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RESEARCH ARTICLES

An Analogue of the Titchmarsh Theorem for the Fourier Transform on Locally Compact Vilenkin Groups*

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Abstract—In this paper for functions on locally compact Vilenkin groups, we prove an analogue of one classical Titchmarsh theorem on the image under the Fourier transform of a set of functions satisfying the Lipschitz condition in L^2 .

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Key words: *harmonic analysis on Vilenkin groups, Titchmarsh theorem, modulus of continuity, Fourier transform on groups, Vilenkin groups, Lipschitz condition.*

1. INTRODUCTION AND FORMULATION OF THE MAIN RESULTS

In this article, using the Fourier transform on a locally compact Vilenkin group, we obtain an analogue of one classical Titchmarsh theorem on description of the image under the Fourier transform of a class of functions satisfying the Lipschitz condition in L^2. We now give the exact statement of this theorem.

Suppose that $f(x)$ is a function in the $L^2(\mathbb{R})$ space (all functions below are complex-valued), $\|\cdot\|_{L^2(\mathbb{R})}$ is the norm of $L^2(\mathbb{R})$, and α is an arbitrary number in the interval $(0, 1)$.

Definition 1.1. *A function $f(x)$ belongs to the Lipschitz class $Lip(\alpha, 2)$ if*

$$\|f(x-t) - f(x)\|_{L^2(\mathbb{R})} = O(t^\alpha)$$

as $t \to 0$.

Theorem 1.2 ([1, Theorem 85]). *If $f(x) \in L^2(\mathbb{R})$ and $\widehat{f}(\lambda)$ is its Fourier transform then the conditions*

$$f \in Lip(\alpha, 2), \qquad 0 < \alpha < 1,$$

and

$$\int\limits_{|\lambda|\geq r} |\widehat{f}(\lambda)|^2 \, d\lambda = O(r^{-2\alpha})$$

as $r \to \infty$ are equivalent.

There are many analogues of Theorem **1.2**: for the Fourier transform on noncompact Riemannian rang 1 symmetric spaces, in particular for the Fourier transform on the Lobachevsky plane; for the Fourier-Jacobi transform; for the Fourier-Dunkl transform and etc. (for example, see [2–5]). For the Fourier transform on the group $\mathbb{Q}_p$ of p-adic numbers an analogue of Theorem **1.2** was proved in [6]. In this paper we obtain an analogue of Theorem **1.2** for the Fourier transform on an arbitrary locally

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compact Vilenkin group. Let us present necessary definitions from harmonic analysis on locally compact Abelian groups (see, for example, [7] and [8]).

Let G be a locally compact Abelian group. A character of G is a continuous complex-valued function $\chi(x)$ on G such that $|\chi(x)| = 1$ and $\chi(x+y) = \chi(x)\chi(y)$ for any $x, y \in G$. Let Γ be the set of all characters of G. The set Γ equipped with the compact-open topology and the operation of point-wise multiplication of characters becomes an LCA-group which is said to be the dual group of G. We note that the group operation in the group G is always written additively and the operation in the dual group Γ is written multiplicatively.

Definition 1.3. *A locally compact Abelian group G is said to be a locally compact Vilenkin group if there exists a strictly decreasing sequence of compact open subgroups $\{G_n\}_{n\in\mathbb{Z}}$ such that $\bigcup_{n\in\mathbb{Z}} G_n = G$ and $\bigcap_{n\in\mathbb{Z}} G_n = \{0\}$.*

The factor group G_n/G_{n+1} is a finite Abelian group. Let d_n be the order of the group G_n/G_{n+1}, then $d_n \geq 2$. Note that in the definition of a Vilenkin group is often added the condition $\sup\{d_n : n \in \mathbb{Z}\} < \infty$ (see, for example, [9, 10]), but in the present paper this condition is not required. Examples of locally compact Vilenkin groups are the group $\mathbb{Q}_p$ of p-adic numbers and, more generally, the additive group K^+ of any local field K (see [11]), the groups $\mathbb{Q}_p^d = \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p$ (d times) and $(K^+)^d = K^+ \times \cdots \times K^+$.

We note, that in the papers on the wavelet theory on groups is often used the following definition of Vilenkin group (see, for example, [12–14]). Let $m \geq 2$ be integer and $\mathbb{Z}/m\mathbb{Z}$ be the additive groups of integers modulo m. The Vilenkin group G consists of the sequences $x = (x_j)$, where $j \in \mathbb{Z}$, $x_j \in \mathbb{Z}/m\mathbb{Z}$ and there exists at most a finite number of negative j such that $x_j \neq 0$. The group operation on G is defined as the coordinatewise addition modulo m. The topology on G is introduced via the complete system of neighborhoods of zero

$$G_n = \{(x_j) \in G : x_j = 0 \text{ for } j \leq n\}, \quad n \in \mathbb{Z}.$$

For the case $m = 2$ the group G is the locally compact Cantor dyadic group.

These Vilenkin groups are the special case of the Vilenkin groups in sence of Definition **1.3**.

In what follows G is an arbitrary locally compact Vilenkin group, Γ its dual group. For any $n \in \mathbb{Z}$ let Γ_n be the annihilator of G_n, that is

$$\Gamma_n = \{\chi \in \Gamma : \chi(x) = 1 \text{ for any } x \in G_n\}.$$

It follows from the properties of dual groups and the annihilators of subgroups (see [7, (23.24), (23.29)]) that Γ_n is a compact open subgroup of Γ, the sequence of subgroups $\{\Gamma_n\}_{n\in\mathbb{Z}}$ is strictly increasing, $\bigcap_{n\in\mathbb{Z}} \Gamma_n = \{1\}$ and $\bigcup_{n\in\mathbb{Z}} \Gamma_n = \Gamma$.

We choose Haar measures dx on G and $d\chi$ on Γ so that

$$\int\limits_{G_0} dx = \int\limits_{\Gamma_0} d\chi = 1.$$

We denote by $\mu(A)$ the Haar measure of a subset $A \subset G$, and by $\lambda(B)$ the Haar measure of a subset $B \subset \Gamma$.

For every $n \in \mathbb{Z}$ we define the number m_n by

$$m_n := \begin{cases} d_1 d_2 \ldots d_n & \text{if } n > 0, \\ 1 & \text{if } n = 0, \\ d_0^{-1} d_{-1}^{-1} \ldots d_{-n+1}^{-1} & \text{if } n < 0. \end{cases} \tag{1.1}$$

Then

$$\mu(G_n) = \frac{1}{m_n}, \quad \lambda(\Gamma_n) = m_n. \tag{1.2}$$

Let $L^p(G)$, $1 \le p < \infty$, be a Banach space of all measurable $\mathbb{C}$-valued functions $f(x)$ on G with finite norm

$$\|f\|_p = \|f\|_{L^p(G)} := \left(\int\limits_G |f(x)|^p \, dx \right)^{1/p}.$$

Similarly, let $L^p(\Gamma)$ be a Banach space of all measurable $\mathbb{C}$-valued functions $g(\chi)$ on Γ with finite norm

$$\|g\|_p = \|g\|_{L^p(\Gamma)} := \left(\int\limits_\Gamma |g(\chi)|^p \, d\chi \right)^{1/p}.$$

As usual, functions from the spaces L^p are considered up to their values on a set of measure 0.

For any function $f(x) \in L^1(G)$, by the Fourier transform of f we mean the function $\widehat{f}(\xi)$ on Γ defined by the formula

$$\widehat{f}(\chi) := \int\limits_G f(x)\, \chi(x)\, dx, \qquad \chi \in \Gamma. \tag{1.3}$$

If $f \in L^2(G)$, then its Fourier transform $\widehat{f}(\xi)$ can be defined as the limit in $L^2(G)$ of a sequence of the functions

$$\widehat{f}_n(\chi) := \int\limits_{G_n} f(x)\, \chi(x)\, dx \tag{1.4}$$

as $n \to \infty$. The Fourier transform $F : f(x) \mapsto \widehat{f}(\chi)$ is a linear isomorphism of the space $L^2(G)$ into the space $L^2(\Gamma)$, and for any function $f \in L^2(G)$ we have the Parseval's identity

$$\|F(f)\|_{L^2(\Gamma)} = \|f\|_{L^2(G)}. \tag{1.5}$$

For a function $f(x)$ on G and for any $h \in G$ let

$$(\tau_h f)(x) := f(x - h). \tag{1.6}$$

The operator τ_h is called the translation operator. If $f \in L^2(G)$ and $F(f)(\chi) = \widehat{f}(\chi)$ is its Fourier transform, then we have:

$$F(\tau_h f)(\chi) = \chi(h)\, \widehat{f}(\chi). \tag{1.7}$$

For $f \in L^2(G)$ and $n \in \mathbb{N}$ let

$$\omega_2(f; n) := \sup\{\|f - \tau_h f\|_2 : h \in G_n\}. \tag{1.8}$$

The sequence of numbers $\{\omega_2(f; n)\}_{n \in \mathbb{N}}$ is called the modulus continuity of f in the space $L^2(G)$.

Let $\omega = \{\omega_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers monotonously decreasing to zero (that is (i) $\omega_n \ge 0$; (ii) $\omega_n \ge \omega_{n+1}$ $\quad \forall n \in \mathbb{N}$; (iii) $\omega_n \to 0$ as $n \to \infty$).

Definition 1.4. *A function $f(x)$ belongs to the space $H_2^\omega(G)$, if $f \in L^2(G)$ and for some constant $c = c(f) > 0$ we have*

$$\omega_2(f; n) \le c\, \omega_n, \qquad n \in \mathbb{N}. \tag{1.9}$$

Let $\omega = \{\omega_n\}_{n \in \mathbb{N}}$ and $\omega' = \{\omega'_n\}_{n \in \mathbb{N}}$ be sequences of real numbers monotonously decreasing to zero. The sequences ω and ω' will be called equivalent if we have

$$c_1\, \omega_n \le \omega'_n \le c_2\, \omega_n, \qquad n \in \mathbb{N}$$

for some positive constants c_1 and c_2. It can be proved (see section 2) that for any nonzero sequence ω the space $H_2^\omega(\mathbb{Q}_p)$ is nonzero, and $H_2^\omega(G_p) = H_2^{\omega'}(G)$ if and only if the sequences ω and ω' are equivalent.

The main results of the paper are the next theorems.

Theorem 1.5. *For every $f \in L^2(G)$ we have the inequality*

$$\left(\int\limits_{\Gamma\setminus\Gamma_n} |\widehat{f}(\chi)|^2\,d\chi\right)^{1/2} \le \frac{1}{\sqrt{2}}\omega_2(f;n), \qquad n \in \mathbb{N}, \tag{1.10}$$

where constant $\frac{1}{\sqrt{2}}$ in (1.10) *is exact.*

The following theorem is an analogue of the Tichmarsh theorem.

Theorem 1.6. *Let $\omega = \{\omega_n\}_{n\in\mathbb{N}}$ be any sequence of real numbers monotonously decreasing to zero. Then the next conditions are equivalent:*

$$f \in H_2^{\omega}(G) \tag{1.11}$$

and

$$\left(\int\limits_{\Gamma\setminus\Gamma_n} |\widehat{f}(\chi)|^2\,d\chi\right)^{1/2} \le c\,\omega_n, \qquad n \in \mathbb{N}, \tag{1.12}$$

where $c = c(f)$ is some positive constant.

For the case when G is the group of p-adic numbers the Theorems **1.5** and **1.6** were proved in [6]. Also we note that in the special case when G is the Cantor dyadic group and $\omega_n = 2^{-\alpha n}$, $\alpha > 0$, the results of the Theorem **1.6** follow from the description of Lipschitz classes in terms of the best approximations of functions by Walsh polynomials (see, for example, [15], p. 189).

2. PROOFS OF THEOREMS 1.5 AND 1.6

Lemma 2.1. *Let χ be a character of group G, $n \in \mathbb{Z}$. Then*

$$\int\limits_{G_n} \chi(x)\,dx = \begin{cases} \mu(G_n), & \text{if } \chi \in \Gamma_n, \\ 0, & \text{if } \chi \notin \Gamma_n. \end{cases}$$

Proof.

Let $I_n = \int\limits_{G_n} \chi(x)\,dx$. If $\chi \in \Gamma_n$ then

$$I_n = \int\limits_{G_n} 1\,dx = \mu(G_n).$$

If $\chi \notin \Gamma_n$ then $\chi(x_0) \ne 1$ for some element $x_0 \in G_n$. It follows from invariance of the Haar measure that

$$\int\limits_{G_n} \chi(x)\,dx = \int\limits_{G_n} \chi(x + x_0)\,dx = \int\limits_{G_0} \chi(x)\,\chi(x_0)\,dx = \chi(x_0)\int\limits_{G_n} \chi(x)\,dx.$$

Then $I_n = \chi(x_0)I_n$ and hence $I_n = 0$. □

Proof of Theorem **1.5**

1) Let $f \in L^2(G)$, $h \in G_n$, $n \in \mathbb{N}$. By definition of the modulus of continuity we have

$$\omega_2(f;n) := \sup\{\|f - \tau_h f\|_2 : h \in G_n\}. \tag{2.1}$$

It follows from (1.7) that

$$F(f - \tau_h f)(\xi) = (1 - \chi_p(\xi h))\,\widehat{f}(\xi), \tag{2.2}$$

then, using the Parseval's identity (1.5), we have

$$\|f - \tau_h f\|_2^2 = \int_{\Gamma} |1 - \chi(h)|^2 \, |\widehat{f}(\chi)|^2 \, d\chi. \tag{2.3}$$

If $\chi \in \Gamma_n$, $h \in G_n$, then $\chi(h) = 1$. Hence, equality (2.3) can be rewritten in the form

$$\|f - \tau_h f\|_2^2 = \int_{\Gamma \setminus \Gamma_n} |1 - \chi(h)|^2 \, |\widehat{f}(\chi)|^2 \, d\chi. \tag{2.4}$$

Integrating the equality (2.4) with respect to $h \in G_n$, we obtain

$$\int_{G_n} \|f - \tau_h f\|_2^2 \, dh = \int_{\Gamma \setminus \Gamma_n} \Big(\int_{G_n} |1 - \chi(h)|^2 \, dh \Big) |\widehat{f}(\chi)|^2 \, d\chi. \tag{2.5}$$

It follows from $|\chi(h)| = 1$ that

$$|1 - \chi(h)|^2 = 2 - 2 \, \mathrm{Re}\, \chi(h). \tag{2.6}$$

It follows from Lemma **2.1** that

$$\int_{G_n} \chi(h) \, dh = 0 \quad \text{if } \chi \in \Gamma \setminus \Gamma_n, \tag{2.7}$$

hence it follows from (2.6) and (2.7) that

$$\int_{G_n} |1 - \chi(h)|^2 \, dh = \int_{G_n} (2 - \mathrm{Re}\, \chi(h)) \, dh = 2 \int_{G_n} dh = 2\mu(G_n). \tag{2.8}$$

From (2.5) and (2.8) it follows that

$$\int_{G_n} \|f - \tau_h f\|_2^2 \, dh = 2\mu(G_n) \int_{\Gamma \setminus \Gamma_n} |\widehat{f}(\chi)|^2 \, d\chi. \tag{2.9}$$

On the other hand, since $\|f - \tau_h f\|_2 \le \omega_2(f; n)$ for $h \in G_n$, then

$$\int_{G_n} \|f - \tau_h f\|_2^2 \, dh \le (\omega_2(f; n))^2 \int_{G_n} dh = \mu(G_n) \ (\omega_2(f; n))^2 \, . \tag{2.10}$$

It follows from (2.9) and (2.10) that

$$2\mu(G_n) \int_{\Gamma \setminus \Gamma_n} |\widehat{f}(\chi)|^2 \, d\chi \le \mu(G_n) \ (\omega_2(f; n))^2 \, ,$$

which implies that the inequality (1.10) holds.

2) We claim that the constant $\frac{1}{\sqrt{2}}$ in (1.10) is exact.

For any $n \in \mathbb{Z}$ and $a \in G$ let $G_n(a) := a + G_n = \{x \in G : x - a \in G_n\}$. In particular, $G_n(0) = G_n$. For every $s \in \mathbb{N}$ we define the function φ_s on G by

$$\varphi_s(x) := \begin{cases} 1 & \text{if } x \in G_s, \\ 0 & \text{if } x \notin G_s, \end{cases}$$

that is, φ_s is the characteristic function of the subset G_s. Then $\|\varphi_s\|_2^2 = \mu(G_s)$ and

$$(\tau_h \varphi_s)(x) = \varphi_s(x - h) = \begin{cases} 1 & \text{if } x \in G_s(h), \\ 0 & \text{if } x \notin G_s(h). \end{cases}$$

Note that $G_s(h) = G_s$ if $h \in G_s$ and $G_s(h) \cap G_s = \emptyset$ if $h \notin G_s$, which implies that

$$\|\varphi_s - \tau_h\varphi_s\|_2^2 = \begin{cases} 0 & \text{if } h \in G_s, \\ 2\mu(G_s) & \text{if } h \notin G_s. \end{cases} \tag{2.11}$$

It follows from the definition of the modulus continuity (see (1.8)) and from (2.11) that

$$(\omega_2(\varphi_s; n))^2 = \begin{cases} 0 & \text{if } n \geq s, \\ 2\mu(G_s) & \text{if } n < s. \end{cases} \tag{2.12}$$

If $n \geq s$ then it follows from (2.11) that

$$\int\limits_{G_n} \|\varphi_s - \tau_h\varphi_s\|_2^2 \, dh = 0, \tag{2.13}$$

and if $n < s$, taking into account (2.12), we have

$$\begin{aligned} \int\limits_{G_n} \|\varphi_s - \tau_h\varphi_s\|_2^2 \, dh &= \int\limits_{G_n \setminus G_s} 2\mu(B_{-s}) \, dh \\ &= 2\mu(G_s)(\mu(G_n) - \mu(G_s)) = (\omega_2(\varphi_s; n))^2 \, (\mu(G_n) - \mu(G_s)). \end{aligned} \tag{2.14}$$

On the other hand, it follows from (2.9) that

$$\int\limits_{G_n} \|\varphi_s - \tau_h\varphi_s\|_2^2 \, dh = 2\mu(G_n) \int\limits_{\Gamma \setminus \Gamma_n} |\widehat{\varphi}_s(\chi)|^2 \, d\chi,$$

which implies, taking to account (2.13), (2.14) and (1.2), that

$$\int\limits_{\Gamma \setminus \Gamma_n} |\widehat{\varphi}_s(\chi)|^2 \, d\chi = \begin{cases} \frac{1}{2}\left(1 - \frac{m_n}{m_s}\right) (\omega_2(\varphi_s; n))^2 & \text{if } n < s, \\ 0 & \text{if } n \geq s, \end{cases} \tag{2.15}$$

where m_n and m_s are defined in (1.1). Since $\frac{m_n}{m_s} \leq 2^{n-s}$, then it follows from (2.15) that, for any $n \in \mathbb{N}$ and $\varepsilon > 0$, for sufficiently large s we have the inequality

$$\left(\int\limits_{\Gamma \setminus \Gamma_n} |\widehat{\varphi}_s(\chi)|^2 \, d\chi \right)^{1/2} \geq \frac{1}{\sqrt{2}}(1 - \varepsilon) \, \omega_2(\varphi_s; n),$$

which implies that the constant $\frac{1}{\sqrt{2}}$ in (1.10) is exact. □

We note that the proof of the inequality (1.10) from Theorem **1.5** is similar to the proof of N. Ya. Vilenkin and A. I. Rubinstein in [16], where they proved an analogue of some S. B. Stechkin inequality for Fourier-Vilenkin series on zero-dimensional compact Abelian groups (see, also, [17, Th. 4.3]).

Proposition 2.1. *Let $\{\omega_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers monotonously decreasing to zero. Then there exists a function $f \in L^2(G)$ such that*

$$\omega_2(f; n) = \omega_n \qquad \forall n \in \mathbb{N}. \tag{2.16}$$

Proof.

A compact Abelian group U is said to be a compact Vilenkin group if there exists a strictly decreasing sequence of compact open subgroups $\{U_n\}_{n\in\mathbb{N}}$ such that $\bigcap_{n=1}^{\infty} U_n = \{0\}$. Using the system $\{U_n\}_{n\in\mathbb{N}}$, for any function $g \in L^2(U)$ its modulus of continuity defines as in (1.8), that is,

$$\omega_2(g; n) := \sup\{\|g - \tau_h g\|_2 : h \in U_n\},$$

where τ_h is is the translation operator (see (1.6)).

It was proved by Rubinstein [18] that for any compact Vilenkin group U and for any sequence $\{\omega_n\}_{n\in\mathbb{N}}$ of real numbers monotonously decreasing to zero, there exists a function $g \in L^2(U)$ such that $\omega_2(g;n) = \omega_n$ for any $n \in \mathbb{N}$. If G is a locally compact Vilenkin group then its subgroup G_0 is a compact Vilenkin group with the sequence of subgroups $\{G_n\}_{n\in\mathbb{N}}$. Then there exists a function $g \in L^2(G_0)$ such that $\omega_2(g;n) = \omega_n$ for any $n \in \mathbb{N}$.

We define a function f on G by the formula

$$f(x) = \begin{cases} g(x) & \text{if } x \in G_0, \\ 0 & \text{if } x \notin G_0. \end{cases}$$

Then $f \in L^2(G)$ and $\omega_2(f;n) = \omega_n$ for any $n \in \mathbb{N}$. □

Corollary 2.1. *For any nonzero sequence $\{\omega_n\}_{n\in\mathbb{N}}$ of real numbers monotonously decreasing to zero the space $H_2^\omega(G)$ is nonzero.*

Proposition 2.2. *Let $\omega = \{\omega_n\}_{n\in\mathbb{N}}$ and $\omega' = \{\omega'_n\}_{n\in\mathbb{N}}$ be sequences of real numbers monotonously decreasing to zero. Then $H_2^\omega(G) = H_2^{\omega'}(G)$ if and only if the sequences ω and ω' are equivalent.*

Proof. It is obvious that if the sequences ω are ω' are equivalent then $H_2^\omega(G) = H_2^{\omega'}(G)$. Suppose that the sequences ω and ω' are not equivalent. For definiteness let $\sup\{\frac{\omega_n}{\omega'_n} : n \in \mathbb{N}\} = +\infty$ (we assume that $\frac{0}{0} = 0$ and $\frac{a}{0} = +\infty$ if $a > 0$). By Proposition 2.1 there exists a function $f \in L^2(G)$ such that $\omega_2(f;n) = \omega_n$ for any $n \in \mathbb{N}$. It is obvious that $f \in H_2^\omega(G)$. Suppose that $f \in H_2^{\omega'}(G)$, then we have $\omega_n = \omega_2(f;n) \le c\omega'_n$, $n \in \mathbb{N}$, whence $\omega_n/\omega'_n \le c$, which is impossible. Hence $f \notin H_2^{\omega'}(G)$ and $H_2^\omega(G) \neq H_2^{\omega'}(G)$.

Proof of Theorem **1.6**

It follows from Theorem **1.5** that (1.11) entails (1.12).

Let $f \in L^2(G)$ and we assume that (1.12) holds. Arguing as in the proof of Theorem **1.5**, we obtain that for any $h \in G_n$ the equality (2.4) holds. It follows from (2.4), using the inequalities $|1 - \chi(h)| \le 2$ and (1.12), that

$$\|f - \tau_h f\|_2^2 = \int\limits_{\Gamma\setminus\Gamma_n} |1 - \chi(h)|^2 |\widehat{f}(\chi)|^2 \, d\chi \le 4 \int\limits_{\Gamma\setminus\Gamma_n} |\widehat{f}(\chi)|^2 \, d\chi \le 4c^2\omega_n^2 \tag{2.17}$$

for $h \in G_n$, $n \in \mathbb{N}$. Taking in (2.17) the supremum over all $h \in G_n$, we obtain that

$$\omega_2(f;n) \le 2c\omega_n, \qquad n \in \mathbb{N},$$

that is the condition (1.11) holds.

Hence the conditions (1.11) and (1.12) are equivalent. □

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