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RESEARCH ARTICLES

Freudenthal Duality in Gravity: from Groups of Type E_7 to Pre-Homogeneous Spaces*

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Abstract—Freudenthal duality can be defined as an anti-involutive, non-linear map acting on symplectic spaces. It was introduced in four-dimensional Maxwell-Einstein theories coupled to a non-linear sigma model of scalar fields. In this short review, I will consider its relation to the U-duality Lie groups of type E_7 in extended supergravity theories, and comment on the relation between the Hessian of the black hole entropy and the pseudo-Euclidean, rigid special (pseudo)Kähler metric of the pre-homogeneous spaces associated to the U-orbits.

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1. FREUDENTHAL DUALITY

We start and consider the following Lagrangian density in four dimensions (*cfr. e.g.* [1]):

$$\mathcal{L} = -\frac{R}{2} + \frac{1}{2} g_{ij}(\varphi)\,\partial_\mu \varphi^i \partial^\mu \varphi^j + \frac{1}{4} I_{\Lambda\Sigma}(\varphi)\, F^\Lambda_{\mu\nu} F^{\Sigma|\mu\nu} + \frac{1}{8\sqrt{-G}} R_{\Lambda\Sigma}(\varphi)\,\epsilon^{\mu\nu\rho\sigma} F^\Lambda_{\mu\nu} F^\Sigma_{\rho\sigma}, \tag{1.1}$$

describing Einstein gravity coupled to Maxwell (Abelian) vector fields and to a non-linear sigma model of scalar fields (with no potential); note that $\mathcal{L}$ may – but does not necessarily need to – be conceived as the bosonic sector of $D = 4$ (*ungauged*) supergravity theory. Out of the Abelian two-form field strengths F^Λ's, one can define their duals G_Λ, and construct a symplectic vector :

$$H := \left(F^\Lambda, G_\Lambda\right)^T, \quad {}^*G_{\Lambda|\mu\nu} := 2\frac{\delta \mathcal{L}}{\delta F^{\Lambda|\mu\nu}}. \tag{1.2}$$

We then consider the simplest solution of the equations of motion deriving from $\mathcal{L}$, namely a static, spherically symmetric, asymptotically flat, dyonic extremal black hole with metric [2]

$$ds^2 = -e^{2U(\tau)}dt^2 + e^{-2U(\tau)}\left[\frac{d\tau^2}{\tau^4} + \frac{1}{\tau^2}\left(d\theta^2 + \sin\theta d\psi^2\right)\right], \tag{1.3}$$

where $\tau := -1/r$. Thus, the two-form field strengths and their duals can be fluxed on the two-sphere at infinity S^2_∞ in such a background, respectively yielding the electric and magnetic charges of the black hole itself, which can be arranged in a symplectic vector $\mathcal{Q}$:

$$p^\Lambda \ := \frac{1}{4\pi}\int_{S^2_\infty} F^\Lambda, \quad q_\Lambda := \frac{1}{4\pi}\int_{S^2_\infty} G_\Lambda, \tag{1.4}$$

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$$\mathcal{Q} \ : \ = \left(p^{\Lambda}, q_{\Lambda}\right)^{T}. \tag{1.5}$$

Then, by exploiting the symmetries of the background (1.3), the Lagrangian (1.1) can be dimensionally reduced from $D = 4$ to $D = 1$, obtaining a 1-dimensional effective Lagrangian ($' := d/d\tau$) [3]:

$$\mathcal{L}_{D=1} = \left(U'\right)^{2} + g_{ij}\left(\varphi\right)\varphi^{i\prime}\varphi^{j\prime} + e^{2U}V_{BH}\left(\varphi, \mathcal{Q}\right), \tag{1.6}$$

where the so-called “effective black hole potential" is defined as [3]

$$V_{BH}\left(\varphi, \mathcal{Q}\right) := -\frac{1}{2}\mathcal{Q}^{T}\mathcal{M}\left(\varphi\right)\mathcal{Q}, \tag{1.7}$$

in terms of the symplectic and symmetric matrix [1]

$$\mathcal{M} \ : \ = \begin{pmatrix} \mathbb{I} & -R \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ -R & \mathbb{I} \end{pmatrix} = \begin{pmatrix} I + RI^{-1}R & -RI^{-1} \\ & \\ -I^{-1}R & I^{-1} \end{pmatrix}, \tag{1.8}$$

$$\mathcal{M}^{T} \ = \ \mathcal{M}; \quad \mathcal{M}\Omega\mathcal{M} = \Omega, \tag{1.9}$$

where $\mathbb{I}$ denotes the identity, and $R\left(\varphi\right)$ and $I\left(\varphi\right)$ are the scalar-dependent matrices occurring in (1.1); moreover, Ω stands for the symplectic metric ($\Omega^{2} = -\mathbb{I}$). Note that, regardless of the invertibility of $R\left(\varphi\right)$ and as a consequence of the physical consistence of the kinetic vector matrix $I\left(\varphi\right)$, $\mathcal{M}$ is negative-definite; thus, the effective black hole potential (1.7) is positive-definite.

By virtue of the matrix $\mathcal{M}$, one can introduce a (scalar-dependent) *anti-involution* $\mathcal{S}$ in any Maxwell-Einstein-scalar theory described by (1.1) with a symplectic structure Ω, as follows :

$$\mathcal{S}\left(\varphi\right) \ : \ = \Omega\mathcal{M}\left(\varphi\right); \tag{1.10}$$

$$\mathcal{S}^{2}\left(\varphi\right) \ = \ \Omega\mathcal{M}\left(\varphi\right)\Omega\mathcal{M}\left(\varphi\right) = \Omega^{2} = -\mathbb{I}; \tag{1.11}$$

in turn, this allows to define an anti-involution on the dyonic charge vector $\mathcal{Q}$, which has been called (scalar-dependent) *Freudenthal duality* [4–6]:

$$\mathfrak{F}\left(\mathcal{Q};\varphi\right) \ : \ = -\mathcal{S}\left(\varphi\right)\mathcal{Q}; \tag{1.12}$$

$$\mathfrak{F}^{2} \ = \ -\mathbb{I}, \quad \left(\forall\left\{\varphi\right\}\right). \tag{1.13}$$

By recalling (1.7) and (1.10), the action of $\mathfrak{F}$ on $\mathcal{Q}$, defining the so-called (φ-dependent) Freudenthal dual of $\mathcal{Q}$ itself, can be related to the symplectic gradient of the effective black hole potential V_{BH} :

$$\mathfrak{F}\left(\mathcal{Q};\varphi\right) = \Omega\frac{\partial V_{BH}\left(\varphi, \mathcal{Q}\right)}{\partial\mathcal{Q}}. \tag{1.14}$$

Through the attractor mechanism [7], all this enjoys an interesting physical interpretation when evaluated at the (unique) event horizon of the extremal black hole (1.3) (denoted below by the subscript “H"); indeed

$$\partial_{\varphi}V_{BH} \ = \ 0 \Leftrightarrow \lim_{\tau\rightarrow-\infty}\varphi^{i}\left(\tau\right) = \varphi^{i}_{H}\left(\mathcal{Q}\right); \tag{1.15}$$

$$S_{BH}\left(\mathcal{Q}\right) \ = \ \frac{A_{H}}{4} = \pi\left.V_{BH}\right|_{\partial_{\varphi}V_{BH}=0} = -\frac{\pi}{2}\mathcal{Q}^{T}\mathcal{M}_{H}\left(\mathcal{Q}\right)\mathcal{Q}, \tag{1.16}$$

where S_{BH} and A_{H}, respectively, denote the Bekenstein-Hawking entropy [8] and the area of the horizon of the extremal black hole, and the matrix horizon value $\mathcal{M}_{H}$ is defined as

$$\mathcal{M}_{H}\left(\mathcal{Q}\right) := \lim_{\tau\rightarrow-\infty}\mathcal{M}\left(\varphi\left(\tau\right)\right). \tag{1.17}$$

Correspondingly, one can define the (scalar-independent) horizon Freudenthal duality $\mathfrak{F}_{H}$ as the horizon limit of (1.12) :

$$\widetilde{\mathcal{Q}} \equiv \mathfrak{F}_{H}\left(\mathcal{Q}\right) := \lim_{\tau\rightarrow-\infty}\mathfrak{F}\left(\mathcal{Q};\varphi\left(\tau\right)\right) = -\Omega\mathcal{M}_{H}\left(\mathcal{Q}\right)\mathcal{Q} = \frac{1}{\pi}\Omega\frac{\partial S_{BH}\left(\mathcal{Q}\right)}{\partial\mathcal{Q}}. \tag{1.18}$$

Remarkably, the (horizon) Freudenthal dual of $\mathcal{Q}$ is nothing but ($1/\pi$ times) the symplectic gradient of the Bekenstein-Hawking black hole entropy S_{BH}; this latter, from dimensional considerations, is only constrained to be an homogeneous function of degree two in $\mathcal{Q}$. As a result, $\widetilde{\mathcal{Q}} = \widetilde{\mathcal{Q}}(\mathcal{Q})$ is generally a complicated (non-linear) function, homogeneous of degree one in $\mathcal{Q}$.

It can be proved that the entropy S_{BH} itself is invariant along the flow in the charge space $\mathcal{Q}$ defined by the symplectic gradient (or, equivalently, by the horizon Freudenthal dual) of $\mathcal{Q}$ itself :

$$S_{BH}(\mathcal{Q}) = S_{BH}(\mathfrak{F}_H(\mathcal{Q})) = S_{BH}\left(\frac{1}{\pi}\Omega\frac{\partial S_{BH}(\mathcal{Q})}{\partial \mathcal{Q}}\right) = S_{BH}\left(\widetilde{\mathcal{Q}}\right). \tag{1.19}$$

It is here worth pointing out that this invariance is pretty remarkable: the (semi-classical) Bekenstein-Hawking entropy of an extremal black hole turns out to be invariant under a generally non-linear map acting on the black hole charges themselves, and corresponding to a symplectic gradient flow in their corresponding vector space.

For other applications and instances of Freudenthal duality, see [9–11].

2. GROUPS OF TYPE E_7

The concept of Lie groups *of type* E_7 as introduced in the 60s by Brown [12], and then later developed, *e.g.* by [13–17].

Starting from a pair $(G, \mathbf{R})$ made of a Lie group G and its faithful representation $\mathbf{R}$, the three axioms defining $(G, \mathbf{R})$ as a group of type E_7 read as follows :

1. Existence of a (unique) symplectic invariant structure Ω in $\mathbf{R}$:
$$\exists!\Omega \equiv \mathbf{1} \in \mathbf{R} \times_a \mathbf{R}, \tag{2.1}$$
which then allows to define a symplectic product $\langle \cdot, \cdot \rangle$ among two vectors in the representation space $\mathbf{R}$ itself :
$$\langle Q_1, Q_2 \rangle := Q_1^M Q_2^N \Omega_{MN} = -\langle Q_2, Q_1 \rangle. \tag{2.2}$$
2. Existence of (unique) rank-4 completely symmetric invariant tensor (K-tensor) in $\mathbf{R}$:
$$\exists! K \equiv \mathbf{1} \in (\mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R})_s, \tag{2.3}$$
which then allows to define a degree-4 invariant polynomial I_4 in $\mathbf{R}$ itself :
$$I_4 := K_{MNPQ} Q^M Q^N Q^P Q^Q. \tag{2.4}$$
3. Defining a triple map T in $\mathbf{R}$ as
$$T \;:\; \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}; \tag{2.5}$$
$$\langle T(Q_1, Q_2, Q_3), Q_4 \rangle \;:\; = K_{MNPQ} Q_1^M Q_2^N Q_3^P Q_4^Q, \tag{2.6}$$
it holds that
$$\langle T(Q_1, Q_1, Q_2), T(Q_2, Q_2, Q_2) \rangle = \langle Q_1, Q_2 \rangle K_{MNPQ} Q_1^M Q_2^N Q_2^P Q_2^Q. \tag{2.7}$$
This property makes a group of type E_7 amenable to a description as an automorphism group of a *Freudenthal triple system* (or, equivalently, as the conformal groups of the underlying Jordan triple system – whose a Jordan algebra is a particular case –).

All electric-magnetic duality (U-duality[1]) groups of $\mathcal{N} \geqslant 2$-extended $D = 4$ supergravity theories with symmetric scalar manifolds are of type E_7. Among these, degenerate groups of type E_7 are those in

[1] Here U-duality is referred to as the "continuous" symmetries of [18]. Their discrete versions are the U-duality non-perturbative string theory symmetries introduced by Hull and Townsend [19].

which the K-tensor is actually reducible, and thus I_4 is the square of a quadratic invariant polynomial I_2. In fact, in general, in theories with electric-magnetic duality groups of type E_7 holds that

$$S_{BH} = \pi\sqrt{|I_4(\mathcal{Q})|} = \pi\sqrt{|K_{MNPQ}\mathcal{Q}^M\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^Q|}, \tag{2.8}$$

whereas in the case of degenerate groups of type E_7 it holds that $I_4(\mathcal{Q}) = (I_2(\mathcal{Q}))^2$, and therefore the latter formula simplifies to

$$S_{BH} = \pi\sqrt{|I_4(\mathcal{Q})|} = \pi\,|I_2(\mathcal{Q})|\,. \tag{2.9}$$

Simple, non-degenerate groups of type E_7 relevant to $\mathcal{N} \geqslant 2$-extended $D = 4$ supergravity theories with symmetric scalar manifolds are reported in Table 1.

Semi-simple, non-degenerate groups of type E_7 of the same kind are given by $G = SL(2,\mathbb{R}) \times SO(2,n)$ and $G = SL(2,\mathbb{R}) \times SO(6,n)$, with $\mathbf{R} = (\mathbf{2},\mathbf{2}+\mathbf{n})$ and $\mathbf{R} = (\mathbf{2},\mathbf{6}+\mathbf{n})$, respectively relevant for $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supergravity.

Moreover, degenerate (simple) groups of type E_7 relevant to the same class of theories are $G = U(1,n)$ and $G = U(3,n)$, with complex fundamental representations $\mathbf{R} = \mathbf{n}+\mathbf{1}$ and $\mathbf{R} = \mathbf{3}+\mathbf{n}$, respectively relevant for $\mathcal{N} = 2$ and $\mathcal{N} = 3$ supergravity [16].

The classification of groups of type E_7 is still an open problem, even if some progress have been recently made *e.g.* in [28] (in particular, *cfr.* Table D therein).

In all the aforementioned cases, the scalar manifold is a *symmetric* cosets $\frac{G}{H}$, where H is the maximal compact subgroup (with symmetric embedding) of G. Moreover, the K-tensor can generally be expressed as [17]

$$K_{MNPQ} = -\frac{n(2n+1)}{6d}\left[t^{\alpha}_{MN}t_{\alpha|PQ} - \frac{d}{n(2n+1)}\Omega_{M(P}\Omega_{Q)N}\right], \tag{2.10}$$

where $\dim\mathbf{R} = 2n$ and $\dim G = d$, and t^{α}_{MN} denotes the symplectic representation of the generators of G itself. Thus, the horizon Freudenthal duality can be expressed in terms of the K-tensor as follows [4]:

$$\mathfrak{F}_H(\mathcal{Q})_M \equiv \widetilde{\mathcal{Q}}_M = \frac{\partial\sqrt{|I_4(\mathcal{Q})|}}{\partial\mathcal{Q}^M} = \epsilon\frac{2}{\sqrt{|I_4(\mathcal{Q})|}}K_{MNPQ}\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^Q, \tag{2.11}$$

where $\epsilon := I_4/|I_4|$; note that the horizon Freudenthal dual of a given symplectic dyonic charge vector $\mathcal{Q}$ is well defined only when $\mathcal{Q}$ is such that $I_4(\mathcal{Q}) \neq 0$. Consequently, the invariance (1.19) of the black hole entropy under the the horizon Freudenthal duality can be recast as the invariance of I_4 itself :

$$I_4(\mathcal{Q}) = I_4\left(\widetilde{\mathcal{Q}}\right) = I_4\left(\Omega\frac{\partial\sqrt{|I_4(\mathcal{Q})|}}{\partial\mathcal{Q}}\right). \tag{2.12}$$

In absence of “flat directions" at the attractor points (namely, of unstabilized scalar fields at the horizon of the black hole), and for $I_4 > 0$, the expression of the matrix $\mathcal{M}_H(\mathcal{Q})$ at the horizon can be computed to read

$$\mathcal{M}_{H|MN}(\mathcal{Q}) = -\frac{1}{\sqrt{I_4}}\left(2\widetilde{\mathcal{Q}}_M\widetilde{\mathcal{Q}}_N - 6K_{MNPQ}\mathcal{Q}^P\mathcal{Q}^Q + \mathcal{Q}_M\mathcal{Q}_N\right), \tag{2.13}$$

and it is invariant under horizon Freudenthal duality :

$$\mathfrak{F}_H(\mathcal{M}_H)_{MN} := \mathcal{M}_{H|MN}(\widetilde{\mathcal{Q}}) = \mathcal{M}_{H|MN}(\mathcal{Q}). \tag{2.14}$$

Table 1. Simple, non-degenerate groups G related to Freudenthal triple systems $\mathfrak{M}(J_3)$ on simple rank-3 Jordan algebras J_3. In general, $G \cong Conf(J_3) \cong Aut(\mathfrak{M}(J_3))$ (see, *e.g.* [20–22] for a recent introduction, and a list of Refs.). $\mathbb{O}$, $\mathbb{H}$, $\mathbb{C}$ and $\mathbb{R}$, respectively, denote the four division algebras of octonions, quaternions, complex and real numbers, and $\mathbb{O}_s$, $\mathbb{H}_s$, $\mathbb{C}_s$ are the corresponding split forms. Note that the G related to split forms $\mathbb{O}_s$, $\mathbb{H}_s$, $\mathbb{C}_s$ is the *maximally non-compact* (*split*) real form of the corresponding compact Lie group. $M_{1,2}(\mathbb{O})$ is the Jordan triple system generated by 2×1 vectors over $\mathbb{O}$ [23]. Note that the STU model, based on $J_3 = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$, has a *semi-simple* G_4, but its *triality symmetry* [24] renders it "effectively simple". The $D = 5$ uplift of the T^3 model based on $J_3 = \mathbb{R}$ is the *pure* $\mathcal{N} = 2$, $D = 5$ supergravity. $J_3^{\mathbb{H}}$ is related to both 8 and 24 supersymmetries, because the corresponding supergravity theories are "*twin*", namely they share the very same bosonic sector [23, 25–27].

J_3	G_4	$\mathbf{R}$	$\mathcal{N}$
$J_3^{\mathbb{O}}$	$E_{7(-25)}$	$\mathbf{56}$	2
$J_3^{\mathbb{O}_s}$	$E_{7(7)}$	$\mathbf{56}$	8
$J_3^{\mathbb{H}}$	$SO^*(12)$	$\mathbf{32}$	2, 6
$J_3^{\mathbb{H}_s}$	$SO(6,6)$	$\mathbf{32}$	0
$J_3^{\mathbb{C}}$	$SU(3,3)$	$\mathbf{20}$	2
$J_3^{\mathbb{C}_s}$	$SL(6,\mathbb{R})$	$\mathbf{20}$	0
$M_{1,2}(\mathbb{O})$	$SU(1,5)$	$\mathbf{20}$	5
$J_3^{\mathbb{R}}$	$Sp(6,\mathbb{R})$	$\mathbf{14'}$	2
$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ (STU)	$[SL(2,\mathbb{R})]^3$	$(\mathbf{2},\mathbf{2},\mathbf{2})$	2
$\mathbb{R}$ (T^3)	$SL(2,\mathbb{R})$	$\mathbf{4}$	2

3. DUALITY ORBITS, RIGID SPECIAL KÄHLER GEOMETRY AND PRE-HOMOGENEOUS VECTOR SPACES

For $I_4 > 0$, $\mathcal{M}_H(\mathcal{Q})$ given by (2.13) is one of the two possible solutions to the set of equations [29]

$$\begin{cases} M^T(\mathcal{Q})\,\Omega M(\mathcal{Q}) = \epsilon\Omega; \\ M^T(\mathcal{Q}) = M(\mathcal{Q}); \\ \mathcal{Q}^T M(\mathcal{Q})\,\mathcal{Q} = -2\sqrt{|I_4(\mathcal{Q})|}, \end{cases} \tag{3.1}$$

which describes symmetric, purely $\mathcal{Q}$-dependent structures at the horizon; they are symplectic or anti-symplectic, depending on whether $I_4 > 0$ or $I_4 < 0$, respectively. Since in the class of (super)gravity $D = 4$ theories discussed the sign of I_4 actually determines a stratification of the representation space $\mathbf{R}$ of charges into distinct orbits of the action of G into $\mathbf{R}$ itself (usually named duality orbits), the symplectic or anti-symplectic nature of the solutions to the system (3.1) is G-invariant, and supported by the various duality orbits of G (in particular, by the so-called "large" orbits, for which I_4 is non-vanishing).

One of the two possible solutions to the system (3.1) reads [29]

$$\begin{aligned} M_+(\mathcal{Q}) &= -\frac{1}{\sqrt{|I_4|}}\left(2\widetilde{\mathcal{Q}}_M\widetilde{\mathcal{Q}}_N - 6\epsilon K_{MNPQ}\mathcal{Q}^P\mathcal{Q}^Q + \epsilon\mathcal{Q}_M\mathcal{Q}_N\right); \\ \mathfrak{F}_H(M_+)_{MN} &:= M_{+|MN}(\widetilde{\mathcal{Q}}) = \epsilon M_{+|MN}(\mathcal{Q}). \end{aligned}$$

For $\epsilon = +1 \Leftrightarrow I_4 > 0$, it thus follows that

$$M_+(\mathcal{Q}) = \mathcal{M}_H(\mathcal{Q}), \tag{3.2}$$

as anticipated.

On the other hand, the other solution to system (3.1) reads [29]

$$M_-(\mathcal{Q}) = \frac{1}{\sqrt{|I_4|}}\left(\widetilde{\mathcal{Q}}_M\widetilde{\mathcal{Q}}_N - 6\epsilon K_{MNPQ}\mathcal{Q}^P\mathcal{Q}^Q\right); \tag{3.3}$$

$$\mathfrak{F}_H(M_-)_{MN} := M_{-|MN}(\widetilde{\mathcal{Q}}) = \epsilon M_{-|MN}(\mathcal{Q}). \tag{3.4}$$

By recalling the definition of I_4 (2.4), it is then immediate to realize that $M_-(\mathcal{Q})$ is the (opposite of the) Hessian matrix of ($1/\pi$ times) the black hole entropy S_{BH} :

$$M_{-|MN}(\mathcal{Q}) = -\partial_M\partial_N\sqrt{|I_4|} = -\frac{1}{\pi}\partial_M\partial_N S_{BH}. \tag{3.5}$$

The matrix $M_-(\mathcal{Q})$ is the (opposite of the) pseudo-Euclidean metric of a non-compact, non-Riemannian rigid special Kähler manifold related to the duality orbit of the black hole electromagnetic charges (to which $\mathcal{Q}$ belongs), which is an example of pre-homogeneous vector space (PVS) [30]. In turn, the nature of the rigid special manifold may be Kähler or pseudo-Kähler, depending on the existence of a $U(1)$ or $SO(1,1)$ connection[2].

In order to clarify this statement, let us make two examples within maximal $\mathcal{N} = 8$, $D = 4$ supergravity. In this theory, the electric-magnetic duality group is $G = E_{7(7)}$, and the representation in which the e.m. charges sit is its fundamental $\mathbf{R} = \mathbf{56}$. The scalar manifold has rank-7 and it is the real symmetric coset[3] $G/H = E_{7(7)}/SU(8)$, with dimension 70.

[2] For a thorough introduction to special Kähler geometry, see *e.g.* [31].

[3] To be more precise, it is worth mentioning that the actual relevant coset manifold is $E_{7(7)}/[SU(8)/\mathbb{Z}_2]$, because spinors transform according to the double cover of the stabilizer of the scalar manifold (see *e.g.* [32, 33], and Refs. therein).

Table 2. Non-generic, nor irregular PVS with simple G, of type 2 (in the complex ground field). To avoid discussing the finite groups appearing, the list presents the Lie algebra of the isotropy group rather than the isotropy group itself [34]. The interpretation (of suitable real, non-compact slices) in $D=4$ theories of Einstein gravity is added; remaining cases will be investigated in a forthcoming publication

G	V	n	isotropy alg.	degree	interpr. $D=4$
$SL(2,\mathbb{C})$	$S^3\mathbb{C}^2$	1	0	4	$\mathcal{N}=2, \mathbb{R}\,(T^3)$
$SL(6,\mathbb{C})$	$\Lambda^3\mathbb{C}^6$	1	$\mathfrak{sl}(3,\mathbb{C})^{\oplus 2}$	4	$\mathcal{N}=2, J_3^{\mathbb{C}}$ $\mathcal{N}=0, J_3^{\mathbb{C}_s}$ $\mathcal{N}=5, M_{1,2}(\mathbb{O})$
$SL(7,\mathbb{C})$	$\Lambda^3\mathbb{C}^7$	1	$\mathfrak{g}_2^{\mathbb{C}}$	7	
$SL(8,\mathbb{C})$	$\Lambda^3\mathbb{C}^8$	1	$\mathfrak{sl}(3,\mathbb{C})$	16	
$SL(3,\mathbb{C})$	$S^2\mathbb{C}^3$	2	0	6	
$SL(5,\mathbb{C})$	$\Lambda^2\mathbb{C}^5$	3	$\mathfrak{sl}(2,\mathbb{C})$	5	
		4	0	10	
$SL(6,\mathbb{C})$	$\Lambda^2\mathbb{C}^6$	2	$\mathfrak{sl}(2,\mathbb{C})^{\oplus 3}$	6	
$SL(3,\mathbb{C})^{\otimes 2}$	$\mathbb{C}^3\otimes\mathbb{C}^3$	2	$\mathfrak{gl}(1,\mathbb{C})^{\oplus 2}$	6	
$Sp(6,\mathbb{C})$	$\Lambda_0^3\mathbb{C}^6$	1	$\mathfrak{sl}(3,\mathbb{C})$	4	$\mathcal{N}=2, J_3^{\mathbb{R}}$
$Spin(7,\mathbb{C})$	$\mathbb{C}^8$	1	$\mathfrak{g}_2^{\mathbb{C}}$	2	
		2	$\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{so}(2,\mathbb{C})$	2	
		3	$\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{so}(3,\mathbb{C})$	2	
$Spin(9,\mathbb{C})$	$\mathbb{C}^{16}$	1	$\mathfrak{spin}(7,\mathbb{C})$	2	
$Spin(10,\mathbb{C})$	$\mathbb{C}^{16}$	2	$\mathfrak{g}_2^{\mathbb{C}}\oplus\mathfrak{sl}(2,\mathbb{C})$	2	
		3	$\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{so}(3,\mathbb{C})$	4	
$Spin(11,\mathbb{C})$	$\mathbb{C}^{32}$	1	$\mathfrak{sl}(5,\mathbb{C})$	4	
$Spin(12,\mathbb{C})$	$\mathbb{C}^{32}$	1	$\mathfrak{sl}(6,\mathbb{C})$	4	$\mathcal{N}=2,6, J_3^{\mathbb{H}}$ $\mathcal{N}=0, J_3^{\mathbb{H}_s}$
$Spin(14,\mathbb{C})$	$\mathbb{C}^{64}$	1	$\mathfrak{g}_2^{\mathbb{C}}\oplus\mathfrak{g}_2^{\mathbb{C}}$	8	
$G_2^{\mathbb{C}}$	$\mathbb{C}^7$	1	$\mathfrak{sl}(3,\mathbb{C})$	2	
		2	$\mathfrak{gl}(2,\mathbb{C})$	2	
$E_6^{\mathbb{C}}$	$\mathbb{C}^{27}$	1	$\mathfrak{f}_4^{\mathbb{C}}$	3	
		2	$\mathfrak{so}(8,\mathbb{C})$	6	
$E_7^{\mathbb{C}}$	$\mathbb{C}^{56}$	1	$\mathfrak{e}_6^{\mathbb{C}}$	4	$\mathcal{N}=2, J_3^{\mathbb{O}}$ $\mathcal{N}=8, J_3^{\mathbb{O}_s}$

1. The unique duality orbit determined by the G-invariant constraint $I_4 > 0$ is the 55-dimensional non-symmetric coset

$$\mathcal{O}_{I_4>0} = \frac{E_{7(7)}}{E_{6(2)}}. \tag{3.6}$$

By customarily assigning positive (negative) signature to non-compact (compact) generators, the pseudo-Euclidean signature of $\mathcal{O}_{I_4>0}$ is $(n_+, n_-) = (30, 25)$. In this case, $M_-(\mathcal{Q})$ given by (3.5) is the 56-dimensional metric of the non-compact, non-Riemannian rigid special Kähler non-symmetric manifold

$$\mathbf{O}_{I_4>0} = \frac{E_{7(7)}}{E_{6(2)}} \times \mathbb{R}^+, \tag{3.7}$$

with signature $(n_+, n_-) = (30, 26)$, thus with character $\chi := n_+ - n_- = 4$. Through a conification procedure (amounting to modding out[4] $\mathbb{C} \cong SO(2) \times SO(1,1) \cong U(1) \times \mathbb{R}^+$, one can obtain the corresponding 54-dimensional non-compact, non-Riemannian special Kähler symmetric manifold

$$\mathbf{O}_{I_4>0}/\mathbb{C} \cong \widehat{\mathbf{O}}_{I_4>0} = \frac{E_{7(7)}}{E_{6(2)} \times U(1)}. \tag{3.8}$$

2. The unique duality orbit determined by the G-invariant constraint $I_4 < 0$ is the 55-dimensional non-symmetric coset

$$\mathcal{O}_{I_4<0} = \frac{E_{7(7)}}{E_{6(6)}}, \tag{3.9}$$

with pseudo-Euclidean signature given by $(n_+, n_-) = (28, 27)$, thus with character $\chi = 0$. In this case, $M_-(\mathcal{Q})$ given by (3.5) is the 56-dimensional metric of the non-compact, non-Riemannian rigid special pseudo-Kähler non-symmetric manifold

$$\mathbf{O}_{I_4<0} = \frac{E_{7(7)}}{E_{6(6)}} \times \mathbb{R}^+, \tag{3.10}$$

with signature $(n_+, n_-) = (28, 28)$. Through a "pseudo-conification" procedure (amounting to modding out $\mathbb{C}_s \cong SO(1,1) \times SO(1,1) \cong \mathbb{R}^+ \times \mathbb{R}^+$, one can obtain the corresponding 54-dimensional non-compact, non-Riemannian special pseudo-Kähler symmetric manifold

$$\mathbf{O}_{I_4<0}/\mathbb{C}_s \cong \widehat{\mathbf{O}}_{I_4<0} = \frac{E_{7(7)}}{E_{6(6)} \times SO(1,1)}. \tag{3.11}$$

Eqs. (3.7) and (3.10) are non-compact, real forms of $\frac{E_7}{E_6} \times GL(1)$, which is the type 29 in the classification of regular, pre-homogeneous vector spaces (PVS) worked out by Sato and Kimura in [34]. From its definition, a PVS is a finite-dimensional vector space V together with a subgroup G of $GL(V)$, such that G has an open dense orbit in V. PVS are subdivided into two types (type 1 and type 2), according to whether there exists an homogeneous polynomial on V which is invariant under the semi-simple (reductive) part of G itself. For more details, see *e.g.* [30, 35, 36].

In the case of $\frac{E_7}{E_6} \times GL(1)$, V is provided by the fundamental representation space $\mathbf{R} = \mathbf{56}$ of $G = E_7$, and there exists a quartic E_7-invariant polynomial I_4 (2.4) in the $\mathbf{56}$; $H = E_6$ is the isotropy (stabilizer) group.

Amazingly, simple, non-degenerate groups of type E_7 (relevant to $D = 4$ Einstein (super)gravities with symmetric scalar manifolds) *almost* saturate the list of irreducible PVS with unique G-invariant polynomial of degree 4 (*cfr.* Table 2); in particular, the parameter n characterizing each PVS can be interpreted as the number of centers of the regular solution in the (super)gravity theory with electric-magnetic duality (U-duality) group given by G. This topic will be considered in detail in a forthcoming publication.

[4] The signature along the $\mathbb{R}^+$-direction is negative [29].

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