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RESEARCH ARTICLES

Bounds of p-Adic Weighted Hardy-Cesàro Operators and Their Commutators on p-Adic Weighted Spaces of Morrey Types*

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Abstract—In this paper we aim to investigate the boundedness of the p-adic weighted Hardy-Cesàro operators and their commutators on weighted functional spaces of Morrey type. In each case, we obtain the corresponding operator norms.

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1. INTRODUCTION

Theories of functions and operators from $\mathbb{Q}_p^n$ into $\mathbb{R}$ or $\mathbb{C}$ play an important role in the p-adic quantum mechanics, in p-adic analysis [6, 8, 22, 28, 29]. p-Adic analysis and non-Archimedean geometry can be used not only for the description of geometry at small distances, but also for describing chaotic behavior of complicated systems such as spin glasses and fractals in the framework of traditional theoretical and mathematical physics (see [13, 19, 20, 28, 29] and the references therein). As far as we know, the studies of the p-adic Hardy operators and p-adic Hausdorff operators are also useful for p-adic analysis [8–10, 17, 27, 30, 31].

Let us give a brief history of results on these operators. In 1984, C. Carton-Lebrun and M. Fosset [3] considered a Hausdorff operator of special kind, which is called the weighted Hardy operator U_ψ, such as the following

$$U_\psi f(x) = \int_0^1 f(tx)\psi(t)dt, \qquad x \in \mathbb{R}^n. \tag{1.1}$$

The authors showed the boundedness of U_ψ on Lebesgue spaces and $BMO(\mathbb{R}^n)$ space. In 2001, J. Xiao [32] obtained that U_ψ is bounded on $L^p(\mathbb{R}^n)$ if and only if

$$\int_0^1 t^{-n/p}\psi(t)dt < \infty. \tag{1.2}$$

Meanwhile, the corresponding operator norm was worked out. The result seems to be of interest as it is related closely to the classical Hardy integral inequality. In addition, J. Xiao also obtained the $BMO(\mathbb{R}^n)$-bounds of U_ψ, which sharpened and extended the main result of C. Carton-Lebrun and

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M. Fosset in [3]. In 2006, K.S. Rim and J. Lee [27] proved analogue results of J. Xiao on p-adic fields. They introduce the p-adic form of U_ψ as the following

$$U_\psi^p f(x) = \int_{\mathbb{Z}_p^\star} f(tx)\psi(t)dt, \qquad x \in \mathbb{Q}_p^n. \tag{1.3}$$

Here $\mathbb{Q}_p$ is the field of all p-adic numbers and $\mathbb{Z}_p^\star = \{x \in \mathbb{Q}_p : \ 0 < |x|_p \leq 1\}$.

In 2012, N. M. Chuong and H. D. Hung [11] introduced the weighted Hardy-Cesàro operator, a more general form of U_ψ in the real case as

Definition 1.1. *Let $\psi : [0,1] \to [0,\infty)$, $s : [0,1] \to \mathbb{R}$ be measurable functions. The weighted Hardy-Cesàro operator $U_{\psi,s}$, associated to the parameter curve $s(x,t) := s(t)x$, is defined by*

$$U_{\psi,s} f(x) = \int_0^1 f\left(s(t)x\right)\psi(t)dt, \tag{1.4}$$

for all measurable complex valued functions f on $\mathbb{R}^n$.

With certain conditions on functions s and ω, the authors [11] proved $U_{\psi,s}$ is bounded on weighted Lebesgue spaces and weighted BMO spaces. The corresponding operator norms are worked out, too. The authors also give a necessary condition on the weight function ψ, for the boundedness of the commutators of operator $U_{\psi,s}$ on $L^r_\omega\left(\mathbb{R}^n\right)$ with symbols in $BMO_\omega\left(\mathbb{R}^n\right)$.

Motivated from above, H. D. Hung [17] considered the form of Hardy-Cesàro operator in p-adic analysis

$$U_{\psi,s}^p f(x) = \int_{\mathbb{Z}_p^\star} f\left(s(t)x\right)\psi(t)dt, \tag{1.5}$$

where $s : \mathbb{Z}_p^\star \to \mathbb{Q}_p$ and $\psi : \mathbb{Z}_p^\star \to [0;\infty)$ are measurable functions. In [17], by applying the boundedness of $U_{\psi,s}$ on p-adic weighted Lebesgue spaces, the author gives an interesting relations between p-adic Hardy operators and discrete Hardy inequalities on the real field.

For further informations on p-adic operators of Hardy type, we refer readers to [9, 11, 17, 27, 30, 31, 34, 36] and references therein. Notice that the classical Morrey spaces were introduced by C. B. Morrey in [26] to investigate the local behavior of solutions to second order elliptic partial differential equations. Moreover, it is well-known that Morrey spaces are useful to study the boundedness of Hardy-Littlewood maximal operator, the fractional integral operator and singular integral operators in the Morrey spaces (see [1, 5, 21]). The weighted Morrey spaces were firstly introduced by Y. Komori and S. Shirai with applications in studying classical operators of harmonic analysis. In p-adic cases, recently, some authors pay much attention to the (weighted) spaces of Morrey type in p-adic settings and use it to study the boundedness of p-adic fractional integral operators, p-adic weighted Hardy operators U_ψ^p (for examples see [7, 17, 31, 36]). As pointed out in [17], the boundedness of p-adic weighted Hardy-Cesàro operator has an interesting and important application in discrete Hardy inequalities, in this paper we study the bounds of p-adic weighted Hardy-Cesàro operator on p-adic weighted spaces of Morrey type. More concretely, we obtain the sharp bounds of those operators and their commutators on p-adic central Morrey spaces and p-adic central BMO spaces. Specially, our results are able to have applications to discrete Hardy inequalities.

Our paper is organized as followed. In Section 2 we give the notation and definitions that we shall use in the sequel. We define the weighted Morrey spaces $L_\omega^{q,\lambda}\left(\mathbb{Q}_p^n\right)$, the weighted central Morrey spaces $\dot{B}_\omega^{q,\lambda}\left(\mathbb{Q}_p^n\right)$ and the p-adic weighted central BMO spaces $CMO_\omega^{q,\lambda}\left(\mathbb{Q}_p^n\right)$. Some useful lemmas for the proofs of main theorems are proved. In Section 3 we state the main results on the boundedness of $U_{\psi,s}^p$ on the above weighted spaces. We also work out the norms of $U_{\psi,s}^p$ on such spaces. We note here that, our results generalize those obtained in [34, 36], where the authors proved such results only for U_ψ^p but without weights for functional spaces. In Section 4, by generalizing Lemma 15 of [36], we obtain the sufficient and necessary results for the boundedness of commutator operators $U_{\psi,s}^{p,b}$ with symbols in the weighted central Morrey spaces and in weighted central BMO spaces. Those will generalize such results obtained in [34, 36].

2. BASIC NOTIONS AND LEMMAS

Let p be a prime in $\mathbb{Z}$ and let $r \in \mathbb{Q}^\star$. Write $r = p^\gamma \frac{a}{b}$ where a and b are integers not divisible by p. Define the p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}$ by $|r|_p = p^{-\gamma}$ and $|0|_p = 0$. The absolute value $|\cdot|_p$ gives a metric on $\mathbb{Q}$ defined by $d_p(x, y) = |x - y|_p$. We denote by $\mathbb{Q}_p$ the completion of $\mathbb{Q}$ with respect to the metric d. $\mathbb{Q}_p$ with natural operations and topology induced by the metric d_p is a locally compact, non-discrete, complete and totally disconnected field. A non-zero element x of $\mathbb{Q}_p$, is uniquely represented as a canonical form $x = p^\gamma \left(x_0 + x_1 p + x_2 p^2 + \cdots\right)$ where $x_j \in \{0, 1, ..., p-1\}$ and $x_0 \neq 0$. We then have $|x|_p = p^{-\gamma}$. Define $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : \ |x|_p \leq 1\}$ and $\mathbb{Z}_p^\star = \mathbb{Z}_p \setminus \{0\}$.

$\mathbb{Q}_p^n = \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p$ contains all n-tuples of $\mathbb{Q}_p$. The norm on $\mathbb{Q}_p^n$ is $|x|_p = \max\limits_{1 \leq k \leq n} |x_k|_p$ for $x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n$. The space $\mathbb{Q}_p^n$ is complete metric locally compact and totally disconnected space. For each $a \in \mathbb{Q}_p$ and $x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n$, we denote $ax = (ax_1, \ldots, ax_n)$. For $\gamma \in \mathbb{Z}$, we denote B_γ as a γ-ball of $\mathbb{Q}_p^n$ with center at 0, containing all x with $|x|_p \leq p^\gamma$, and $S_\gamma = B_\gamma \setminus B_{\gamma-1}$ its boundary. Also, for $a \in \mathbb{Q}_p^n$, $B_\gamma(a)$ consists of all x with $x - a \in B_\gamma$, and $S_\gamma(a)$ consists of all x with $x - a \in S_\gamma$.

Since $\mathbb{Q}_p^n$ is a locally-compact commutative group with respect to addition, there exists the Haar measure dx on the additive group of $\mathbb{Q}_p^n$ normalized by $\int_{B_0} dx = 1$. Then $d(ax) = |a|_p^n dx$ for all $a \in \mathbb{Q}_p^\star$, $|B_\gamma(x)| = p^{n\gamma}$ and $|S_\gamma(x)| = p^{n\gamma}\left(1 - p^{-n}\right)$.

Let ω be any weight function on $\mathbb{Q}_p^n$, that is a nonnegative, locally integrable function from $\mathbb{Q}_p^n$ into $\mathbb{R}$. Let $L^r_\omega\left(\mathbb{Q}_p^n\right)$ $(1 \leq r < \infty)$ be the space of complex-valued functions f on $\mathbb{Q}_p^n$ so that

$$\|f\|_{L^r_\omega(\mathbb{Q}_p^n)} = \left(\int_{\mathbb{Q}_p^n} |f(x)|^r \omega(x) dx\right)^{1/r} < \infty.$$

For further readings on p-adic analysis, see [29]. Here, some often used computational principles are worth mentioning at the outset. Firstly, if $f \in L^1_\omega\left(\mathbb{Q}_p\right)$ we can write

$$\int\limits_{\mathbb{Q}_p^n} f(x)\omega(x)dx = \sum_{\gamma \in \mathbb{Z}} \int\limits_{S_\gamma} f(y)\omega(y)dy. \tag{2.1}$$

Secondly, we also often use the fact that

$$\int\limits_{\mathbb{Q}_p^n} f\left(ax\right) dx = \frac{1}{|a|_p^n} \int\limits_{\mathbb{Q}_p^n} f(x)dx, \tag{2.2}$$

if $a \in \mathbb{Q}_p^n \setminus \{0\}$ and $f \in L^1(\mathbb{Q}_p^n)$.

The weighted BMO spaces $BMO_\omega(\mathbb{R}^n)$ was firstly introduced by B. Muckenhoupt and R. Wheeden [24], where they proved that $BMO_\omega(\mathbb{R}^n)$ is the dual of weighted Hardy spaces. The p-adic BMO type spaces appeared in some recent papers (cf. [7, 9, 17, 27, 34, 36]), where they were used to study the boundedness of p-adic operators of Hardy type. The p-adic weighted spaces $BMO_\omega\left(\mathbb{Q}_p^n\right)$ are defined as the follows.

$$\|f\|_{BMO_\omega(\mathbb{Q}_p^n)} = \sup_B \frac{1}{\omega(B)} \int_B |f(x) - f_{B,\omega}|\omega(x)dx < \infty, \tag{2.3}$$

where supremum is taken over all ball B of $\mathbb{Q}_p^n$. Here, $\omega(B) = \int_B \omega(x)dx$, and $f_{B,\omega}$ is the mean value of f on B with weight ω:

$$f_{B,\omega} = \frac{1}{\omega(B)} \int_B f(x)\omega(x)dx. \tag{2.4}$$

In whole paper, to be simple we denote $f_{\gamma,\omega} = f_{B_\gamma,\omega}$.

Let $s : \mathbb{Z}_p^\star \to \mathbb{Q}_p$ and $\psi : \mathbb{Z}_p^\star \to \mathbb{R}_+$ be measurable functions and $\omega : \mathbb{Q}_p^n \to \mathbb{R}_+$ be a locally integrable function. For a function f on $\mathbb{Q}_p^n$, we define the p-adic weighted Hardy-Cesàro operator $U^p_{\psi,s}$ on $\mathbb{Q}_p^n$ as

$$U^p_{\psi,s} f(x) = \int_{\mathbb{Z}_p^\star} f\left(s(t)x\right)\psi(t)dt. \tag{2.5}$$

We shall consider the class of weights $\mathcal{W}_\alpha$, which consists of all nonnegative locally integrable function ω on $\mathbb{Q}_p^n$ so that $\omega(tx) = |t|_p^\alpha \omega(x)$ for all $x \in \mathbb{Q}_p^n$ and $t \in \mathbb{Q}_p^\star$ and $0 < \int_{S_0} \omega(x)dx < \infty$. It is easy to see that $\omega(x) = |x|_p^\alpha$ is in $\mathcal{W}_\alpha$ if and only if $\alpha > -n$. It is given in [17] that, for any $\omega \in \mathcal{W}_\alpha$, then $U^p_{\psi,s}$ is bounded on $BMO_\omega\left(\mathbb{Q}_p^n\right)$ if and only if $\int_{\mathbb{Z}_p^\star} \psi(t)dt$ is finite (see theorem 3.3 [17]). The following lemma will be useful in the sequel.

Lemma 2.1. *Let $\omega \in \mathcal{W}_\alpha$, $\alpha > -n$. Then for any $\gamma \in \mathbb{Z}$, we have*

$$\omega(B_\gamma) = p^{(n+\alpha)\gamma} \cdot \omega(B_0) \quad \textit{and} \quad \omega(S_\gamma) = p^{(n+\alpha)\gamma} \cdot \omega(S_0).$$

Since the proof of Lemma 2.1 is elementary, it will be omitted. The next lemma is proved in [17].

Lemma 2.2 ([17], Lemma 6.1). *If ω belongs to $\mathcal{W} = \bigcup_{\alpha > -n} \mathcal{W}_\alpha$, then* $\log|x|_p \in BMO_\omega\left(\mathbb{Q}_p^n\right)$.

It is well-known that Morrey spaces are useful to study the local behavior of solutions to second-order elliptic partial differential equations and the boundedness of Hardy-Littlewood maximal operator, the fractional integral operators, singular integral operators (see [1, 5, 21]). We notice that the weighted Morrey spaces in Euclide settings were firstly introduced by Y. Komori and S. Shirai [21], where they used them to study the boundedness of some important classical operators in harmonic analysis like Hardy-Littlewood maximal operator, Calderón-Zygmund operators. Their p-adic versions are given as follows.

Definition 2.3. *Let ω be a weight function on $\mathbb{Q}_p^n$, $1 \le q \le \infty$, and λ be real numbers such that $-\frac{1}{q} \le \lambda < \infty$. The weighted p-adic Morrey space $L^{q,\lambda}_\omega(\mathbb{Q}_p^n)$ is defined as the set of all functions $f \in L^q_{\omega,loc}(\mathbb{Q}_p^n)$ so that $\|f\|_{L^{q,\lambda}_\omega(\mathbb{Q}_p^n)} < \infty$, where*

$$\|f\|_{L^{q,\lambda}_\omega(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} \sup_{a\in\mathbb{Q}_p^n} \left(\frac{1}{\omega\left(B_\gamma\left(a\right)\right)^{1+\lambda q}} \int_{B_\gamma(a)} |f(x)|^q \omega(x)dx \right)^{\frac{1}{q}}. \tag{2.6}$$

With the norm $\|\cdot\|_{L^{q,\lambda}_\omega(\mathbb{Q}_p^n)}$, $L^{q,\lambda}_\omega(\mathbb{Q}_p^n)$ becomes a Banach space. It is easy from Definition 2.3 that $L^{q,-\frac{1}{q}}_\omega\left(\mathbb{Q}_p^n\right) = L^q_\omega\left(\mathbb{Q}_p^n\right)$. Here we restrict our consideration in case when λ belongs to $\left[-\frac{1}{q}, \infty\right)$ since the fact that $L^{q,\lambda}_\omega\left(\mathbb{Q}_p^n\right) = \{0\}$ for any $\lambda < -\frac{1}{q}$. For some recent developments of Morrey spaces and their related function spaces on $\mathbb{R}^n$, we refer the reader to [33]. One of useful example for functions from p-adic weighted Morrey spaces is given in the following lemma.

Lemma 2.4. *Let $1 < q < \infty, -\frac{1}{q} \le \lambda \le 0$ and $\omega \in \mathcal{W}_\alpha$, where $\alpha > -n$. If $f_0(x) = |x|_p^{(n+\alpha)\lambda}$ then $f_0 \in L^{q,\lambda}_\omega(\mathbb{Q}_p^n)$ and $\|f_0\|_{L^{q,\lambda}_\omega(\mathbb{Q}_p^n)} > 0$.*

Proof. Let $a \in \mathbb{Q}_p^n$ and $\gamma \in \mathbb{Z}$, we put

$$I_{a,\gamma} = \frac{1}{\omega\left(B_\gamma\left(a\right)\right)^{1+\lambda q}} \int_{B_\gamma(a)} |f_0(x)|^q \omega(x)dx.$$

Since $f_0(x) > 0$ almost everywhere $x \in \mathbb{Q}_p^n$, it is enough to prove $I_{a,\gamma} \le C$, where C is a positive constant that does not depend on a, γ. We consider two cases. First case if $|a|_p = p^{\gamma'} > p^\gamma$. For each

$x \in B_\gamma(a)$, then $|x|_p = \max\{|a|_p, |x-a|_p\} = |a|_p$. This implies $B_\gamma(a) \subset S_{\gamma'}$. As a consequence, we have that $I_{a,\gamma}$ equals to

$$\begin{aligned}\frac{1}{\omega\left(B_\gamma\left(a\right)\right)^{1+\lambda q}}\int_{B_\gamma(a)}|x|^{(n+\alpha)\lambda q}\omega(x)dx &= \left(|a|_p^{-(n+\alpha)}\omega\left(B_\gamma\left(a\right)\right)\right)^{-\lambda q}\\ &\leq \left(|a|_p^{-(n+\alpha)}\omega\left(S_{\gamma'}\right)\right)^{-\lambda q}\\ &=(\omega(S_0))^{-\lambda q} < \infty.\end{aligned}$$

Now we consider the left case, when $|a|_p \leq p^\gamma$. In that case, $B_\gamma(a) = B_\gamma$. Similarly, we get

$$I_{a,\gamma} \leq \left(p^{-(n+\alpha)\gamma}\omega(B_\gamma)\right)^{-\lambda q} = \left(\omega(B_0)\right)^{-\lambda q}.$$

Thus, we obtain that $I_{a,\gamma} \leq \max\left\{(\omega(S_0))^{-\lambda q}, (\omega(B_0))^{-\lambda q}\right\}$ for any $(a,\gamma) \in \mathbb{Q}_p^n \times \mathbb{Z}$. This completes the proof of the lemma. □

In [2], J. Alvarez et al. studied the relationships between central BMO spaces and Morrey spaces. Furthermore, they introduced λ-central bounded mean oscillation spaces and central Morrey spaces, respectively. Next, we introduce their p-adic versions. Here we shall consider their p-adic weighted versions and we will prove that such spaces are useful to study the boundedness of $U^p_{\psi,s}$.

Definition 2.5. *Let λ, q be real numbers so that $1 < q < \infty$. We define the p-adic weighted central Morrey space $\dot{B}^{q,\lambda}_\omega\left(\mathbb{Q}_p^n\right)$ by the set of all functions f on $\mathbb{Q}_p^n$ which $f \in L^q_{\omega,loc}(\mathbb{Q}_p^n)$ such that $\|f\|_{\dot{B}^{q,\lambda}_\omega(\mathbb{Q}_p^n)} < \infty$, where*

$$\|f\|_{\dot{B}^{q,\lambda}_\omega(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}}\left(\frac{1}{\omega\left(B_\gamma\right)^{1+\lambda q}}\int_{B_\gamma}|f(x)|^q\omega(x)dx\right)^{\frac{1}{q}}. \tag{2.7}$$

It is clear that $L^{q,\lambda}_\omega(\mathbb{Q}_p^n)$ is continuously embedded in $\dot{B}^{q,\lambda}_\omega\left(\mathbb{Q}_p^n\right)$ for all $1 < q < \infty, \lambda \in \mathbb{R}$. Moreover, $\dot{B}^{q,\lambda}_\omega\left(\mathbb{Q}_p^n\right)$ is a Banach space and reduce to zero when $\lambda < -\frac{1}{q}$. We remark that if $1 < q_1 < q_2 < \infty$, then $\dot{B}^{q_2,\lambda}_\omega\left(\mathbb{Q}_p^n\right) \subset \dot{B}^{q_1,\lambda}_\omega\left(\mathbb{Q}_p^n\right)$ for $\lambda \in \mathbb{R}$. Indeed, this follows by applying Hölder's inequality. On the other hand, while $b_0(x) = \log|x|_p \in BMO_\omega\left(\mathbb{Q}_p^n\right)$, $b_0(x) \notin \dot{B}^{q,\lambda}_\omega\left(\mathbb{Q}_p^n\right)$ where $\lambda \leq 0$. To see this, just note that $\frac{1}{\omega(B_\gamma)^{1+\lambda q}}\int_{B_\gamma}|b_0(x)|^q\omega(x)dx \sim \gamma^q$ and $\omega\left(B_\gamma\right)^{1+\lambda q} \sim p^{(n+\alpha)(1+\lambda q)\gamma}$ when $\gamma \to \infty$.

In proving the boundedness of commutators, we will need the following lemma.

Lemma 2.6. *Let $1 < q < \infty, -\frac{1}{q} \leq \lambda \leq 0$ and $\omega \in \mathcal{W}_\alpha$, $\alpha > -n$. Then the function $f_0(x) = |x|_p^{(n+\alpha)\lambda}$ belongs to $\dot{B}^{q,\lambda}_\omega(\mathbb{Q}_p^n)$.*

Proof. From Lemma 2.4, f_0 belongs to $L^{q,\lambda}_\omega\left(\mathbb{Q}_p^n\right)$. Since $L^{q,\lambda}_\omega\left(\mathbb{Q}_p^n\right)$ is continuously included in $\dot{B}^{q,\lambda}_\omega\left(\mathbb{Q}_p^n\right)$, we get that $f_0 \in \dot{B}^{q,\lambda}_\omega\left(\mathbb{Q}_p^n\right)$. □

The spaces of bounded central mean oscillation $CMO^q\left(\mathbb{R}^d\right)$ appears naturally when considering the dual spaces of the homogeneous Herz type Hardy spaces (see [2, 4, 15, 25]). The p-adic weighted central BMO spaces are defined as follows.

Definition 2.7. *Let $\lambda < \frac{1}{n}$ and $1 < q < \infty$ be two real numbers. The p-adic weighted space $CMO^{q,\lambda}_\omega\left(\mathbb{Q}_p^n\right)$ is defined as the set of all function $f \in L^q_{\omega,loc}\left(\mathbb{Q}_p^n\right)$ such that*

$$\|f\|_{CMO^{q,\lambda}_\omega(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}}\left(\frac{1}{\omega\left(B_\gamma\right)^{1+\lambda q}}\int_{B_\gamma}|f(x) - f_{\gamma,\omega}|^q\omega(x)dx\right)^{\frac{1}{q}} < \infty. \tag{2.8}$$

It is clear that, $CMO^{q,\lambda}_{\omega}\left(\mathbb{Q}_p^n\right)$ becomes a Banach space if we identify functions that differ from a constant. When $\lambda = 0$, $CMO^{q,\lambda}_{\omega}\left(\mathbb{Q}_p^n\right)$ reduce to $CMO^{q}_{\omega}\left(\mathbb{Q}_p^n\right)$ with corresponding norm is

$$\|f\|_{CMO^q_{\omega}(\mathbb{Q}_p^n)} := \sup_{\gamma\in\mathbb{Z}} \left(\frac{1}{\omega\left(B_\gamma\right)} \int_{B_\gamma} |f(x) - f_{\gamma,\omega}|^q \omega(x)dx\right)^{\frac{1}{q}}.$$

On the other hand, it follows from Definition 2.7 that, $\dot{B}^{q,\lambda}_{\omega}\left(\mathbb{Q}_p^n\right)$ are Banach spaces continuously included in $CMO^{q}_{\omega}\left(\mathbb{Q}_p^n\right)$ spaces. By a simple argument one can see that $CMO^{q,\lambda}_{\omega}\left(\mathbb{Q}_p^n\right)$ reduces to the constant functions when $\lambda < -\frac{1}{q}$. In the sequel, we need the following result.

Lemma 2.8. *Assume that $\omega \in \mathcal{W}_\alpha$, $\alpha > -n$. Then, for any $1 < q < \infty$, there exists a positive constant C_q such that*

$$\|f\|_{CMO^q_{\omega}(\mathbb{Q}_p^n)} \leq C_q \|f\|_{BMO_{\omega}(\mathbb{Q}_p^n)}. \tag{2.9}$$

To prove the lemma, a usual way is to show that functions in BMO are locally exponentially integrable. Since this fact is based on the theory of Calderón-Zygmund decompositions in p-adic settings, which are systematically introduced in [9, 17, 18], we leave the proof of Lemma 2.8 to the readers (also see [18] for a proof in case $\omega \equiv 1$).

3. BOUNDS OF $U^p_{\psi,s}$ ON WEIGHTED SPACES OF MORREY TYPE

This section will be devoted to state and prove results on the bounds of $U^p_{\psi,s}$ on p-adic weighted spaces of Morrey type. Throughout the whole paper, $s : \mathbb{Z}_p^\star \to \mathbb{Q}_p$ will denote a measurable function. By ω we will denote a weight from $\mathcal{W}_\alpha$, where $\alpha > -n$. We also denote by ψ a nonnegative and measurable function on $\mathbb{Z}_p^\star$.

Theorem 3.1. *Let $1 < q < \infty, -\frac{1}{q} \leq \lambda \leq 0$ be real numbers. Let ψ be a nonnegative, measurable function on $\mathbb{Z}_p^\star$. Then, $U^p_{\psi,s}$ is bounded on $L^{q,\lambda}_{\omega}(\mathbb{Q}_p^n)$ if and only if*

$$\mathcal{A} := \int_{\mathbb{Z}_p^\star} |s(t)|_p^{(n+\alpha)\lambda} \psi(t)dt < \infty. \tag{3.1}$$

Moreover, in that case, the operator norm of $U^p_{\psi,s}$ on $L^{q,\lambda}_{\omega}(\mathbb{Q}_p^n)$ equals to $\mathcal{A}$.

Proof. Suppose that $\mathcal{A}$ is finite. Let $f \in L^{q,\lambda}_{\omega}(\mathbb{Q}_p^n)$. Using Minkowski's inequality (see [16]) and p-adic change of variable (2.2), we have:

$$\begin{aligned}
&\left(\frac{1}{\omega\left(B_\gamma\left(a\right)\right)^{1+\lambda q}} \int_{B_\gamma(a)} |U^p_{\psi,s} f(x)|^q \omega(x)dx\right)^{\frac{1}{q}} \\
&= \left(\frac{1}{\omega\left(B_\gamma\left(a\right)\right)^{1+\lambda q}} \int_{B_\gamma(a)} \left|\int_{\mathbb{Z}_p^\star} f(s(t)x)\psi(t)dt\right|^q \omega(x)dx\right)^{\frac{1}{q}} \\
&\leq \int_{\mathbb{Z}_p^\star} \left(\frac{1}{\omega\left(s(t)B_\gamma\left(a\right)\right)^{1+\lambda q}} \int_{s(t)B_\gamma(a)} |f(y)|^q \omega(y)dy\right)^{\frac{1}{q}} . |s(t)|_p^{(n+\alpha)\lambda} . \psi(t)dt \\
&\leq \|f\|_{L^{q,\lambda}_{\omega}(\mathbb{Q}_p^n)} . \int_{\mathbb{Z}_p^\star} |s(t)|_p^{(n+\alpha)\lambda} . \psi(t)dt \\
&< \infty.
\end{aligned}$$

Thus, if $\mathcal{A}$ is finite then, $U^p_{\psi,s}$ is bounded on $L^{q,\lambda}_\omega(\mathbb{Q}^n_p)$ and

$$\|U^p_{\psi,s}\|_{L^{q,\lambda}_\omega(\mathbb{Q}^n_p)\to L^{q,\lambda}_\omega(\mathbb{Q}^n_p)} \le \mathcal{A}. \tag{3.2}$$

On the other hand, assume that $U^p_{\psi,s}$ is bounded on $L^{q,\lambda}_\omega(\mathbb{Q}^n_p)$. Take $f_0(x) = |x|_p^{(n+\alpha)\lambda}$, applying Lemma 2.4 , we have $f_0 \in L^{q,\lambda}_\omega(\mathbb{Q}^n_p)$ and $\|f_0\|_{L^{q,\lambda}_\omega(\mathbb{Q}^n_p)} > 0$. Note that, $U^p_{\psi,s}f_0(x) = f_0(x)\cdot\mathcal{A}$. So it follows that

$$\begin{aligned}\|U^p_{\psi,s}f_0\|_{L^{q,\lambda}_\omega(\mathbb{Q}^n_p)} &= \mathcal{A}\cdot\|f_0\|_{L^{q,\lambda}_\omega(\mathbb{Q}^n_p)}\\ &\le \|U^p_{\psi,s}\|_{L^{q,\lambda}_\omega(\mathbb{Q}^n_p)\to L^{q,\lambda}_\omega(\mathbb{Q}^n_p)}.\|f_0\|_{L^{q,\lambda}_\omega(\mathbb{Q}^n_p)}.\end{aligned}$$

Hence,

$$\mathcal{A} \le \|U^p_{\psi,s}\|_{L^{q,\lambda}_\omega(\mathbb{Q}^n_p)\to L^{q,\lambda}_\omega(\mathbb{Q}^n_p)}. \tag{3.3}$$

From (3.2) and (3.3), we deduce the desired result. □

Theorem 3.2. *Let $1 < q < \infty, -\frac{1}{q} \le \lambda \le 0$. Then, $U^p_{\psi,s}$ is bounded on $\dot{B}^{q,\lambda}_\omega\left(\mathbb{Q}^n_p\right)$ if and only if $\mathcal{A}$ is finite. Moreover,*

$$\|U^p_{\psi,s}\|_{\dot{B}^{q,\lambda}_\omega\left(\mathbb{Q}^n_p\right)\to\dot{B}^{q,\lambda}_\omega\left(\mathbb{Q}^n_p\right)} = \mathcal{A}. \tag{3.4}$$

Proof. From the proof of Theorem 3.1, with $a = 0$, we obtain that

$$\left(\frac{1}{\omega\left(B_\gamma\right)^{1+\lambda q}}\int_{B_\gamma}\left|U^p_{\psi,s}f(x)\right|^q\omega(x)dx\right)^{\frac{1}{q}} \le \mathcal{A}\cdot\|f\|_{\dot{B}^{q,\lambda}_\omega\left(\mathbb{Q}^n_p\right)},$$

for all $f \in \dot{B}^{q,\lambda}_\omega\left(\mathbb{Q}^n_p\right)$. This implies that $U^p_{\psi,s}$ is bounded on $\dot{B}^{q,\lambda}_\omega\left(\mathbb{Q}^n_p\right)$ if $\mathcal{A}$ is finite. The converse is similar to the proof of Theorem 3.1 since $f_0 \in L^{q,\lambda}_\omega\left(\mathbb{Q}^n_p\right)$, implies that $f_0 \in \dot{B}^{q,\lambda}_\omega\left(\mathbb{Q}^n_p\right)$, and the result follows immediately. □

We note here that Theorem 3.2 has a nice application to discrete Hardy inequalities. In fact when $\lambda = -\frac{1}{q}$ then $\dot{B}^{q,\lambda}_\omega\left(\mathbb{Q}^n_p\right)$ reduces to $L^q_\omega\left(\mathbb{Q}^n_p\right)$, thus as pointed out in ([17]), Corollary 3.2 in [17] is a corollary of this Theorem.

Theorem 3.3. *Let $1 < q < \infty, 0 \le \lambda < \frac{1}{n}$. Then $U^p_{\psi,s}$ is bounded on $CMO^{q,\lambda}_\omega\left(\mathbb{Q}^n_p\right)$ if and only if $\mathcal{A}$ is finite. Moreover,*

$$\|U^p_{\psi,s}\|_{CMO^{q,\lambda}_\omega\left(\mathbb{Q}^n_p\right)\to CMO^{q,\lambda}_\omega\left(\mathbb{Q}^n_p\right)} = \mathcal{A}. \tag{3.5}$$

Proof. Suppose that $\mathcal{A}$ is finite, and $f \in CMO^{q,\lambda}_\omega\left(\mathbb{Q}^n_p\right)$. Let γ be any integer number. Using Fubini theorem (see [29]) and change of variable, for any $f \in CMO^{q,\lambda}_\omega\left(\mathbb{Q}^n_p\right)$, we have

$$\begin{aligned}\left(U^p_{\psi,s}f\right)_{\gamma,\omega} &= \frac{1}{\omega\left(B_\gamma\right)}\int_{B_\gamma}\left(\int_{\mathbb{Z}^\star_p} f(s(t)x)\psi(t)dt\right)\omega(x)dx\\ &= \int_{\mathbb{Z}^\star_p} f_{s(t)B_\gamma,\omega}\psi(t)dt.\end{aligned}$$

Applying Minkowski's inequality, we get

$$\left(\frac{1}{\omega\left(B_\gamma\right)^{1+\lambda q}}\int_{B_\gamma}\left|U^p_{\psi,s}f(x)-\left(U^p_{\psi,s}f\right)_{\gamma,\omega}\right|^q\omega(x)dx\right)^{\frac{1}{q}}$$
$$=\left(\frac{1}{\omega\left(B_\gamma\right)^{1+\lambda q}}\int_{B_\gamma}\left|\int_{\mathbb{Z}_p^\star}\left(f(s(t)x)-f_{s(t)B_\gamma,\omega}\right)\psi(t)dt\right|^q\omega(x)dx\right)^{\frac{1}{q}}$$
$$\le\int_{\mathbb{Z}_p^\star}\left(\frac{1}{\omega\left(B_\gamma\right)^{1+\lambda q}}\int_{s(t)B_\gamma}\left|f(y)-f_{s(t)B_\gamma,\omega}\right|^q\omega(y)dy\right)^{\frac{1}{q}}|s(t)|_p^{(n+\alpha)\lambda}.\psi(t)dt$$
$$\le\|f\|_{CMO^{q,\lambda}_\omega(\mathbb{Q}_p^n)}\mathcal{A}$$
$$<\infty.$$

Therefore, $U^p_{\psi,s}$ is bounded on $CMO^{q,\lambda}_\omega\left(\mathbb{Q}_p^n\right)$ and

$$\|U^p_{\psi,s}\|_{CMO^{q,\lambda}_\omega(\mathbb{Q}_p^n)\to CMO^{q,\lambda}_\omega(\mathbb{Q}_p^n)}\le\mathcal{A}. \tag{3.6}$$

Conversely, if $U^p_{\psi,s}$ is bounded on $CMO^{q,\lambda}_\omega\left(\mathbb{Q}_p^n\right)$. Take $f_0(x)=|x|_p^{(n+\alpha)\lambda}$ then by Lemma 2.6, $f_0\in CMO^{q,\lambda}_\omega\left(\mathbb{Q}_p^n\right)$. On the other hand,

$$\left(\frac{1}{\omega\left(B_\gamma\right)^{1+\lambda q}}\int_{B_\gamma}\left|U^p_{\psi,s}f_0(x)-\left(U^p_{\psi,s}f_0\right)_{B_\gamma,\omega}\right|^q\omega(x)dx\right)^{\frac{1}{q}}$$
$$=\left(\frac{1}{\omega\left(B_\gamma\right)^{1+\lambda q}}\int_{B_\gamma}\left|f_0(x)-(f_0)_{B_\gamma,\omega}\right|^q\omega(x)dx\right)^{\frac{1}{q}}\mathcal{A}.$$

Therefore, $\|U^p_{\psi,s}f_0\|_{CMO^{q,\lambda}_\omega(\mathbb{Q}_p^n)}=\|f_0\|_{CMO^{q,\lambda}_\omega(\mathbb{Q}_p^n)}\mathcal{A}$, which implies immediately that

$$\|U^p_{\psi,s}\|_{CMO^{q,\lambda}_\omega(\mathbb{Q}_p^n)\to CMO^{q,\lambda}_\omega(\mathbb{Q}_p^n)}\ge\mathcal{A}. \tag{3.7}$$

Thus $\mathcal{A}$ is finite. From this together with (3.6) and (3.7), the proof of Theorem 3.3 will be completely demonstrated. □

4. CHARACTERIZATIONS OF WEIGHT FUNCTIONS FOR COMMUTATORS

More recently, a great attention was paid to the study on commutators of operators. A well-known result of R. R. Coifman, R. Rochberg and G. Weiss [12] states that the commutator $T_bf=bTf-T(bf)$ (where T is a Calderón-Zygmund singular integral operator) is bounded on $L^p(\mathbb{R}^n)$, $1<p<\infty$, if and only if $b\in BMO(\mathbb{R}^n)$. Many results have been generalized to commutators of other operators, not only Calderón-Zygmund singular integral operators. In p-adic settings, commutators of integral operators of Hardy type were recently investigated in various papers (see e.g. [7, 17, 34–36] and references therein). Recently, H. D. Hung [17] considered the commutator of $U^p_{\psi,s}$ as follows

$$U^{p,b}_{\psi,s}f:=bU_{\psi,s}f-U_{\psi,s}(bf). \tag{4.1}$$

In [17], the author gave a necessary condition on ψ so that the $U^{p,b}_{\psi,s}$ is bounded on weighted Lebesgue spaces with symbols in $BMO_\omega(\mathbb{Q}_p^n)$. In case $s(t)=t$, $U^{p,b}_{\psi,s}$ reduce to $U^{p,b}_\psi$ the commutator of U^p_ψ, which is considered by Q. Y. Wu, Z. W. Fu and L. Mi [34–36] where they proved the boundedness of $U^{p,b}_\psi$ on p-adic central Morrey and BMO spaces. This section aims to extend known results in [34–36] to $U^{p,b}_{\psi,s}$ operator and in case of weighted spaces.

Theorem 4.1. *Let q, q_1, q_2 be real numbers such that $1 < q < q_1 < \infty$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ and $-\frac{1}{q_1} \leq \lambda < 0$. Let $s : \mathbb{Z}_p^\star \to \mathbb{Q}_p$ be a measurable function such that $s(t) \neq 0$ almost everywhere $t \in \mathbb{Z}_p^\star$. We assume that $b \in CMO_\omega^{q_2}\left(\mathbb{Q}_p^n\right)$. If both $\mathcal{A}, \mathcal{B}$ are finite then the commutator $U_{\psi,s}^{p,b}$ is determined as a bounded operator from $\dot{B}_\omega^{q_1,\lambda}\left(\mathbb{Q}_p^n\right)$ to $\dot{B}_\omega^{q,\lambda}\left(\mathbb{Q}_p^n\right)$. Conversely, if $U_{\psi,s}^{p,b}$ is bounded $\dot{B}_\omega^{q_1,\lambda}\left(\mathbb{Q}_p^n\right)$ to $\dot{B}_\omega^{q_2,\lambda}\left(\mathbb{Q}_p^n\right)$ then $\mathcal{B}_\star$ is finite. Here and after,*

$$\mathcal{B} = \int_{\mathbb{Z}_p^\star} |s(t)|_p^{(n+\alpha)\lambda} \cdot \left|\log_p |s(t)|_p\right| \cdot \psi(t)dt, \tag{4.2}$$

and

$$\mathcal{B}_\star = \left|\int_{\mathbb{Z}_p^\star} |s(t)|_p^{(n+\alpha)\lambda} \cdot \log_p |s(t)|_p \cdot \psi(t)dt\right|. \tag{4.3}$$

Moreover,

$$\begin{aligned}\left(\frac{\|f_0\|_{\dot{B}_\omega^{q,\lambda}(\mathbb{Q}_p^n)}}{\|f_0\|_{\dot{B}_\omega^{q_1,\lambda}(\mathbb{Q}_p^n)}} \cdot \mathcal{B}_\star\right) \cdot \|b\|_{CMO_\omega^{q_2,\lambda_2}(\mathbb{Q}_p^n)} &\leq \|U_{\psi,s}^{p,b}\|_{\dot{B}_\omega^{q_1,\lambda}(\mathbb{Q}_p^n)\to\dot{B}_\omega^{q,\lambda}(\mathbb{Q}_p^n)} \\ &\leq \left(2\mathcal{A} + p^{n+\alpha}\mathcal{B}\right) \cdot \|b\|_{CMO_\omega^{q_2}(\mathbb{Q}_p^n)}.\end{aligned}$$

We notice that $\log_p |s(t)|_p$ is integer for any $t \in \mathbb{Z}_p^\star$. So if $|s(t)|_p \neq 1$ almost everywhere on $\mathbb{Z}_p^\star$, then $\mathcal{B} \geq \mathcal{A}$. On the other hand, if $|s(t)|_p \geq 1$ a.e $t \in \mathbb{Z}_p^\star$ or $|s(t)|_p \leq 1$ a.e $t \in \mathbb{Z}_p^\star$ then $\mathcal{B}_\star = \mathcal{B}$. These imply the following interesting corollary.

Corollary 4.2. *Let q, q_1, q_2 be real numbers such that $1 < q < q_1 < \infty$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ and $-\frac{1}{q_1} \leq \lambda < 0$. Let $s : \mathbb{Z}_p^\star \to \mathbb{Q}_p$ be a measurable function such that $|s(t)|_p > 1$ a.e $t \in \mathbb{Z}_p^\star$ or $|s(t)|_p < 1$ a.e $t \in \mathbb{Z}_p^\star$. We assume that $b \in CMO_\omega^{q_2}\left(\mathbb{Q}_p^n\right)$. Then the commutator $U_{\psi,s}^{p,b}$ is determined as a bounded operator from $\dot{B}_\omega^{q_1,\lambda}\left(\mathbb{Q}_p^n\right)$ to $\dot{B}_\omega^{q,\lambda}\left(\mathbb{Q}_p^n\right)$ if and only if $\mathcal{B}$ is finite.*

We note here that $\mathcal{A} < \infty$ does not imply $\mathcal{B} < \infty$. Indeed, we can find an easy counterexample as the following: let $s(t) = pt$, $\psi(t) = \frac{1}{|pt|_p^{1+(n+\alpha)\lambda}\left(\log_p |pt|_p\right)^2}$, then (2.1) and (2.2) imply

$$\mathcal{A} = \int_{\mathbb{Z}_p^\star} \frac{1}{|pt|_p \left(\log_p |pt|_p\right)^2} dt = \sum_{k\leq 0} \int_{S_k} \frac{dt}{p^{k-1}(k-1)^2} = \sum_{k\leq 0} \frac{1}{(k-1)^2} \cdot (p-1) < \infty,$$

and

$$\mathcal{B} = \int_{\mathbb{Z}_p^\star} \frac{1}{|pt|_p \left|\log_p |pt|_p\right|} dt = \sum_{k\leq 0} \int_{S_k} \frac{dt}{p^{k-1}|k-1|} = \sum_{k\leq 0} \frac{1}{|k-1|} \cdot (p-1) = \infty.$$

Proof. In order to prove Theorem 4.1, firstly we prove the following key lemma.

Lemma 4.3. *Suppose that b is a function in $CMO_\omega^{q,\lambda}\left(\mathbb{Q}_p^n\right)$ and γ, γ' are integer numbers. Here $\lambda \in \mathbb{R}$ so that $\lambda \leq \frac{1}{n}$, $1 < q < \infty$ and $\omega \in \mathcal{W}_\alpha$, with $\alpha > -n$. Then*

$$\left|b_{B_\gamma,\omega} - b_{B_{\gamma'},\omega}\right| \leq p^{n+\alpha} \cdot \left|\gamma' - \gamma\right| \cdot \max\{\omega\left(B_\gamma\right)^\lambda, \omega\left(B_{\gamma'}\right)^\lambda\} \cdot c_\lambda \cdot \|b\|_{CMO_\omega^{q,\lambda}}.$$

Here and after

$$c_\lambda = \begin{cases} 1 & \text{if} \quad \lambda = 0 \\ (n+\alpha)\ln p \cdot \frac{p^{(n+\alpha)\lambda}}{\left|p^{(n+\alpha)\lambda}-1\right|} \cdot |\lambda| & \text{if} \quad \lambda \neq 0. \end{cases}$$

Proof of Lemma 4.3. It is clear that is enough to prove the lemma for $\gamma' > \gamma$. Applying Hölder's inequality, we have

$$\begin{aligned}\left|b_{B_{\gamma+1},\omega} - b_{B_\gamma,\omega}\right| &\le \frac{1}{\omega(B_\gamma)} \int_{B_\gamma} \left|b(x) - b_{B_{\gamma+1},\omega}\right| \omega(x)dx \\ &\le \frac{1}{\omega(B_\gamma)} \int_{B_{\gamma+1}} \left|b(x) - b_{B_{\gamma+1},\omega}\right| \omega(x)dx \\ &\le \frac{\omega(B_{\gamma+1})^{\frac{q-1}{q}}}{\omega(B_\gamma)} \left(\int_{B_{\gamma+1}} \left|b(x) - b_{B_{\gamma+1},\omega}\right|^q \omega(x)dx\right)^{1/q} \\ &= p^{n+\alpha} \cdot \omega(B_{\gamma+1})^\lambda \cdot \|b\|_{CMO^{q,\lambda}_\omega(\mathbb{Q}_p^n)}.\end{aligned}$$

Therefore,

$$\left|b_{B_{\gamma+1},\omega} - b_{B_\gamma,\omega}\right| \le p^{n+\alpha} \cdot \omega(B_{\gamma+1})^\lambda \cdot \|b\|_{CMO^{q,\lambda}_\omega(\mathbb{Q}_p^n)}. \tag{4.4}$$

Now we have

$$\begin{aligned}\left|b_{B_{\gamma'},\omega} - b_{B_\gamma,\omega}\right| &\le \sum_{k=\gamma}^{\gamma'-1} \left|b_{B_{k+1},\omega} - b_{B_k,\omega}\right| \\ &\le p^{n+\alpha}\|b\|_{CMO^{q,\lambda}_\omega(\mathbb{Q}_p^n)} \cdot \sum_{k=\gamma}^{\gamma'-1} \omega(B_{k+1})^\lambda \\ &= p^{n+\alpha}\|b\|_{CMO^{q,\lambda}_\omega(\mathbb{Q}_p^n)} \cdot \omega(B_{\gamma'})^\lambda \sum_{j=0}^{\gamma'-\gamma-1} p^{-(n+\alpha)\lambda j}.\end{aligned}$$

Therefore, it suffices to prove lemma in case $\lambda \neq 0$. For the first case when $\lambda > 0$, by using the elementary inequality $1 - e^{-x} \le x$ in case $x = (n+\alpha)\lambda(\gamma' - \gamma)\ln p$, we obtain

$$\begin{aligned}\left|b_{B_{\gamma'},\omega} - b_{B_\gamma,\omega}\right| &\le p^{n+\alpha} \cdot \frac{p^{(n+\alpha)\lambda}}{p^{(n+\alpha)\lambda} - 1} \cdot (n+\alpha)\lambda(\gamma' - \gamma)\ln p\|b\|_{CMO^{q,\lambda}_\omega(\mathbb{Q}_p^n)} \cdot \omega(B_{\gamma'})^\lambda \\ &= p^{n+\alpha} \cdot \left|\gamma' - \gamma\right| \cdot \max\{\omega(B_\gamma)^\lambda, \omega(B_{\gamma'})^\lambda\} \cdot c_\lambda \cdot \|b\|_{CMO^{q,\lambda}_\omega(\mathbb{Q}_p^n)}.\end{aligned}$$

For the rest case when $\lambda < 0$, the proof is similar, so we omit it.

Now we shall prove Theorem 4.1. We use the ideas of [12, 14, 17, 34, 36]. Let us assume that both $\mathcal{A}$ and $\mathcal{B}$ are finite. By Minkowski's inequality and change of variable, we have

$$\begin{aligned}&\left(\frac{1}{\omega(B_\gamma)^{1+\lambda q}} \int_{B_\gamma} \left|U^{p,b}_{\psi,s} f(x)\right|^q \omega(x)dx\right)^{1/q} \\ \le\ &\left(\frac{1}{\omega(B_\gamma)^{1+\lambda q}} \int_{B_\gamma} \left(\int_{\mathbb{Z}_p^\star} (|b(x) - b(s(t)x)|)\,|f(s(t)x)|\,\psi(t)dt\right)^q \omega(x)dx\right)^{1/q} \\ \le\ &\left(\frac{1}{\omega(B_\gamma)^{1+\lambda q}} \int_{B_\gamma} \left(\int_{\mathbb{Z}_p^\star} \left(\left|b(x) - b_{B_\gamma,\omega}\right|\right)|f(s(t)x)|\,\psi(t)dt\right)^q \omega(x)dx\right)^{1/q} \\ &+ \left(\frac{1}{\omega(B_\gamma)^{1+\lambda q}} \int_{B_\gamma} \left(\int_{\mathbb{Z}_p^\star} \left(\left|b_{B_\gamma,\omega} - b_{s(t)B_\gamma,\omega}\right|\right)|f(s(t)x)|\,\psi(t)dt\right)^q \omega(x)dx\right)^{1/q} \\ &+ \left(\frac{1}{\omega(B_\gamma)^{1+\lambda q}} \int_{B_\gamma} \left(\int_{\mathbb{Z}_p^\star} \left(\left|b(s(t)x) - b_{s(t)B_\gamma,\omega}\right|\right)|f(s(t)x)|\,\psi(t)dt\right)^q \omega(x)dx\right)^{1/q}\end{aligned}$$

$=:I_1 + I_2 + I_3.$

To estimate I_1 and I_3, we use the fact that $\omega(tB_\gamma) = |t|_p^{n+\alpha}\omega(B_\gamma)$, and get by Minkowski's inequality and p-adic change of variable

$$I_1 \le \omega(B_\gamma)^{-\frac{1}{q}-\lambda} \int_{\mathbb{Z}_p^\star} \left(\int_{B_\gamma} \left| b(x) - b_{B_\gamma,\omega} \right|^{q_2} \omega(x)dx \right)^{1/q_2} \cdot \left(\int_{B_\gamma} |f(s(t)x)|^{q_1} \omega(x)dx \right)^{1/q_1}$$

$$= \left(\frac{1}{\omega(B_\gamma)} \int_{B_\gamma} \left| b(x) - b_{B_\gamma,\omega} \right|^{q_2} \omega(x)dx \right)^{1/q_2}$$

$$\times \int_{\mathbb{Z}_p^\star} \left(\frac{1}{\omega(s(t)B_\gamma)^{1+\lambda q_1}} \int_{s(t)B_\gamma} |f(y)|^{q_1} \omega(y)dy \right)^{1/q_1} |s(t)|_p^{(n+\alpha)\lambda} \psi(t)dt$$

$$\le \|b\|_{CMO_\omega^{q_2}(\mathbb{Q}_p^n)} \cdot \|f\|_{\dot{B}_\omega^{q_1,\lambda}(\mathbb{Q}_p^n)} \int_{\mathbb{Z}_p^\star} |s(t)|_p^{(n+\alpha)\lambda} \psi(t)dt.$$

Hence

$$I_1 \le \|b\|_{CMO_\omega^{q_2}(\mathbb{Q}_p^n)} \cdot \|f\|_{\dot{B}_\omega^{q_1,\lambda}(\mathbb{Q}_p^n)} \int_{\mathbb{Z}_p^\star} |s(t)|_p^{(n+\alpha)\lambda} \psi(t)dt. \tag{4.5}$$

Similarly to estimate I_1, we can deduce that

$$I_3 \le \|b\|_{CMO_\omega^{q_2}(\mathbb{Q}_p^n)} \cdot \|f\|_{\dot{B}_\omega^{q_1,\lambda}(\mathbb{Q}_p^n)} \int_{\mathbb{Z}_p^\star} |s(t)|_p^{(n+\alpha)\lambda} \psi(t)dt. \tag{4.6}$$

For the term I_2, applying Hölder's inequality, we have

$$I_2 \le \int_{\mathbb{Z}_p^\star} \left(\frac{1}{\omega(B_\gamma)^{1+\lambda q}} \int_{B_\gamma} |f(s(t)x)|^{q} \omega(x)dx \right)^{1/q} \cdot \left| b_{B_\gamma,\omega} - b_{s(t)B_\gamma,\omega} \right| \psi(t)dt$$

$$\le \int_{\mathbb{Z}_p^\star} \left(\frac{1}{\omega(B_\gamma)^{1+\lambda q_1}} \int_{B_\gamma} |f(s(t)x)|^{q_1} \omega(x)dx \right)^{1/q_1} \cdot \left| b_{B_\gamma,\omega} - b_{s(t)B_\gamma,\omega} \right| \psi(t)dt$$

$$\le \|f\|_{\dot{B}_\omega^{q_1,\lambda}(\mathbb{Q}_p^n)} \cdot \int_{\mathbb{Z}_p^\star} |s(t)|^{(n+\alpha)\lambda} \cdot \left| b_{B_\gamma,\omega} - b_{s(t)B_\gamma,\omega} \right| \cdot \psi(t)dt.$$

From the hypothesis of the theorem it follows that, for almost everywhere $t \in \mathbb{Z}_p^\star$, there exists an integer γ' such that $|s(t)|_p = p^{\gamma'}$. Using Lemma 4.3 with $\lambda = 0$, we get

$$\begin{aligned} \left| b_{B_\gamma,\omega} - b_{s(t)B_\gamma,\omega} \right| &= \left| b_{B_\gamma,\omega} - b_{B_{\gamma+\gamma'},\omega} \right| \\ &\le p^{n+\alpha} \cdot |\gamma'| \cdot \|b\|_{CMO_\omega^{q_2}(\mathbb{Q}_p^n)} \\ &= p^{n+\alpha} \cdot \|b\|_{CMO_\omega^{q_2}(\mathbb{Q}_p^n)} \left| \log_p |s(t)|_p \right|. \end{aligned}$$

Therefore we obtain

$$I_2 \le p^{n+\alpha} \|b\|_{CMO_\omega^{q_2}(\mathbb{Q}_p^n)} \cdot \|f\|_{\dot{B}_\omega^{q_1,\lambda}(\mathbb{Q}_p^n)} \int_{\mathbb{Z}_p^\star} |s(t)|_p^{(n+\alpha)\lambda} \left| \log_p |s(t)|_p \right| \psi(t)dt. \tag{4.7}$$

Combine (4.5), (4.6) and (4.7), we obtain that $I_1 + I_2 + I_3$ is not greater than

$$\left(2\mathcal{A} + p^{n+\alpha}\mathcal{B}\right) \cdot \|b\|_{CMO_\omega^{q_2}(\mathbb{Q}_p^n)} \cdot \|f\|_{\dot{B}_\omega^{q_1,\lambda}(\mathbb{Q}_p^n)}.$$

Thus, $U_{\psi,s}^{p,b}$ is bounded from $\dot{B}_\omega^{q_1,\lambda}(\mathbb{Q}_p^n)$ to $\dot{B}_\omega^{q,\lambda}(\mathbb{Q}_p^n)$. Moreover,

$$\|U_{\psi,s}^{p,b}\|_{\dot{B}_\omega^{q_1,\lambda}(\mathbb{Q}_p^n) \to \dot{B}_\omega^{q,\lambda}(\mathbb{Q}_p^n)} \le \left(2\mathcal{A} + p^{n+\alpha}\mathcal{B}\right) \cdot \|b\|_{CMO_\omega^{q_2}(\mathbb{Q}_p^n)}.$$

Now we assume that $U^{p,b}_{\psi,s}$ is bounded from $\dot{B}^{q_1,\lambda}_{\omega}\left(\mathbb{Q}^n_p\right)$ to $\dot{B}^{q,\lambda}_{\omega}\left(\mathbb{Q}^n_p\right)$. Take $b_0(x)=\log_p|x|_p$ then from lemma 2.2 and lemma 2.8 imply $b_0\in CMO^{q_2}_{\omega}\left(\mathbb{Q}^n_p\right)$. Since $\left|U^{p,b}_{\psi,s}f_0(x)\right|=f_0(x)\cdot\mathcal{B}_\star$, by Lemma 2.6 we get

$$\begin{aligned}\|U^{p,b}_{\psi,s}f_0\|_{\dot{B}^{q,\lambda}_{\omega}(\mathbb{Q}^n_p)} &= \|f_0\|_{\dot{B}^{q,\lambda}_{\omega}(\mathbb{Q}^n_p)}\cdot\mathcal{B}_\star\\ &=\left(\frac{\|f_0\|_{\dot{B}^{q,\lambda}_{\omega}(\mathbb{Q}^n_p)}}{\|f_0\|_{\dot{B}^{q_1,\lambda}_{\omega}(\mathbb{Q}^n_p)}}\cdot\mathcal{B}_\star\right)\|f_0\|_{\dot{B}^{q_1,\lambda}_{\omega}(\mathbb{Q}^n_p)}.\end{aligned}$$

Therefore we obtain that $\mathcal{B}_\star$ is finite. □

Theorem 4.4. *Let* $1<q<q_1<\infty$, $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}, -\frac{1}{q}<\lambda<0, -\frac{1}{q_1}<\lambda_1<0, 0<\lambda_2<\frac{1}{n}$ *and* $\lambda=\lambda_1+\lambda_2$. *Let* $s:\mathbb{Z}^\star_p\to\mathbb{Q}_p$ *be a measurable function such that* $s(t)\neq 0$ *almost everywhere. If* $\mathcal{C}$ *is finite, then for any* $b\in CMO^{q_2,\lambda_2}_{\omega}(\mathbb{Q}^n_p)$, *the corresponding commutator* $U^{p,b}_{\psi,s}$ *is bounded from* $\dot{B}^{q_1,\lambda_1}_{\omega}(\mathbb{Q}^n_p)$ *to* $\dot{B}^{q,\lambda}_{\omega}(\mathbb{Q}^n_p)$ *and we have*

$$\|U^{p,b}_{\psi,s}\|_{\dot{B}^{q_1,\lambda_1}_{\omega}(\mathbb{Q}^n_p)\to\dot{B}^{q,\lambda}_{\omega}(\mathbb{Q}^n_p)}\le(2+p^{n+\alpha}c_{\lambda_2})\cdot\mathcal{C}\cdot\|b\|_{CMO^{q_2,\lambda_2}_{\omega}(\mathbb{Q}^n_p)}\cdot\|f\|_{\dot{B}^{q_1,\lambda_1}_{\omega}(\mathbb{Q}^n_p)}.$$

Here and after, $\mathcal{C}=\int_{\mathbb{Z}^\star_p}\max\{1,|s(t)|^{(n+\alpha)\lambda_2}\}|s(t)|_p^{(n+\alpha)\lambda_1}\cdot|\log|s(t)|_p|\,\psi(t)dt.$

Proof. Using the similar arguments in the proof of Theorem 4.1, for each $\gamma\in\mathbb{Z}$, we will arrive at

$$\left(\frac{1}{\omega\left(B_\gamma\right)^{1+\lambda q}}\int_{B_\gamma}|U^{p,b}_{\psi,s}f(x)|^q\omega(x)dx\right)^{\frac{1}{q}}\le I_1+I_2+I_3,$$

where both I_1 and I_3 are not greater than

$$\|b\|_{CMO^{q_2,\lambda_2}_{\omega}(\mathbb{Q}^n_p)}\cdot\|f\|_{\dot{B}^{q_1,\lambda_1}_{\omega}(\mathbb{Q}^n_p)}\cdot\int_{\mathbb{Z}^\star_p}|s(t)|_p^{(n+\alpha)\lambda_1}\cdot\psi(t)dt.$$

The estimate for I_2,

$$\begin{aligned}I_2&=\left(\frac{1}{\omega\left(B_\gamma\right)^{1+\lambda q}}\int_{B_\gamma}\left(\int_{\mathbb{Z}^\star_p}\left|(b_{s(t)B_\gamma,\omega}-b_{B_\gamma,\omega})f(s(t)x)\right|\psi(t)dt\right)^q\omega(x)dx\right)^{\frac{1}{q}}\\ &\le\int_{\mathbb{Z}^\star_p}\left(\frac{1}{\omega\left(B_\gamma\right)^{1+\lambda q}}\int_{B_\gamma}|f(s(t)x)|^q\omega(x)dx\right)^{\frac{1}{q}}|b_{s(t)B_\gamma,\omega}-b_{B_\gamma,\omega}|\psi(t)dt\\ &\le\int_{\mathbb{Z}^\star_p}\left(\frac{1}{\omega\left(B_\gamma\right)^{1+\lambda_1 q_1}}\int_{B_\gamma}|f(s(t)x)|^{q_1}\omega(x)dx\right)^{\frac{1}{q_1}}\left(\frac{1}{\omega\left(B_\gamma\right)^{1+\lambda_2 q_2}}\int_{B_\gamma}\omega(x)dx\right)^{\frac{1}{q_2}}\\ &\quad\times|b_{s(t)B_\gamma,\omega}-b_{B_\gamma,\omega}|\psi(t)dt\\ &=\omega\left(B_\gamma\right)^{-\lambda_2}\int_{\mathbb{Z}^\star_p}\left(\frac{1}{\omega\left(s(t)B_\gamma\right)^{1+\lambda_1 q_1}}\int_{s(t)B_\gamma}|f(y)|^{q_1}\omega(y)dy\right)^{\frac{1}{q_1}}\\ &\quad\times|s(t)|_p^{(n+\alpha)\lambda_1}|b_{s(t)B_\gamma,\omega}-b_{B_\gamma,\omega}|\psi(t)dt\\ &\le\|f\|_{\dot{B}^{q_1,\lambda_1}_{\omega}(\mathbb{Q}^n_p)}\omega(B_\gamma)^{-\lambda_2}\int_{\mathbb{Z}^\star_p}|s(t)|_p^{(n+\alpha)\lambda_1}|b_{s(t)B_\gamma,\omega}-b_{B_\gamma,\omega}|\psi(t)dt.\end{aligned}$$

For each $t \in \mathbb{Z}_p^\star$ such that $s(t) \neq 0$, there exists $\gamma' = \gamma'(t) \in \mathbb{Z}$ so that $|s(t)|_p = p^{\gamma'}$. From Lemma 4.3, we get

$$\begin{aligned}&|b_{B_\gamma,\omega} - b_{s(t)B_\gamma,\omega}| = |b_{B_\gamma,\omega} - b_{B_{\gamma+\gamma'},\omega}|\\ &\leq p^{n+\alpha} c_{\lambda_2} |\gamma'| \|b\|_{CMO_\omega^{q_2,\lambda_2}(\mathbb{Q}_p^n)} \max\{\omega(B_\gamma)^{\lambda_2}, \omega(B_{\gamma'+\gamma})^{\lambda_2}\}.\end{aligned}$$

Thus,

$$\begin{aligned}I_2 &\leq p^{n+\alpha} c_{\lambda_2} \|b\|_{CMO_\omega^{q_2,\lambda_2}(\mathbb{Q}_p^n)} \|f\|_{\dot{B}_\omega^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\\ &\quad\times \int_{\mathbb{Z}_p^\star} \max\{1, p^{\gamma'(n+\alpha)\lambda_2}\} |s(t)|_p^{(n+\alpha)\lambda_1} \left|\log_p |s(t)|_p\right| \psi(t)dt\\ &\leq p^{n+\alpha} c_{\lambda_2} \|b\|_{CMO_\omega^{q_2,\lambda_2}(\mathbb{Q}_p^n)} \|f\|_{\dot{B}_\omega^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\\ &\quad\times \int_{\mathbb{Z}_p^\star} \max\{1, |s(t)|_p^{(n+\alpha)\lambda_2}\} |s(t)|_p^{(n+\alpha)\lambda_1} \left|\log_p |s(t)|_p\right| \psi(t)dt.\end{aligned}$$

Hence, we obtain

$$I_1 + I_2 + I_3 \leq (2 + p^{n+\alpha} c_{\lambda_2}) \mathcal{C} \|b\|_{CMO_\omega^{q_2,\lambda_2}(\mathbb{Q}_p^n)} \|f\|_{\dot{B}_\omega^{q_1,\lambda_1}(\mathbb{Q}_p^n)}.$$

So, we have proved that

$$\|U_{\psi,s}^{p,b}\|_{\dot{B}_\omega^{q,\lambda}(\mathbb{Q}_p^n)} \leq (2 + p^{n+\alpha} c_{\lambda_2}) \cdot \|b\|_{CMO_\omega^{q_2,\lambda_2}(\mathbb{Q}_p^n)} \cdot \mathcal{C} \cdot \|f\|_{\dot{B}_\omega^{q_1,\lambda_1}(\mathbb{Q}_p^n)}.$$

This completes the proof of Theorem 4.4. □

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