ISSN 2070-0466, p-Adic Numbers, Ultrametric Analysis and Applications, 2017, Vol. 9, No. 1, pp. 15–21. © Pleiades Publishing, Ltd., 2017.

RESEARCH ARTICLES

The p-Adic Order of the k-Fibonacci and k-Lucas Numbers*

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Received October 1, 2016

Abstract—Let $(F_{k,n})_n$ and $(L_{k,n})_n$ be the k-Fibonacci and k-Lucas sequence, respectively, which satisfies the same recursive relation $a_{n+1} = ka_n + a_{n-1}$ with initial values $F_{k,0} = 0$, $F_{k,1} = 1$, $L_{k,0} = 2$ and $L_{k,1} = k$. In this paper, we characterize the p-adic orders $\nu_p(F_{k,n})$ and $\nu_p(L_{k,n})$ for all primes p and all positive integers k.

DOI: 10.1134/S2070046617010022

Key words: *k-Fibonacci sequence, p-adic order, rank of apparition of prime p in the k-Fibonacci sequence.*

1. INTRODUCTION

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. Let k be a positive integer and denote $(F_{k,n})_{n\geq 0}$, the *k-Fibonacci sequence* whose terms satisfy the recurrence relation

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2}, \tag{1.1}$$

with initial conditions $F_{k,0} = 0$ and $F_{k,1} = 1$. In the same way, the companion *k-Lucas sequence* $(L_{k,n})_{n\geq 0}$ is defined by satisfying the same recursive relation with initial values $L_{k,0} = 2$ and $L_{k,1} = k$.

The above sequences are among the several generalizations of Fibonacci and Lucas numbers (case $k = 1$) and they were extensively studied in the series of papers due to Falcon and Plaza [2–5].

The p-adic order, $\nu_p(r)$, of r is the exponent of the highest power of a prime p which divides r. The p-adic order of Fibonacci numbers and Lucas numbers was completely characterized, see [7–9, 14].

In this paper we characterized the p-adic order of the k-Fibonacci numbers and k-Lucas numbers. Our main results are

Theorem 1.1. *(i) If k is an even integer, then*

$$\nu_2(L_{k,n}) = \begin{cases} 1, & \text{if } n \equiv 0 \pmod 2; \\ \nu_2(k), & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

(ii) If k is an odd integer, then

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$$\nu_2(L_{k,n}) = \begin{cases} 0, & \text{if } n \equiv 1,2 \pmod 3; \\ 2, & \text{if } n \equiv 3 \pmod 6; \\ 1, & \text{if } n \equiv 0 \pmod 6. \end{cases}$$

Theorem 1.2. *(i) If k is an even integer, then*

$$\nu_2(F_{k,n}) = \begin{cases} 0, & \text{if } n \equiv 1 \pmod 2; \\ \nu_2(k) + \nu_2(n) - 1, & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

(ii) If k is an odd integer, then

$$\nu_2(F_{k,n}) = \begin{cases} 0, & \text{if } n \equiv 1,2 \pmod 3; \\ 1, & \text{if } n \equiv 3 \pmod 6; \\ 3, & \text{if } n \equiv 6 \pmod{12}; \\ \nu_2(n) + 2, & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 1.3. *For $p \neq 2$ and $p \mid k^2 + 4$, we have that $\nu_p(F_{k,n}) = \nu_p(n)$ and $\nu_p(L_{k,n}) = 0$.*

Let $z := z(k,p)$ be the smallest positive index for which $F_{k,n} \equiv 0 \pmod p$. This index is often called the *rank of apparition of* p in the k-Fibonacci sequence (the z function for $k = 1$ was extensively studied in the series of papers [10–13]). The order of p in $F_{k,z}$ will be denoted by $e := e(k,p)$.

Theorem 1.4. *For $p \neq 2$ and $p \nmid k^2 + 4$, we have that*

(i) $$\nu_p(F_{k,n}) = \begin{cases} \nu_p(n) + \nu_p(F_{k,z}), & \text{if } n \equiv 0 \pmod z; \\ 0, & \text{if } n \not\equiv 0 \pmod z. \end{cases}$$

(ii) $$\nu_p(L_{k,n}) = \begin{cases} \nu_p(n) + \nu_p(F_{k,z}), & \text{if } n \not\equiv 0 \pmod z \text{ and } 2n \equiv 0 \pmod z; \\ 0, & \text{otherwise}. \end{cases}$$

2. AUXILIARY RESULTS

In this section, we shall provide some useful results in order to prove the theorems. The first result provides an addition formula for k-Fibonacci numbers (see [2]).

Lemma 2.1. $F_{k,n+m} = F_{k,n+1}F_{k,m} + F_{k,n}F_{k,m-1}$.

By using Lemma 2.1 is possible to show the next lemma.

Lemma 2.2. *If $n \mid m$ then $F_{k,n} \mid F_{k,m}$.*

Lemma 2.3. *$F_{k,n} \equiv 0 \pmod p$ if and only if $z \mid n$, where $z := z(k,p)$.*

Proof. If $z \mid n$ then we get that $F_{k,z} \mid F_{k,n}$, by Lemma 2.2. Since $p \mid F_{k,z}$, we have that $p \mid F_{k,n}$. Conversely, if $z \nmid n$, thus $n = tz + r$ for some integer t and $0 \leq r < z$, by using the addition formula of Lemma 2.1, we get that $F_{k,tz+1}F_{k,r} \equiv 0 \pmod{p}$. Since $\gcd(F_{k,tz+1}, F_{k,tz}) = 1$, we infer that $p \mid F_{k,r}$ and by the minimality of z we arrive at $r = 0$. □

Lemma 2.4. *It holds that*

$$F_{k,an} \equiv aF_{k,n}F_{k,n+1}^{a-1} \pmod{F_{k,n}^2}$$

and

$$F_{k,an+1} \equiv F_{k,n+1}^{a} \pmod{F_{k,n}^2},$$

where a is a positive integer.

Proof. Firstly, by the recurrence relation we get that $F_{k,n+1} \equiv F_{k,n-1} \pmod{F_{k,n}}$. Then, by using the addition formula of Lemma 2.1, we obtain

$$F_{k,2n} = F_{k,n}F_{k,n+1} + F_{k,n-1}F_{k,n} \equiv 2F_{k,n}F_{k,n+1} \pmod{F_{k,n}^2}.$$

Similarly,

$$F_{k,2n+1} = F_{k,n+1}^2 + F_{k,n}^2 \equiv F_{k,n+1}^2 \pmod{F_{k,n}^2}.$$

Now, suppose that

$$F_{k,an} \equiv aF_{k,n}F_{k,n+1}^{a-1} \pmod{F_{k,n}^2}$$

and

$$F_{k,an+1} \equiv F_{k,n+1}^{a} \pmod{F_{k,n}^2},$$

are true.Then let us prove for the case $a+1$.

$$\begin{aligned} F_{k,(a+1)n} &= F_{k,an+n} = F_{k,an+1}F_{k,n} + F_{k,an}F_{k,n-1} \\ &\equiv F_{k,n+1}^{a}F_{k,n} + aF_{k,n}F_{k,n+1}^{a-1}F_{k,n-1} \pmod{F_{k,n}^2} \\ &\equiv (a+1)F_{k,n}F_{k,n+1}^{a} \pmod{F_{k,n}^2}, \end{aligned}$$

and similarly to $F_{k,(a+1)n+1}$. □

Lemma 2.5. *If p is an odd prime, then*

(i) $L_{k,p} \equiv k \pmod{p}$;

(ii) $F_{k,p} \equiv \left(\frac{k^2+4}{p}\right) \pmod{p}$, *where, as usual,* $\left(\frac{\cdot}{p}\right)$ *denotes the Legendre symbol.*

Proof. By using the combinatorial formulas for k-Lucas number and k-Fibonacci number (see [5]) we get that

$$\begin{aligned} 2^{p-1}L_{k,p} &= \sum_{i=0}^{\lfloor \frac{p}{2} \rfloor} \binom{p}{2i} k^{p-2i}(k^2+4)^i \\ &\equiv k^p \equiv k \pmod{p}, \end{aligned}$$

where in the last congruence we used the Fermat Little Theorem. Then, since $p > 2$ we get that $L_{k,p} \equiv k \pmod{p}$.

Similarly, by Euler criterion, we get that

$$2^{p-1}F_{k,p} = \sum_{i=0}^{\frac{p-1}{2}} \binom{p}{2i+1} k^{p-1-2i}(k^2+4)^i$$

$$\equiv \ (k^2+4)^{\frac{p-1}{2}} \equiv \left(\frac{k^2+4}{p}\right) \pmod p,$$

then, by Fermat Little Theorem we arrive at $F_{k,p} \equiv \left(\frac{k^2+4}{p}\right) \pmod p$. □

Lemma 2.6. $F_{k,an} = 2^{1-a}F_{k,n}(MF_{k,n}^2 + aL_{k,n}^{a-1})$*, where* M *is an integer.*

Proof. By solving the Binet's formulas $F_{k,n} = (\sigma_1^n - \sigma_2^n)/(\sigma_1 - \sigma_2)$ and $L_{k,n} = \sigma_1^n + \sigma_2^n$ (see [2, 5]) for σ_1^n and σ_2^n in terms of $F_{k,n}$ and $L_{k,n}$ (where σ_1 and σ_2 satisfy $x^2 = kx + 1$), expanding $(\sigma_1^n)^a$ and $(\sigma_2^n)^a$ by the binomial theorem, and recombining, we obtain the relation

$$\begin{aligned} F_{k,an} &= (\sigma_1^{an} - \sigma_2^{an})/(\sigma_1 - \sigma_2) \\ &= 2^{-a}[(L_{k,n} + (\sigma_1 - \sigma_2)F_{k,n})^a - (L_{k,n} - (\sigma_1 - \sigma_2)F_{k,n})^a]/(\sigma_1 - \sigma_2) \\ &= 2^{1-a}\sum_{j\ odd}\binom{a}{j}(\sigma_1 - \sigma_2)^{(j-1)/2}F_{k,n}^j L_{k,n}^{a-j} = 2^{1-a}F_{k,n}(MF_{k,n}^2 + aL_{k,n}^{a-1}), \end{aligned}$$

where M is an integer depending on k, a and n. □

Now, we are ready to deal with the proof of the theorems.

3. PROOF OF THEOREM 1.1

(i) Let's prove simultaneously that

$$\nu_2(L_{k,2m}) = 1 \text{ and } \nu_2(L_{k,2m+1}) = \nu_2(k)$$

by induction on m.

When $m = 0$ then $\nu_2(L_{k,0}) = \nu_2(2) = 1$ and $\nu_2(L_{k,1}) = \nu_2(k)$. Now assuming that the result is true for all $l < m$ and knowing that $\nu_2(a+b) = \min\{\nu_2(a), \nu_2(b)\}$ if $\nu_2(a) \neq \nu_2(b)$, we have by recurrence relation, that

$$\nu_2(L_{k,2m}) = \nu_2(kL_{k,2(m-1)+1} + L_{k,2(m-1)}).$$

Now, by the induction hypothesis and since k is even

$$\nu_2(kL_{k,2(m-1)+1}) = \nu_2(k) + \nu_2(L_{k,2(m-1)+1}) = 2\nu_2(k) \geq 2 > \nu(L_{k,2(m-1)}) = 1,$$

then

$$\nu_2(L_{k,2m}) = \nu_2(L_{k,2(m-1)}) = 1. \tag{3.1}$$

Since $\nu_2(kL_{k,2m}) = \nu_2(k) + \nu_2(L_{k,2m}) = \nu_2(k) + 1$ by (3.1) and by induction hypothesis $\nu_2(L_{k,2(m-1)+1}) = \nu_2(k)$, we have that

$$\begin{aligned} \nu_2(L_{k,2m+1}) &= \nu_2(kL_{k,2m} + L_{k,2(m-1)+1}) \\ &= \nu_2(L_{k,2(m-1)+1}) = \nu_2(k). \end{aligned}$$

This concludes the proof when k is even.

(ii) Now, if k is an odd integer, it is equivalent to prove that

$$\nu_2(L_{k,6m+i}) = \begin{cases} 0, & \text{if } i = 1, 2, 4, 5, \\ 1, & \text{if } i = 0, \\ 2, & \text{if } i = 3. \end{cases} \tag{3.2}$$

We proceed by simultaneously induction on m. When $m = 0$ the results are easily checked using that k is odd. Now, we assume that the equation (3.2) is true for all $l < m$. To prove that (3.2) is true for m we need to use the recurrence relations until to be able to use the fact that $\nu_2(a+b)=\min\{\nu_2(a),\nu_2(b)\}$ if $\nu_2(a)\neq\nu_2(b)$ and the induction hypotheses. Therefore,

$$\begin{aligned}\nu_2(L_{k,6m}) &= \nu_2((k^2+1)L_{k,6(m-1)+4}+kL_{k,6(m-1)+3})\\ &= \nu_2((k^2+1)L_{k,6(m-1)+4})=1,\end{aligned}$$

because $\nu_2(k^2+1)=\nu_2(2(2t^2+2t+1))=1$, since $k=2t+1$ is odd. Moreover,

$$\begin{aligned}\nu_2(L_{k,6m+1}) &= \nu_2(kL_{k,6m}+L_{k,6(m-1)+5})\\ &= \nu_2(kL_{k,6(m-1)+5})=0\end{aligned}$$

and similarly we can prove that $\nu_2(L_{k,6m+2})=\nu_2(L_{k,6m+4})=\nu_2(L_{k,6m+5})=0$.

Finally, using the recurrence relation and noting that for $k=2t+1$, we have

$$\nu_2(k^5+4k^3+3k)\geq 3>0=\nu_2(k^4+3k^2+1),$$

then

$$\begin{aligned}\nu_2(L_{k,6m+3}) &= \nu_2((k^5+4k^3+3k)L_{k,6(m-1)+4}+(k^4+3k^2+1)L_{k,6(m-1)+3})\\ &= \nu_2((k^4+3k^2+1)L_{k,6(m-1)+3})=2,\end{aligned}$$

which ends the proof.

4. PROOF OF THEOREM 1.2

(i) Firstly, let us prove that $F_{k,2m-1}\equiv 1 \pmod 2$ for all $m\geq 1$ by induction on m. The basis step, i.e., when $m=1$, is easily checked. Suppose that the result is true for $m=j$ and let us prove for $m=j+1$.

Note that, using that k is even, we have that $F_{k,2j+1}=kF_{k,2j}+F_{k,2j-1}\equiv 1 \pmod 2$. Thus, the result is true for $n\equiv 1 \pmod 2$ and k even.

Now we will prove that $\nu_2(F_{k,n})=\nu_2(k)+\nu_2(n)-1$ when $n\equiv 0 \pmod 2$. Since n is even we have that $\nu_2(n)\geq 1$ and we will proceed by induction on $\nu_2(n)$.

For the basis step, when $\nu_2(n)=1$, we have that $n=2(2m+1)$ for some non-negative integer m. Then, by using the known fact that $F_{k,2l}=F_{k,l}L_{k,l}$ (see in [1]) we can show that $F_{k,n}=F_{k,2m+1}L_{k,2m+1}$.

Thus

$$\nu_2(F_{k,n})=\nu_2(F_{k,2m+1})+\nu_2(L_{k,2m+1})$$

and by using the case proved previously and the Theorem 1.1, we conclude that the basis step is true.

Suppose that for $\nu_2(n)=j$ the result is true. Then, for $\nu_2(n)=j+1$ we have that $n=2^{j+1}(2m+1)$ for some non-negative integer m, and again by the fact $F_{k,2l}=F_{k,l}L_{k,l}$ and by the Theorem 1.1 and using the induction hypothesis we conclude that

$$\begin{aligned}\nu_2(F_{k,n}) &= \nu_2(F_{k,2^j(2m+1)})+\nu_2(L_{k,2^j(2m+1)})\\ &= \nu_2(k)+j=\nu_2(k)+\nu_2(n)-1.\end{aligned}$$

(ii)We will deal with the case where $n\equiv 0 \pmod{12}$, the other cases are similar to the case in which k is odd of Theorem 1.1, so to avoid repetition will leave the details to the reader.

Writing $n=12m$ we will prove by induction on m. The basis step when $m=1$ can be checked after a straightforward calculation. Suppose that $\nu_2(F_{k,12j})=\nu_2(12j)+2$ for all $j\leq m$. Let us check to $j=m+1$.

By Theorem 1.1 we get that

$$\nu_2(F_{k,12(m+1)})=\nu_2(F_{k,6(m+1)})+\nu_2(L_{k,6(m+1)})=\nu_2(F_{k,6(m+1)})+1. \tag{4.1}$$

Then, if 2 divides $m+1$ we can write (4.1) as

$$\nu_2(F_{k,12(m+1)}) = \nu_2(F_{k,12(\frac{m+1}{2})})+1$$

$$\begin{aligned} &= \nu_2\left(12\left(\frac{m+1}{2}\right)\right)+2+1 \\ &= \nu_2(12(m+1))+2, \end{aligned}$$

where in the second equality we use the induction hypothesis since $(m+1)/2 \le m$.

Now, if 2 not divides $m+1$, we get that $6(m+1) \equiv 6 \pmod{12}$ and by the previously cases we arrive that

$$\begin{aligned} \nu_2(F_{k,12(m+1)}) &= \nu_2(F_{k,6(m+1)})+1 \\ &= 3+1 \\ &= \nu_2(12(m+1))+2, \end{aligned}$$

which concludes the proof.

5. PROOF OF THEOREM 1.3

(i) By using the combinatorial formula for k-Fibonacci number, we have that

$$2^{n-1}F_{k,n} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} k^{n-1-2i}(k^2+4)^i, \tag{5.1}$$

where $p \nmid k$, since $p \mid k^2+4$ and $p \ne 2$.

Observe that $\nu_p(\binom{n}{2i+1}k^{n-1-2i}(k^2+4)^i) = \nu_p(\binom{n}{2i+1}(k^2+4)^i)$ and $\binom{n}{2i+1} = n/(2i+1)\binom{n-1}{2i}$, hence $\nu_p(\binom{n}{2i+1}(k^2+4)^i) = \nu_p(n) - \nu_p(2i+1) + \nu_p(\binom{n-1}{2i}(k^2+4)^i) \ge \nu_p(n) - \nu_p(2i+1) + i > \nu_p(n)$, except for $k=0$ when $\nu_p(\binom{n}{2i+1}k^{n-1-2i}(k^2+4)^i) = \nu_p(n)$. Note that the identity 5.1 implies in $\nu_p(F_{k,n}) = \nu_p(n)$.

(ii) By using combinatorial formula for k-Lucas number, we have that

$$2^{n-1}L_{k,n} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} k^{n-2i}(k^2+4)^i, \tag{5.2}$$

since $p \mid \binom{n}{2i}k^{n-2i}(k^2+4)^i$ for all $i \ge 1$, but $p \nmid k^n$, we conclude that $\nu_p(L_{k,n}) = 0$.

6. PROOF OF THEOREM 1.4

Firstly, by Lemma 2.3, if $n \not\equiv 0 \pmod{z}$ then $\nu_p(F_{k,n}) = 0$.

Now, if $n \equiv 0 \pmod{z}$ write $n = czp^\alpha$ with $\gcd(c,p) = 1$ and an integer α. By using Lemma 2.6 for $a = p$ we have that

$$F_{k,czp^\alpha} = 2^{1-p}F_{k,czp^{\alpha-1}}(MF_{k,czp^{\alpha-1}}^2 + pL_{k,czp^{\alpha-1}}^{p-1}).$$

Since $z \mid czp^{\alpha-1}$, by Lemma 2.2, $F_{k,z} \mid F_{k,czp^{\alpha-1}}$ then $p \mid F_{k,czp^{\alpha-1}}$. Moreover, since $\gcd(F_{k,m}, L_{k,m}) = 1$ or 2 and $p \ne 2$ we get that

$$\begin{aligned} \nu_p(F_{k,czp^\alpha}) &= \nu_p(F_{k,czp^{\alpha-1}}) + \nu_p(M'p^2 + pL_{k,czp^{\alpha-1}}^{p-1}) \\ &= \nu_p(F_{k,czp^{\alpha-1}}) + \nu_p(p) \\ &= \nu_p(F_{k,czp^{\alpha-1}}) + 1. \end{aligned}$$

By induction, we arrive at

$$\nu_p(F_{k,czp^\alpha}) = \nu_p(F_{k,cz}) + \alpha. \tag{6.1}$$

Observe that, by Lemma 2.4, we have

$$F_{k,cz} \equiv cF_{k,z}F_{k,z+1}^{c-1} \pmod{F_{k,z}^2},$$

and since $\nu_p(F_{k,z}) = e$, we get that $\nu_p(F_{k,cz}) = e$.

Now, we need show that $\nu_p(n) = \nu_p(czp^\alpha) = \alpha$, i.e, $\gcd(z,p) = 1$.

Since $L_{k,p} = F_{k,p+1} + F_{k,p-1} = kF_{k,p} + 2F_{k,p-1}$ (see [5]) by using Lemma 2.5 we get that

$$F_{k,p-1} \equiv \frac{k\left(1 - \left(\frac{k^2+4}{p}\right)\right)}{2} \pmod{p}$$

and

$$F_{k,p+1} \equiv \frac{k\left(1 + \left(\frac{k^2+4}{p}\right)\right)}{2} \pmod{p}.$$

Then, if $\left(\frac{k^2+4}{p}\right) = 1$ we get that $F_{k,p-1} \equiv 0 \pmod{p}$, and if $\left(\frac{k^2+4}{p}\right) = -1$ we have that $F_{k,p+1} \equiv 0 \pmod{p}$. In other words, $p \mid F_{k,p-\left(\frac{k^2+4}{p}\right)}$. Thus, by Lemma 2.3, we get that $\gcd(z,p) = 1$ which concludes the proof for the k-Fibonacci case.

For p-adic order of $L_{k,n}$, it is enough to write $L_{k,n} = \frac{F_{k,2n}}{F_{k,n}}$ and to use the result for k-Fibonacci sequences.

ACKNOWLEDGEMENT

The fifth author thanks for support to Specific Research Project of Faculty of Science, University of Hradec Kralove, No 2101, 2017.

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