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RESEARCH ARTICLES

Value Sharing Problems for Differential and Difference Polynomials of Meromorphic Functions in a non-Archimedean Field*

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Abstract—In this paper we discuss the uniqueness problem for differential and difference polynomials of the form $(f^{nm}(z)f^{nd}(qz+c))^{(k)}$ for meromorphic functions in a non-Archimedean field.

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1. INTRODUCTION

It is well-known, by the results on Hayman's conjecture, that for a transcendental entire function $f(z)$, the Picard exception values of $f^{(n)}(z)f'(z)$ may only be zero. These results caused increasingly attentions to the value sharing problem of entire and meromorphic functions and their derivatives.

In 1990, H. X. Yi ([21]) proved the following theorem, which answered a question posed by C. C. Yang in 1976 ([20]):

Theorem A ([21]). *Let f and g be two non-constant entire functions. Assume that f, g share 0 CM and $f^{(n)}, g^{(n)}$ share 1 CM and $2\delta(0, f) > 1$, where n is a nonnegative integer. Then either $f^{(n)}g^{(n)} = 1$, or $f = g$.*

In 1997, instead of the n-th derivatives, I. Lahiri ([15]) investigated a more general case of non-linear differential polynomials of meromorphic functions sharing 1 CM.

In this direction, in 2002 C. Y. Fang and M. L. Fang ([6]) proved that, if $n \geq 13$, and for two non-constant meromorphic functions f and g, $f^{(n)}(f-1)^2 f$ and $g^{(n)}(g-1)^2 g$ share 1 CM, then $f = g$. In the last decade the value sharing problem is considered also for difference polynomials of entire and meromorphic functions.

Laine and Yang [14] investigated the value distribution of difference products of entire functions, and obtained the following theorem.

Theorem B. *Let $f(z)$ be a transcendental entire function of finite order, and c be a non-zero complex constant. Then for $n \geq 2$, $f(z)^n f(z+c)$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often.*

X. C.-Qi, L.-Z. Yang and K. Liu ([18]) considered the differential and difference operator of the form $f(z)^{(n)} f(z+c)$, and proved the following theorem.

Theorem C ([18]). *Let f and g be transcendental entire functions with finite order, and c be a nonzero complex constant. If $n \geq 6$, $f(z)^{(n)} f(z+c)$ and $g(z)^{(n)} g(z+c)$ share 1 CM, then $f = tg$ for a constant t that satisfies $t^{n+1} = 1$.*

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In [16], Liu, Liu and Cao improved these results by considering meromorphic functions of fine order.

Many interesting results are obtained also for meromorphic functions in a non-Archimedean field (see [1–5, 9–11, 17]). In [17] J. Ojeda proved that for a transcendental meromorphic function f over $\mathbb{K}$, which is an algebraically closed field of characteristic zero, complete for a non-Archimedean absolute value, the function $f'f^n - 1$ has infinitely many zeros, if $n \geq 3$. For the case $n = 2$, the same fact was established by A. Escassut and J. Ojeda in [4]. In [9–11] similar results are established for differential monomials, difference polynomials, and nth derivatives of p-adic meromorphic functions. K. Boussaf, A. Escassut, J. Ojeda ([2]) studied the uniqueness problem for p-adic meromorphic functions $f'P'(f), g'P'(g)$ sharing a small function. In [1], J.-P. Bezivin, K. Boussaf and A. Escassut, studied the zeros of the derivative of a p-adic meromorphic function.

The purpose of this paper is to establish some results on the unicity and uniqueness problem for differential operators and difference polynomials of meromorphic functions in a non-Archimedean field.

We consider linear composition polynomials of meromorphic functions in a non-Archimedean field and their derivatives of the form $(f^{nm}(z)f^{nd}(qz+c))^{(k)}$. Note that in case $k = 0, m = 1, d = 1, q = 1$ we have a difference operator, which is investigated by Laine and Yang in [14].

Namely, we prove the following theorems.

Theorem 1.1. *Let f and g be two non-constant meromorphic functions on $\mathbb{K}$, $q, c \in \mathbb{K}$, $|q| = 1$, $c \neq 0$, and let n, m, d, k be positive integers, satisfying the conditions $m > d \geq 1, n \geq 2k + \frac{k(m+d)+16}{m-d}$. If $(f^{nm}(z)f^{nd}(qz+c))^{(k)}$ and $(g^{nm}(z)g^{nd}(qz+c))^{(k)}$ share 1 CM, then $f = hg$ with $h^{n(m+d)} = 1$, $h \in \mathbb{K}$.*

Theorem 1.2. *Let f and g be two non-constant meromorphic functions on $\mathbb{K}$, $q, c \in \mathbb{K}$, $|q| = 1$, $c \neq 0$, and let n, m, d, k be positive integers, satisfying the conditions $m > d \geq 1, n \geq 2k + \frac{2k(2m+2d+3)+28}{m-d}$. If $(f^{nm}(z)f^{nd}(qz+c))^{(k)}$ and $(g^{nm}(z)g^{nd}(qz+c))^{(k)}$ share 1 IM, then $f = hg$ with $h^{n(m+d)} = 1$, $h \in \mathbb{K}$.*

The main tool of the proof is the p-adic Nevanlinna theory ([2, 5, 8, 13]). Therefore, in the next section we first establish some properties of the characteristic functions of non-Archimedean meromorphic functions.

2. VALUE DISTRIBUTION OF NON-ARCHIMEDEAN MEROMORPHIC FUNCTIONS

Let us first recall some basic definitions. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, complete for a non-Archimedean absolute value denoted by $|\ .\ |$, and log be a real logarithm function of base $p > 1$, and ln be a real logarithm function of base e.

We denote by $\mathcal{A}(\mathbb{K})$ the ring of entire functions in $\mathbb{K}$, by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions, i.e., the field of fractions of $\mathcal{A}(\mathbb{K})$, and $\widehat{\mathbb{K}} = \mathbb{K} \cup \{\infty\}$.

For $f \in \mathcal{M}(\mathbb{K})$ and $S \subset \widehat{\mathbb{K}}$, we define

$$E_f(S) = \bigcup_{a \in S} \{(z, m)|\ f(z) = a \text{ with multiplicity } m\}.$$

In case $m = 1$ (i.e., ignoring multiplicity) we denote $\overline{E}_f(S)$ (this is the preimages of S). Let $\mathcal{F}$ be a nonempty subset of $\mathcal{M}(\mathbb{K})$. Two functions f, g of $\mathcal{F}$ are said to *share S, counting multiplicity*, (share S CM), if $E_f(S) = E_g(S)$ and to *share S, ignoring multiplicity*, (share S IM), if $\overline{E}_f(S) = \overline{E}_g(S)$.

2.1. Counting Functions of a non-Archimedean Entire Function (see [9], pp. 21-23, [3, 4])

Let f be a non-constant entire function on $\mathbb{K}$ and $b \in \mathbb{K}$. Then we can write f in the form

$$f = \sum_{n=q}^{\infty} b_n (z-b)^n$$

with $b_n \neq 0$, and we put $\omega(f, 0, b) = q$.

For a point $a \in \mathbb{K}$ we define the function $\omega(f, a) : \ \mathbb{K} \to \ \mathbb{N}$ by $\omega(f, a, z) = \omega(f - a, 0, z)$.

Fix a real number ρ_0 with $0 < \rho_0 \leq r$. Take $a \in \mathbb{K}$ and we denote the *counting function of zeroes of* $f - a$, counting multiplicity in the disk $D_r = \{z \in \mathbb{K} : |z| \leq r\}$, i.e. we set

$$N(f, a, r) = \frac{1}{\ln p} \int_{\rho_0}^{r} \frac{n(f, a, x)}{x} dx,$$

where $n(f, a, x)$ is the number of solutions of the equation $f(z) = a$ (counting multiplicity) in the disk $D_x = \{z \in \mathbb{K} : |z| \leq x\}$. For l a positive integer, set

$$N_l(f, a, r) = \frac{1}{\ln p} \int_{\rho_0}^{r} \frac{n_l(f, a, x)}{x} dx,$$

where

$$n_l(f, a, r) = \sum_{|z| \leq r} \min \{\omega(f, a, z), l\}.$$

Let k be a positive integer. Define the function $\omega^{\leq k}(f, 0)$ from $\mathbb{K}$ into $\mathbb{N}$ by

$$\omega^{\leq k}(f, 0, z) = \begin{cases} 0 & \text{if } \ \omega(f, 0, z) > k \\ \omega(f, 0, z) & \text{if } \ \omega(f, 0, z) \leq k, \end{cases}$$

and for a point $a \in \mathbb{K}$ we set

$$\omega^{\leq k}(f, a, z) = \omega^{\leq k}(f - a, 0, z), \ n^{\leq k}(f, a, r) = \sum_{|z| \leq r} \omega^{\leq k}(f, a, z).$$

Define

$$N^{\leq k}(f, a, r) = \frac{1}{\ln p} \int_{\rho_0}^{r} \frac{n^{\leq k}(f, a, x)}{x} dx.$$

Set

$$N_l^{\leq k}(f, a, r) = \frac{1}{\ln p} \int_{\rho_0}^{r} \frac{n_l^{\leq k}(f, a, x)}{x} dx,$$

where

$$n_l^{\leq k}(f, a, r) = \sum_{|z| \leq r} \min \{\omega^{\leq k}(f, a, z), l\}.$$

In a like manner to used for non-constant entire functions on $\mathbb{K}$ we define

$$N^{<k}(f, a, r), N_l^{<k}(f, a, r), N^{>k}(f, a, r), N^{\geq k}(f, a, r), N_l^{\geq k}(f, a, r), N_l^{>k}(f, a, r).$$

2.2. Characteristic Functions of a non-Archimedean Meromorphic Function (see [13, pp. 33-46], [2, 5, 7, 8])

Recall that for a non-constant entire function $f(z)$ on $\mathbb{K}$, represented by the power series

$$f(z)=\sum_{n=0}^{\infty}a_nz^n,$$

we define $|f|_r=\max\{|a_n|r^n, 0\le n<\infty\}$, for each $r>0$.

Now let $f=\dfrac{f_1}{f_2}$ be a non-constant meromorphic function on $\mathbb{K}$, where f_1, f_2 are entire functions on $\mathbb{K}$ having no common zeros, we set $|f|_r=\frac{|f_1|_r}{|f_2|_r}$.

For a point $a\in\mathbb{K}\cup\{\infty\}$ we define the function $\omega(f,a):\ \mathbb{K}\to\ \mathbb{N}$ by

$\omega(f,a,z)=\omega(f_1-af_2,0,z)$ with $a\neq\infty$ and $\omega(f,\infty,z)=\omega(f_2,0,z)$.

Take $a\in\mathbb{K}$. We denote the *counting function of zeroes of* $f-a$, counting multiplicity in the disk $D_r=\{z\in\mathbb{K}:|z|\le r\}$, i.e. we set

$$N(f,a,r)=N(f_1-af_2,0,r),\ \text{and set}\ N(f,\infty,r)=N(f_2,0,r).$$

As in the previous paragraph we define

$$N_l(f,a,r), N^{\le k}(f,a,r), N^{<k}(f,a,r), N_l^{<k}(f,a,r),$$

$$N^{>k}(f,a,r), N^{\ge k}(f,a,r), N_l^{\ge k}(f,a,r), N_l^{>k}(f,a,r).$$

Define the *compensation function of* f by

$$m(f,\infty,r)=\max\left\{0,\log|f|_r\right\},\ \text{and set}\ m(f,a,r)=m(\frac{1}{f-a},\infty,r).$$

Finally, define the *characteristic function of* f by

$$T(f,r)=m(f,\infty,r)+N(f,\infty,r).$$

Then for $a\in\mathbb{K}\cup\{\infty\}$, we have

$$N(f,a,r)+m(f,a,r)=T(f,r)+O(1),\ T(f,r)=T(\frac{1}{f},r)+O(1),$$

$$T(f,r)=\max_{1\le i\le 2}\log|f_i|_r+O(1),\ |f^{(k)}|_r\le\frac{|f|_r}{r^k},\ m(\frac{f^{(k)}}{f},\infty,r)=O(1).$$

The following lemmas were proved in [13, p. 21] (see also [8]).

Lemma 2.1. *Let f be a non-constant entire function on $\mathbb{K}$. Then*

$$T(f,r)-T(f,\rho_0)=N(f,0,r),\ \textit{where}\ \ 0<\rho_0\le r.$$

Notices that $N(f,0,r)$ depends on fixed ρ_0.

Lemma 2.2. *Let f be a non-constant meromorphic function on $\mathbb{K}$ and let $a_1,a_2,...,a_q$ be distinct points of $\mathbb{K}$. Then*

$$(q-1)T(f,r)\le N_1(f,\infty,r)+\sum_{i=1}^{q}N_1(f,a_i,r)-\log r+O(1).$$

3. UNIQUENESS FOR LINEARLY COMPOSITION POLYNOMIALS OF MEROMORPHIC FUNCTIONS AND THEIR n-TH DERIVATIVES

We are going to prove Theorem 1.1, Theorem 1.2.

Lemma 3.1. *[7] Let f be a non-constant meromorphic function on $\mathbb{K}$ and n, k be positive integers, $n \geq k+1$. Then*

$$T(f,r) \leq T((f^n)^{(k)},r) + O(1).$$

Lemma 3.2. *Let f and g be non-constant meromorphic functions on $\mathbb{K}$ and*

$$F = \frac{1}{f-1}, G = \frac{1}{g-1},\ L = \frac{F''}{F'} - \frac{G''}{G'}.$$

1. If $E_f(1) = E_g(1)$ and $L \not\equiv 0$, then

$$T(f,r) \leq N_2(f,\infty,r) + N_2(f,0,r) + N_2(g,\infty,r) + N_2(g,0,r) - \log r + O(1),$$

and the same inequality holds for $T(g,r)$;

2. If $\overline{E}_f(1) = \overline{E}_g(1)$ and $L \equiv 0$, then one of the following three cases holds:

$$i)\ T(f,r) \leq N_1(f,\infty,r) + N_1(f,0,r) + N_1(g,\infty,r) + N_1(g,0,r) - \log r + O(1),$$

and the same inequality holds for $T(g,r)$;

ii) $fg \equiv 1$;

iii) $f \equiv g$.

Proof. 1. Proof of 1. follows immediately from the proof of Lemma 3.5 of [7].

2. By $L \equiv 0$ we have

$$\frac{F''}{F'} \equiv \frac{G''}{G'}.$$

Thus

$$f \equiv \frac{ag+b}{cg+d},$$

where $a, b, c, d \in \mathbb{K}$ satisfying $ad - bc \neq 0$. Then $T(f,r) = T(g,r) + O(1)$.

Next we consider the following subcases:

Subcase 1. $ac \neq 0$. Then

$$f - \frac{a}{c} \equiv \frac{b - \frac{ad}{c}}{cg+d}.$$

By Lemma 2.2

$$T(f,r) \leq N_1(f,\infty,r) + N_1(f,\frac{a}{c},r) + N_1(f,0,r) - \log r + O(1)$$

$$= N_1(f,\infty,r) + N_1(g,\infty,r) + N_1(f,0,r) - \log r + O(1).$$

We get i).

Subcase 2. $a \neq 0, c = 0$. Then $f \equiv \frac{ag+b}{d}$. If $b \neq 0$, then by Lemma 2.2 we give

$$T(f,r) \leq N_1(f,\infty,r) + N_1(f,\frac{b}{d},r) + N_1(f,0,r) - \log r + O(1)$$

$$= N_1(f,\infty,r) + N_1(g,0,r) + N_1(f,0,r) - \log r + O(1).$$

We get i). If $b=0$, then $f\equiv \frac{ag}{d}$. If $\frac{a}{d}=1$, then $f\equiv g$. We obtain iii). If $\frac{a}{d}\neq 1$, then by $\overline{E}_f(1)=\overline{E}_g(1)$ we have $f\neq 1, f\neq \frac{a}{d}$. Note that if f is a meromorphic function on $\mathbb{K}$ that never takes on two points in $\widehat{\mathbb{K}}$, then f is constant(see [8, 19]). From this and $f\neq 1, f\neq \frac{a}{d}$ we obtain a contradiction.

Subcase 3. $a=0, c\neq 0$. Then $f\equiv \frac{b}{cg+d}$. If $d\neq 0$, then by Lemma 2.2 we have

$$T(f,r)\leq N_1(f,\infty,r)+N_1(f,\frac{b}{d},r)+N_1(f,0,r)-\log r+O(1)$$

$$=N_1(f,\infty,r)+N_1(g,0,r)+N_1(f.0,r)-\log r+O(1).$$

We obtain i).

If $d=0$, then $f\equiv \frac{b}{cg}$. If $\frac{b}{c}=1$, then $fg\equiv 1$. We obtain ii).

If $\frac{b}{c}\neq 1$, then by $\overline{E}_f(1)=\overline{E}_g(1)$ we have $f\neq 1, f\neq \frac{b}{c}$. By a similar argument as in **Subcase 2.** we obtain a contradiction.

The proof of Lemma 3.2 is complete. □

Lemma 3.3. *Let f be a non-constant meromorphic function on $\mathbb{K}$ and n,k be positive integers, $n>2k$. Then*

1. $(n-2k)T(f,r)+kN(f,\infty,r)+N(\frac{(f^n)^{(k)}}{f^{n-k}},0,r)\leq T((f^n)^{(k)},r)+O(1)$;

2. $N(\frac{(f^n)^{(k)}}{f^{n-k}},0,r)\leq kT(f,r)+kN_1(f,\infty,r)+O(1)$.

Now we need the following lemmas.

Lemma 3.4. *Let f be a non-constant meromorphic function on $\mathbb{K}$ and $q,c\in\mathbb{K}$, $|q|=1, c\neq 0$. Then*

1. $m(\frac{f(qz+c)}{f(z)},\infty,r)=O(1)$;

2. $m(\frac{f(z)}{f(qz+c)},\infty,r)=O(1)$;

3. $T(f(qz+c),r)=T(f(z),r)+O(1)$;

4. $N(f(qz+c),0,r)=N(f(z),0,r)+O(1)$;

5. $N(f(qz+c),\infty,r)=N(f(z),\infty,r)+O(1)$.

Proof. Set $A=\frac{f(qz+c)}{f(z)}$. Then

1. Suppose $|c|<r$. Because the set of $r\in\mathbb{R}_+$ such that there exists $z\in\mathbb{K}$ with $|z|=r$ is dense in $\mathbb{R}_+$, without loss of generality, one may assume that there exists $z\in\mathbb{K}$ such that $|z|=r$. Then $r=|q||z|=|qz|=|c+qz|$. So $|f(z)|_r=|f(qz+c)|_r$ and $|A|=1$. If $r\leq |c|$, then $|c+qz|\leq\max\{|c|,|qz|\}\leq |c|$. Thus $|A|_r=O(1)$. Therefore $m(A,\infty,r)=\max\{0,\log|A|_r\}=O(1)$.

2. Similarly, we obtain $m(\frac{f(z)}{f(qz+c)},\infty,r)=O(1)$.

3. Let $f=\dfrac{f_1}{f_2}$ be a non-constant meromorphic function on $\mathbb{K}$, where f_1,f_2 are holomorphic functions on $\mathbb{K}$ having no common zeros. Then, by using the arguments similar to the ones in the proof of 1., we have:

If $|c|<r$, then $|f_1(z)|_r=|f_1(qz+c)|_r$ and $|f_2(z)|_r=|f_2(qz+c)|_r$. If $r\leq|c|$, then $|f_1(z)|_r\leq |f_1(z)|_c, |f_1(qz+c)|_r\leq |f_1(z)|_c$, and $|f_2(z)|_r\leq|f_2(z)|_c, |f_2(qz+c)|_r\leq |f_2(z)|_c$. Moreover, $T(f,r)=\max\limits_{1\leq i\leq 2}\log|f_i|_r,\ T(f(qz+c),r)=\max\limits_{1\leq i\leq 2}\log|f_i(qz+c)|_r$. So $T(f(qz+c),r)=T(f(z),r)+O(1)$.

4. Similarly, we see that if $|c|<r$, then $r=|q||z|=|qz|=|c+qz|$, and if $r\leq|c|$, then $|c+qz|\leq\max\{|c|,|qz|\}\leq|c|$. So $N(f(qz+c),0,r)=N(f(z),0,r)+O(1)$.

5. Similarly, we obtain $N(f(qz+c),\infty,r)=N(f(z),\infty,r)+O(1)$. Lemma 3.4 is proved. □

Lemma 3.5. *Let f be a non-constant meromorphic function on $\mathbb{K}$, $|q| = 1$, $c \neq 0$, and n, m, d, k be positive integers, such that $m > d, d \geq 1, n > 2k$. Then 1.*

$$(m-d)T(f,r) \leq T(f^m(z)f^d(qz+c),r) + O(1);$$

2.

$$(n-2k)(m-d)T(f,r) + kN(f^m(z)f^d(qz+c),\infty,r)$$

$$+N(\frac{(f^{nm}(z)f^{nd}(qz+c))^{(k)}}{(f^m(z)f^d(qz+c))^{n-k}},0,r) \leq T((f^{nm}(z)f^{nd}(qz+c))^{(k)},r) + O(1);$$

3. $N(\frac{(f^{nm}(z)f^{nd}(qz+c))^{(k)}}{(f^m(z)f^d(qz+c))^{n-k}},0,r) \leq k(m+d)T(f,r)$

$$+kN_1(f^m(z)f^d(qz+c),\infty,r) + O(1) \leq k(m+d+2)T(f,r) + O(1).$$

Proof. 1. Set $F = f^m(z)f^d(qz+c)$. We have $f^dF = f^{m+d}(z)f^d(qz+c)$ and $f^{m+d}(z) = F.(\frac{f(z)}{f(qz+c)})^d$. Therefore

$$(m+d)T(f,r) = T(f^{m+d},r) + O(1) = T(F.(\frac{f(z)}{f(qz+c)})^d,r) + O(1)$$

$$\leq T(F,r) + T((\frac{f(z)}{f(qz+c)})^d,r) + O(1) \leq T(F,r) + d(T(f,r) + T(f(qz+c),r)) + O(1).$$

From this and 3.4.3, we obtain

$$(m+d)T(f,r) \leq T(F,r) + 2dT(f,r) + O(1).$$

So

$$(m-d)T(f,r) \leq T(f^m(z)f^d(qz+c),r) + O(1).$$

2. By 3.3.1 we have

$$(n-2k)T(f^m(z)f^d(qz+c),r) + kN(f^m(z)f^d(qz+c),\infty,r)$$

$$+N(\frac{(f^{nm}(z)f^{nd}(qz+c))^{(k)}}{(f^m(z)f^d(qz+c))^{n-k}},0,r) \leq T((f^{nm}(z)f^{nd}(qz+c))^{(k)},r) + O(1).$$

Moreover, by 1. we obtain

$$(m-d)T(f,r) \leq T(f^m(z)f^d(qz+c),r) + O(1).$$

From these two inequalities we get 2..

3. By 3.3.2 we have

$$N(\frac{(f^{nm}(z)f^{nd}(qz+c))^{(k)}}{(f^m(z)f^d(qz+c))^{n-k}},0,r) \leq kT(f^m(z)f^d(qz+c),r)$$

$$+kN_1(f^m(z)f^d(qz+c),\infty,r) + O(1).$$

On the other hand

$$kT(f^m(z)f^d(qz+c),r) \leq k(T(f^m(z),r) + T(f^d(qz+c),r) + O(1)$$

$$= k(m+d)T(f,r) + O(1); kN_1(f^m(z)f^d(qz+c),\infty,r) \leq k(N_1(f(z),\infty,r)$$

$$+N_1(f(qz+c),\infty,r)) \leq k(T(f,r) + T(f(qz+c),r)) + O(1) = 2kT(f,r) + O(1).$$

From these inequalities we get 3. ∎

Let f and g be two non-constant meromorphic functions on $\mathbb{K}$ such that $\overline{E}_f(1) = \overline{E}_g(1)$. Let a be a zero of $f-1$ with multiplicity $\omega(f,1,a)$, and be a zero of $g-1$ with multiplicity $\omega(g,1,a)$. We denote by $N_1(f,1,r;\omega(f,1,a) > \omega(g,1,a))$ the counting function of such zeros of $f-1$, where $\omega(f,1,a) > \omega(g,1,a)$ and each zero is counted only with multiplicity 1, by $N_1^{\geq 2}(f,1,r;\omega(f,1,a) = \omega(g,1,a))$ the counting function of such zeros of $f-1$, where $\omega(f,1,a) = \omega(g,1,a) \geq 2$, and each zero is counted only with multiplicity 1. In the same way, we can define $N_1(g,1,r;\omega(g,1,a) > \omega(f,1,a))$, $N_1^{\geq 2}(g,1,r;\omega(g,1,a) = \omega(f,1,a))$.

Lemma 3.6. *Let f and g be two non-constant meromorphic functions on $\mathbb{K}$. If $\overline{E}_f(1) = \overline{E}_g(1)$, then one of the following three relations holds:*

1. $$\begin{aligned} T(f,r) \leq & N_2(f,\infty,r) + N_2(f,0,r)) + N_2(g,\infty,r) + N_2(g,0,r) \\ & + 2(N_1(f,\infty,r) + N_{1,f}(0,r)) + N_1(g,\infty,r) + N_1(g,0,r) - 2\log r + O(1), \end{aligned}$$

and the same inequality holds for $T(g,r)$;

2. $fg \equiv 1$;

3. $f \equiv g$.

Proof. Set

$$F = \frac{1}{f-1}, G = \frac{1}{g-1},$$

$$L = \frac{f^{"}}{f^{\prime}} - 2\frac{f^{\prime}}{f-1} - \frac{g^{"}}{g^{\prime}} + 2\frac{g^{\prime}}{g-1}. \tag{3.1}$$

Then

$$L = \frac{F^{"}}{F^{\prime}} - \frac{G^{"}}{G^{\prime}}. \tag{3.2}$$

Next we consider two following cases:

Case 1. $L \not\equiv 0$. We now consider the poles of L. By (3.2), it is clear that all poles of L are of order 1. Write $f = \frac{f_1}{f_2}$ (resp., $g = \frac{g_1}{g_2}$), where f_1, f_2 (resp., g_1, g_2) are entire functions on $\mathbb{K}$ having no common zeros. Then

$$f^{\prime} = \frac{f_1^{\prime} f_2 - f_2^{\prime} f_1}{f_2^2}, f^{"} = \frac{(f_1^{"} f_2 - f_2^{"} f_1) f_2 - 2f_2^{\prime}(f_1^{\prime} f_2 - f_2^{\prime} f_1)}{f_2^3};$$

$$\frac{f^{"}}{f^{\prime}} = \frac{(f_1^{"} f_2 - f_2^{"} f_1) f_2 - 2f_2^{\prime}(f_1^{\prime} f_2 - f_2^{\prime} f_1)}{f_2(f_1^{\prime} f_2 - f_2^{\prime} f_1)}, \ \frac{f^{\prime}}{f-1} = \frac{(f_1^{\prime} f_2 - f_2^{\prime} f_1)}{f_2(f_1 - f_2)}. \tag{3.3}$$

Similarly,

$$\frac{g^{"}}{g^{\prime}} = \frac{(g_1^{"} g_2 - g_2^{"} g_1) g_2 - 2g_2^{\prime}(g_1^{\prime} g_2 - g_2^{\prime} g_1)}{g_2(g_1^{\prime} g_2 - g_2^{\prime} g_1)}, \ \frac{g^{\prime}}{g-1} = \frac{(g_1^{\prime} g_2 - g_2^{\prime} g_1)}{g_2(g_1 - g_2)}. \tag{3.4}$$

From (3.1), (3.3), (3.4) we see that if a is a pole of L, then $f(a) = \infty$, or $f^{\prime}(a) = 0$, or $f(a) = 1$, or $g(a) = \infty$, or $g^{\prime}(a) = 0$, or $g(a) = 1$. Now let a be a pole of f with $\omega(f,\infty,a) = 1$. Write $f = \frac{f_3}{z-a}$, $f_3(a) \neq 0, f_3(a) \neq \infty$. Then $F = \frac{1}{f-1} = \frac{z-a}{f_3 - (z-a)}$, and

$$F^{\prime} = \frac{f_3 - f_3^{\prime}.(z-a)}{(f_3 - (z-a))^2}, \ F^{"} = \frac{-f_3^{"}.(z-a)(f_3 - (z-a)) - 2(f_3^{\prime} - 1)(f_3 - f_3^{\prime}.(z-a))}{(f_3 - (z-a))^3};$$

$$\frac{F^{"}}{F^{\prime}} = \frac{-f_3^{"}.(z-a)(f_3 - (z-a)) - 2(f_3^{\prime} - 1)(f_3 - f_3^{\prime}.(z-a))}{(f_3 - (z-a))(f_3 - f_3^{\prime}.(z-a))}. \tag{3.5}$$

From (3.5) we get $\frac{F"}{F'}(a) \neq \infty$. Therefore, if a is a common pole of f and g with $\omega(f,\infty,a) = \omega(g,\infty,a) = 1$, then

$$L(a) = \left[\frac{F"}{F'}(a) - \frac{G"}{G'}(a)\right] \neq \infty. \tag{3.6}$$

Now let $f(a) = 1$ with $\omega(f,1,a) = m$. Since $\overline{E}_f(1) = \overline{E}_g(1)$, we have $g(a) = 1$ and $\omega(g,1,a) = n$.

Write

$$F = \frac{F_1}{(z-a)^m}, F_1(a) \neq 0, F_1(a) \neq \infty;\ G = \frac{G_1}{(z-a)^n}, G_1(a) \neq 0, G_1(a) \neq \infty.$$

Then

$$F' = \frac{F_1'.(z-a) - mF_1}{(z-a)^{m+1}};$$

$$F" = \frac{[F_1".(z-a) + (1-m)F_1'](z-a) - (m+1)(F_1'.(z-a) - mF_1)}{(z-a)^{m+2}};$$

$$G' = \frac{G_1'.(z-a) - nG_1}{(z-a)^{n+1}};$$

$$G" = \frac{[G_1".(z-a) + (1-n)G_1'](z-a) - (n+1)(G_1'.(z-a) - nG_1)}{(z-a)^{n+2}};$$

$$\frac{F"}{F'} - \frac{G"}{G'} = \frac{1}{z-a}\Big[\frac{(F_1".(z-a) + (1-m)F_1')(z-a) - (m+1)(F_1'.(z-a) - mF_1)}{F_1'.(z-a) - mF_1}$$

$$-\frac{(G_1".(z-a) + (1-n)G_1')(z-a) - (n+1)(G_1'.(z-a) - nG_1)}{G_1'.(z-a) - nG_1}\Big]$$

From these, if $m = n = 1$, then

$$\frac{F"}{F'} - \frac{G"}{G'} = \frac{1}{z-a}\frac{F_2.(z-a)^2}{(F_1'.(z-a) - F_1)(G_1'.(z-a) - G_1)}$$

$$= \frac{F_2.(z-a)}{(F_1'.(z-a) - F_1)(G_1'.(z-a) - G_1)} \text{ and if } m = n \geq 2 \text{ then ;}$$

$$\frac{F"}{F'} - \frac{G"}{G'} = \frac{1}{z-a}\Big[\frac{(F_1".(z-a) + (1-m)F_1')(z-a) - (m+1)(F_1'.(z-a) - mF_1)}{F_1'.(z-a) - mF_1}$$

$$-\frac{(G_1".(z-a) + (1-m)G_1')(z-a) - (m+1)(G_1'.(z-a) - mG_1)}{G_1'.(z-a) - mG_1}\Big]$$

$$= \frac{F_3}{(F_1'.(z-a) - mF_1)(G_1'.(z-a) - mG_1)}.$$

By these equalities and $F_1(a) \neq 0, F_1(a) \neq \infty, G_1(a) \neq 0, G_1(a) \neq \infty$ and $L \not\equiv 0$, we see that if $m = n = 1$, then $L(a) = 0$, and if $m = n \geq 2$, then $L(a) \neq \infty$.

From this and (3.1)- (3.6) we can see that if a is a pole of L, then

$$f(a) = \infty \text{ with } \omega(f\infty, a) \geq 2, \text{ or } f'(a) = 0, \text{ or } f(a) = 1 \text{ with } \omega(f,1,a) > \omega(g,1,a),$$

or

$$g(a)=\infty \text{ with } \omega(g,\infty,a)\geq 2, \text{ or } g'(a)=0, \text{ or } g(a)=1 \text{ with } \omega(g,1,a)>\omega(f,1,a). \quad (3.7)$$

On the other hand, by (3.1) we have

$$m(L,\infty,r)=O(1), \text{ and } N^{\leq 1}(f,1,r)=N^{\leq 1}(g,1,r)$$
$$\leq N(L,0,r)\leq T(L,r)+O(1)\leq N(L,\infty,r)+O(1). \quad (3.8)$$

Moreover, by Lemma 2.2 ,

$$T(f,r)\leq N_1(f,\infty,r)+N_1(f,0,r)+N_1(f,1,r)-N_0(f',0,r)-\log r+O(1), \quad (3.9)$$

where $N_0(f',0,r)$ is the counting function of those zeros of f' but not that of $f(f-1)$. $N_{1,0}(f',0,r)$ is defined similarly, where each zero of f' is counted with multiplicity 1. Then we have

$$T(g,r)\leq N_1(g,\infty,r)+N_1(g,0,r)+N_1(g,1,r)-N_0(g',0,r)-\log r+O(1). \quad (3.10)$$

From (3.9) and (3.10) we have

$$T(f,r)+T(g,r)\leq N_1(f,\infty,r)+N_1(f,0,r)+N_1(f,1,r)-N_0(f',0,r)+N_1(g,\infty,r)$$
$$+N_1(g,0,r)+N_1(g,1,r)-N_0(g',0,r)-2\log r+O(1). \quad (3.11)$$

Noting that $\overline{E}_f(1)=\overline{E}_g(1), N_1(f,1,r)=N^{\leq 1}(f,1,r)+N_1^{\geq 2}(f,1,r)$ we obtain

$$N_1^{\geq 2}(f,1,r)\leq N_1(f,1,r;\omega(f,1,a)>\omega(g,1,a))$$
$$+N_1(g,1,r;\omega(g,1,a)>\omega(f,1,a))+N_1^{\geq 2}(g,1,r;\omega(g,1,a)=\omega(f,1,a)),$$
$$+N_1(f,1,r)+N_1(1,r)\leq N^{\leq 1}(f,1,r)+N_1(f,1,r;\omega(f,1,a)>\omega(g,1,a))$$
$$+N_1(g,1,r;\omega(g,1,a)>\omega(f,1,a))+N_1^{\geq 2}(g,1,r;\omega(g,1,a)=\omega(f,1,a))+N_1(g,1,r)$$
$$\leq N^{\leq 1}(f,1,r)+N_1(f,1,r;\omega(f,1,a)>\omega(g,1,a))+N(g,1,r)$$
$$\leq N^{\leq 1}(f,1,r)+N_1(f,1,r;\omega(f,1,a)>\omega(g,1,a))+T(g,r).$$

Combining this and (3.11), we obtain

$$T(f,r)\leq N_1(f,\infty,r)+N_1(f,0,r)+N^{\leq 1}(f,1,r)-N_0(f',0,r)+N_1(g,\infty,r)$$
$$+N_1(g,0,r)+N_1(f,1,r;\omega(f,1,a)>\omega(g,1,a))-N_0(g',0,r)-2\log r+O(1). \quad (3.12)$$

Then from (3.1)- (3.8) we deduce that

$$N^{\leq 1}(f,1,r)\leq N(L,\infty,r)\leq N_1^{\geq 2}(f,\infty,r)+N_1^{\geq 2}(g,\infty,r)$$
$$+N_{1,0}(f',0,r)+N_{1,0}(g',0,r)+N_1^{\geq 2}(f,0,r)+N_1^{\geq 2}(g,0,r)$$
$$+N_1(f,1,r;\omega(f,1,a)>\omega(g,1,a))+N_1(g,1,r;\omega(g,1,a)>\omega(f,1,a))+O(1).$$

Combining this and (3.12), we have

$$T(f,r)\leq N_2(f,\infty,r)+N_2(f,0,r)+N_2(g,\infty,r)+N_2(g,0,r)$$
$$+2N_1(f,1,r;\omega(f,1,a)>\omega(g,1,a))+N_1(g,1,r;\omega(g,1,a)>\omega(f,1,a))-2\log r+O(1). \quad (3.13)$$

We denote by $N(f',0,r;f\neq 0)$ the counting function of those zeros of f' which are not the zeros of f, where a zero of f' is counted according to its multiplicity. We get

$$N(f',0,r;f\neq 0)=N(\frac{f'}{f},0,r)\leq T(\frac{f'}{f},r)+O(1)=N(\frac{f'}{f},\infty,r)$$

$$+m(\frac{f^{'}}{f},\infty,r)+O(1)\le N_1(f,\infty,r)+N_1(f,0,r)+O(1),$$

and

$$N(f^{'},0,r)=N(f^{'},0,r;f\ne 0)+N^{\ge 2}(f,0,r).$$

So

$$N(f^{'},0,r;f\ne 0)\le N_1(f,\infty,r)+N_1(f,0,r)+O(1), N(f^{'},0,r)$$

$$\le N_1(f,\infty,r)+N_1(f,0,r)+N^{\ge 2}(f,0,r)+O(1)\le N_1(f,\infty,r)+N(f,0,r)+O(1).$$

On the other hand,

$$(N(f,1,r)-N_1(f,1,r))+(N(f,0,r)-N_1(f,0,r))\le N(f^{'},0,r)$$

and

$$N_1(f,1,r;\omega(f,1,a)>\omega(g,1,a))\le N(f,1,r)-N_1(f,1,r).$$

Therefore,

$$N_1(f,1,r;\omega(f,1,a)>\omega(g,1,a))+(N(f,0,r)-N_1(f,0,r))\le N(f^{'},0,r)\le N_1(f,\infty,r)$$

$$+N(f,0,r)+O(1),\ N_1(f,1,r;\omega(f,1,a)>\omega(g,1,a))\le N_1(f,\infty,r)+N_1(f,0,r)+O(1).$$

Likewise, we have

$$N_1(g,1,r;\omega(g,1,a)>\omega(f,1,a))\le N_1(g,\infty,r)+N_1(g,0,r)+O(1).$$

Combining with (3.13), we obtain 1).

Case 2. $L\equiv 0$. By Case 2. of Lemma 3.2, we obtain the conclusion of Lemma 3.6. □

Now we use the above Lemmas to prove the main results of the paper.

Proof of Theorem 1.1. Set $A=(f^{nm}(z)f^{nd}(qz+c))^{(k)}, B=(g^{nm}(z)g^{nd}(qz+c))^{(k)}$, $C=f^m(z)f^d(qz+c), D=g^m(z)(g^d(qz+c), P=\frac{A}{C^{n-k}}, Q=\frac{B}{D^{n-k}}$. Then $A=(C^n)^{(k)}=C^{n-k}P$, $B=(D^n)^{(k)}=D^{n-k}Q$.

Note that

$$N_1(A,\infty,r)+N_1^{\ge 2}(A,\infty,r)=N_2(A,\infty,r), N_1(A,0,r)+N_1^{\ge 2}(A,0,r)=N_2(A,0,r),$$

$$N_1(B,\infty,r)+N_1^{\ge 2}(B,\infty,r)=N_2(B,\infty,r), N_1(B,0,r)+N_1^{\ge 2}(B,0,r)=N_2(B,0,r).$$

Then, applying Lemma 3.4 to the $(C^n)^{(k)}$, $(D^n)^{(k)}$ we consider the following cases:

Case 1.

$$T(A,r)\le N_2(A,\infty,r)+N_2(A,0,r)+N_2(B,\infty,r)+N_2(B,0,r)-\ \mathrm{logr}+O(1),$$

$$T(B,r)\le N_2(A,\infty,r)+N_2(A,0,r)+N_2(B,\infty,r)+N_2(B,0,r)-\ \mathrm{logr}+O(1). \tag{3.14}$$

We see that if a is a pole of A, then $C(a)=\infty$ with $\omega(A,\infty,a)\ge n+k\ge 2$ and by Lemma 3.4 we obtain $N_1(C,\infty,r)=N_1(f^mf^d(qz+c),\infty,r)\le N_1(f,\infty,r)+N_1(f(qz+c),\infty,r)+O(1)\le T(f,r)+T(f(qz+c),r)+O(1)=2T(f,r)+O(1)$. Similarly, $N_1(C,0,r)\le 2T(f,r)+O(1)$. Therefore, by Lemma 3.5 we get

$$N_2(A,\infty,r)=2N_1(C,\infty,r)\le 4T(f,r)+O(1),\ N_2(A,0,r)\le N_2(C^{n-k},0,r)$$

$$+N(P,0,r)=2N_1(C,0,r)+N(P,0,r)\le 4T(f,r)+N(P,0,r)+O(1)$$

$$\le 4T(f,r)+k(m+d)T(f,r)+kN_1(C,\infty,r)+O(1).$$

Similarly,

$$N_2(B,\infty,r) \le 4T(g,r) + O(1),\ N_2(B,0,r) \le 4T(g,r) + N(Q,0,r) + O(1)$$

$$\le 4T(g,r) + k(m+d)T(g,r) + kN_1(D,\infty,r) + O(1).$$

From the above inequalities and (3.14) we have

$$T(A,r) \le 8(T(f,r)+T(g,r)) + N(P,0,r) + N(Q,0,r) - \mathrm{logr} + O(1)$$

$$\le 8(T(f,r)+T(g,r)) + k(m+d)T(f,r) + kN_1(C,\infty,r) + N(Q,0,r) - \mathrm{logr} + O(1),$$

$$T(B,r) \le 8(T(f,r)+T(g,r)) + N(P,0,r) + N(Q,0,r) - \mathrm{logr} + O(1)$$

$$\le 8(T(f,r)+T(g,r)) + k(m+d)T(g,r) + kN_1(D,\infty,r) + N(P,0,r) - \mathrm{logr} + O(1),$$

$$T(A,r) + T(B,r) \le 16(T(f,r)+T(g,r)) + N(P,0,r) + N(Q,0,r)$$

$$+k(m+d)(T(f,r)+T(g,r)) + k(N_1(C,\infty,r) + N_1(D,\infty,r)) - 2\mathrm{logr} + O(1).$$

By Lemma 3.5 we obtain

$$(n-2k)(m-d)T(f,r) + kN(C,\infty,r) + N(P,0,r) \le T(A,r) + O(1),$$

$$(n-2k)(m-d)T(g,r) + kN(D,\infty,r) + N(Q,0,r) \le T(B,r) + O(1).$$

Combining the above inequalities we get

$$(n-2k)(m-d)(T(f,r)+T(g,r)) + k(N(C,\infty,r) + N(D,\infty,r)) + N(P,0,r) + N(Q,0,r)$$

$$\le (k(m+d)+16)(T(f,r)+T(g,r)) + k(N_1(C,\infty,r) + N(D,\infty,r)) + N(P,0,r)$$

$$+N(Q,0,r) - 2\mathrm{logr} + O(1), [(n-2k)(m-d) - (k(m+d)+16)](T(f,r)+T(g,r)) + 2\mathrm{logr}$$

$\le O(1)$. As $n \ge 2k + \dfrac{k(m+d)+16}{m-d}$, we obtain a contradiction.

Case 2. $(f^{nm}(z)f^{nd}(qz+c))^{(k)}(g^{nm}(z)g^{nd}(qz+c))^{(k)} = (C^n)^{(k)}(D^n)^{(k)} = 1$. We prove $C \ne 0$, $C \ne \infty$, $D \ne 0$, $D \ne \infty$. Assume C has zeros. Let a be such that $\omega(C,0,a) = \alpha$, $\alpha \ge 1$. Then a is a pole of D with $\omega(D,\infty,a) = \beta$, $\beta \ge 1$ such that $n\alpha - k = n\beta + k$ and $n(\alpha - \beta) = 2k$. From this and by $n \ge 2k + \frac{k(m+d)+16}{m-d} > 2k+1$ we obtain a contradiction. By similar arguments we have $D \ne 0$, $C \ne \infty$, $D \ne \infty$. Since C, D are not constant, we have a contradiction.

Case 3. $(f^{nm}(z)f^{nd}(qz+c))^{(k)} = (g^{nm}(z)g^{nd}(qz+c))^{(k)}, (C^n)^{(k)} = (D^n)^{(k)}$. Because f, g are not constant, and by Lemma 3.5 we see that C, D are not constant. Then $C^n = D^n + s, D^n = C^n - s$, where s is a polynomial of degree $< k$. We prove $s \equiv 0$. Assume $s \not\equiv 0$. Then

$$nT(D,r) = T(D^n,r) + O(1) \le T(C^n,r) + T(s,r) + O(1) \le nT(C,r) + (k-1)\mathrm{logr} + O(1).$$

From this and $n \ge 2k + \frac{k(m+d)+16}{m-d} > 2k+1$ we obtain

$$\frac{k-1}{n} < \frac{1}{2},\ T(D,r) \le T(C,r) + \frac{1}{2}\mathrm{logr} + O(1). \tag{3.15}$$

Set $F = \frac{C^n}{s}, G = \frac{D^n}{s}$. Since C, D are not constant, we get

$$Fs = C^n, nT(C,r) = T(C^n,r) \le T(F,r) + T(s,r) + O(1) \le T(F,r) + (k-1)\mathrm{logr} + O(1),$$

$$nT(C,r) - (k-1)\mathrm{logr} \le T(F,r) + O(1), N_1(F,0,r) \le N_1(C,0,r) \le T(C,r) + O(1),$$

$$N_1(D,0,r) \le T(D,r) + O(1) \le T(C,r) + \frac{1}{2}\mathrm{logr} + O(1), N_1(F,\infty,r) \le N_1(C^n,\infty,r)$$

$$+N_1(\frac{1}{s},\infty,r)\le N_1(C,\infty,r)+(k-1)\text{logr}+O(1)\le T(C,r)+(k-1)\text{logr}+O(1).$$

From this and Lemma 2.2, because $F-1=G$ we have

$$nT(C,r)-(k-1)\text{logr}+O(1)\le T(F,r)\le N_1(F,0,r)+N_1(F,\infty,r)+N_1(F,1,r)-\text{logr}$$

$$+O(1)\le T(C,r)+T(C,r)+(k-1)\text{logr}+N_1(G,0,r)-\text{logr}+O(1)\le 2T(C,r)$$

$$+(k-1)\text{logr}+N_1(D,0,r)-\text{logr}+O(1)\le 2T(C,r)+T(C,r)+\frac{1}{2}\text{logr}+(k-1)\text{logr}-\text{logr}$$

$$+O(1).\ \text{Thus,}(n-3)T(C,r)-2(k-1)\text{logr}+\frac{1}{2}\text{logr}\le O(1).$$

On the other hand, since C is not constant we obtain $T(C,r)\ge\text{logr}+O(1)$. So $(n-2k-1)\text{logr}+\frac{1}{2}\text{logr}\le O(1)$. From this and $n\ge 2k+\frac{k(m+d)+16}{m-d}>2k+1$ we obtain a contradiction. So $s\equiv 0$.

Therefore, $C^n=D^n$ and $C=eD, f^m(z)f^d(qz+c)=eg^m(z)g^d(qz+c)$ with $e^n=1$. Set $h=\frac{f}{g}$. Assume h is not constant. Then $h(qz+c)=\frac{f(qz+c)}{g(qz+c)}$ is not constant, and $T(h(qz+c),r)=T(h,r)+O(1)$, $h^m=\frac{e}{h^d(qz+c)}, mT(h,r)=T(h^m,r)+O(1)=T(\frac{e}{h^d(qz+c)},r)+O(1)=dT(h(qz+c),r)+O(1)=dT(h,r)+O(1)$. Thus, $(m-d)T(h,r)=O(1)$. From this and because $m>d$, h is not constant, we obtain a contradiction. So h is constant. By $f^m(z)f^d(qz+c)=eg^m(z)g^d(qz+c), e^n=1$ we deduce that $f=hg$ with $h^{m+d}=e, h^{n(m+d)}=1$. Theorem 1.1 is proved.

Proof of Theorem 1.2. We shall use the notations in the proof of Theorem 1.1. Then, applying Lemma 3.6 to the $(C^n)^{(k)}$, $(D^n)^{(k)}$ we consider the following cases:

Case 1.

$$T(A,r)\le N_2(A,\infty,r)+N_2(A,0,r)+N_2(B,\infty,r)+N_2(B,0,r)$$

$$+2(N_1(A,\infty,r)+N_1(A,0,r))+N_1(B,\infty,r)+N_1(B,0,r)-2\text{logr}+O(1),$$

$$T(B,r)\le N_2(A,\infty,r)+N_2(A,0,r)+N_2(B,\infty,r)+N_2(B,0,r)+2(N_1(B,\infty,r)$$

$$+N_1(B,0,r))+N_1(A,\infty,r)+N_1(A,0,r)-2\text{logr}+O(1).$$

Combining the above inequalities we have

$$T(A,r)+T(B,r)\le 2(N_2(A,\infty,r)+N_2(A,0,r)+N_2(B,\infty,r)+N_2(B,0,r))$$

$$+3(N_1(A,\infty,r)+N_1(A,0,r)+N_1(B,\infty,r)+N_1(B,0,r))-4\text{logr}+O(1).$$

From this, as in Case 1. of Theorem 1.1, and by Lemma 3.5 we have

$$(n-2k)(m-d)(T(f,r)+T(g,r))+k(N(C,\infty,r)+N(D,\infty,r))$$

$$+N(P,0,r)+N(Q,0,r)\le T(A,r)+T(B,r)+O(1); \tag{3.16}$$

$$T(A,r)+T(B,r)\le 16(T(f,r)+T(g,r))+N(P,0,r)+N(Q,0,r)+k(m+d)(T(f,r)$$

$$+T(g,r))+k(N_1(C,\infty,r)+N_1(D,\infty,r))+3(N_1(A,\infty,r)+N_1(A,0,r)$$

$$+N_1(B,\infty,r)+N_1(B,0,r))-4\text{logr}+O(1); \tag{3.17}$$

$$N_1(A,\infty,r)\le 2T(f,r)+O(1), N_1(A,0,r)\le 2T(f,r)+k(m+d+2)T(f,r)+O(1);$$

$$N_1(B,\infty,r)\le 2T(g,r)+O(1), N_1(B,0,r))\le 2T(g,r)+k(m+d+2)T(g,r)+O(1). \tag{3.18}$$

By (3.16), (3.17), (3.18) we obtain

$$(n-2k)(m-d)(T(f,r)+T(g,r)) \leq 16(T(f,r)+T(g,r))+k(m+d)(T(f,r)+T(g,r))$$

$$+3[2T(f,r)+2T(f,r)+k(m+d+2)T(f,r)+2T(g,r)+2T(g,r)+k(m+d+2)T(g,r)]$$

$$-4\mathrm{logr}+O(1); = (28+2k(2m+2d+3))(T(f,r)+T(g,r))-4\mathrm{logr}+O(1).$$

Therefore, $[(n-2k)(m-d)-(28+2k(2m+2d+3))](T(f,r)+T(g,r))+4\mathrm{logr} \leq O(1)$. As $n \geq 2k+\frac{28+2k(2m+2d+3)}{m-d}$, we obtain a contradiction.

Case 2 and Case 3. We use the arguments similar to the ones in the Cases 2 and 3 of Theorem 1.1. So we conclude that $f = hg$ with $h^{n(m+d)} = 1$. Theorem 1.2 is proved.

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