e^{-\frac{a}{2}(\frac{t}{3}+C_1)} \cdot A_3 = -\frac{a^3}{3}\overline{B}^3(-2C_0) e^{-2a(\overline{B}(-2C_0)t+\frac{t}{3}+C_1)}\\
&R_5 = e^{-\frac{a}{2}(\frac{t}{3}+C_1)} \cdot A_4 = \frac{125a^4}{24 \times 16}\overline{B}^4(-2C_0) e^{-\frac{5a}{2}(\overline{B}(-2C_0)t+\frac{t}{3}+C_1)}\\
&R_6 = e^{-\frac{a}{2}(\frac{t}{3}+C_1)} \cdot A_5 = -\frac{1296a^5}{120 \times 32}\overline{B}^5(-2C_0) e^{-3a(\overline{B}(-2C_0)t+\frac{t}{3}+C_1)}\\
&\vdots\\
&R_{k+1} = e^{-\frac{a}{2}(\frac{t}{3}+C_1)} \cdot A_k = \frac{(k+1)^{k-1}(-a)^k}{2^k k!}\overline{B}^k(-2C_0) e^{-\frac{(k+1)a}{2}(\overline{B}(-2C_0)t+\frac{t}{3}+C_1)}.
\end{aligned}$$

Consequently,

$$R(t) = \frac{2C_0}{a} + \sum_{k=0}^{\infty} \frac{(k+1)^{k-1}}{2^k k!} (-a)^k \overline{B}^k (-2C_0) e^{-\frac{(k+1)a}{2}(\overline{B}(-2C_0)t + \frac{t}{3} + C_1)}.$$

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