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RESEARCH ARTICLES

On Solvability of One Integral Equation on Half Line with Chebyshev Polynomial Nonlinearity*

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Abstract—We consider an integral equation on half-line with Chebyshev polynomial nonlinearity, arising in dynamic theory of universe and p-adic string theory. We prove existence of the positive and monotonically increasing continuous solution in class of essentially bounded functions on half-line. We also found two sided estimates for obtained solution, as well as the limit of solution at infinity (Theorem 2.1). We prove uniqueness of a solution in the certain class of functions (Theorem 2.2). We generalize the results for more general integral equation with "double" nonlinearity (Theorem 2.3). At the end we give some examples of functions, describing nonlinearity. Using suggested constructive solution method, we present some results of numerical calculations, having direct application in cosmology.

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1. INTRODUCTION

We consider the following nonlinear integral equation on positive half-line:

$$P_n(f(x)) = \int_0^\infty [K(x-t) - K_0(x+t)]f(t)dt, \quad x \in \mathbb{R}^+ \tag{1.1}$$

with respect to unknown measurable function $f(x)$ belonging to the class:

$$\mathfrak{M} = \{\varphi(x):\ x \in \mathbb{R}^+,\ -1 \le \varphi(x) \le 1\} \tag{1.2}$$

where $n \ge 1,\ n \in \mathbb{N}$. The kernel $K(x)$ defined on $(-\infty, +\infty)$ is the measurable function, satisfying the following conditions:

$$K(x) \ge 0,\ x \in \mathbb{R},\ \int_{-\infty}^{+\infty} K(x)dx = 1. \tag{1.3}$$

The kernel $K_0(x)$ is defined on $(0, +\infty)$ and satisfies the conditions:

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$$0 \le K_0(x+t) \le K(x-t), \ \forall x, t \in \mathbb{R}^+ \ \text{and} \\ K_0(\tau) \downarrow \ \text{by} \ \tau \ \text{on} \ \mathbb{R}^+. \tag{1.4}$$

$P_n(z) = cos(n\, arccos z)-$ is the Chebyshev polynomial.

In the case when $n = 1$, $K_0 = 0$, the equation (1.1) is transformed into well known in literature homogeneous conservative Winer-Hopf integral equation, the study of which are devoted numerous works by local and foreign authors (see [1–5] and references therein).

In the case when $n = 3$, the integral equation (1.1) can be written in the form

$$4f^3(x) - 3f(x) = \int_0^\infty [K(x-t) - K_0(x+t)]f(t)dt, \quad x \in \mathbb{R}^+. \tag{1.5}$$

The equation (1.5) has direct applications in cosmology. In particular the equation (1.5) arises in dynamic theory of universe and p-adic string theory (see [6–8]).

Further we will construct positive, continuous monotonically increasing and bounded solution of equation (1.1) for each $n > 1$ in class of functions $\mathfrak{M}$ (see Theorem 2.1). We also will prove the uniqueness of solution in the certain class of functions (Theorem 2.2).

The results of Theorem 2.1 are generalized for the following integral equation with "double" nonlinearity:

$$P_n(f(x)) = \int_0^\infty [K(x-t) - K_0(x+t)]G(f(t))dt, \quad x \in \mathbb{R}^+. \tag{1.6}$$

Imposing certain conditions on function $G(x)$ describing nonlinearity we will prove existence of positive monotonically increasing and bounded solution, as well as will find limit of solution at infinity (see Theorem 2.3). At the end we will give some examples of functions $G(x)$, and some results of numerical calculations having direct applications in p-adic string theory.

2. BASIC RESULTS

We consider the Chebyshev polynomial on interval $\left[cos\frac{\pi}{2n}, 1\right]$.

Note that

$$P_n(z) \ \uparrow \ \text{in} \ z \ \text{on interval} \ \left[cos\frac{\pi}{2n}, 1\right]. \tag{2.1}$$

Actually the derivative of this function is nonnegative

$$P_n'(z) = \frac{n sin(n\, arccos z)}{\sqrt{1-z^2}} \ge 0 \ \text{if} \ z \in \ \left[cos\frac{\pi}{2n}, 1\right). \tag{2.2}$$

Therefore the statement (2.1) is approved.

We prove that the unit is the unique solution of equation

$$P_n(z) = z \ \text{on interval} \ \left[cos\frac{\pi}{2n}, 1\right]. \tag{2.3}$$

Indeed, if we denote by

$$t = arccos z, \ z \in \left[cos\frac{\pi}{2n}, 1\right] \tag{2.4}$$

then due to monotonicity of function $arccos z$ the equation (2.3) takes the following form in respect to variable t

$$cos nt = cos t, \ t \in \left[0, \frac{\pi}{2n}\right]. \tag{2.5}$$

It is easy to check that unique solution of equation (2.5) on interval $\left[0, \frac{\pi}{2n}\right]$ is zero. Therefore, taking into consideration (2.4) we obtain $z = 1$.

We introduce the following special iteration

$$\begin{gathered} cos(n\, arccos f_{m+1}(x)) = \int_0^\infty [K(x-t) - K_0(x+t)] f_m(t) dt, \quad x \geq 0, \\ f_0(x) = cos\frac{\pi}{2n}, \;\; m = 0, 1, 2, \ldots, \;\; n > 1, \;\; n \in \mathbb{N}. \end{gathered} \tag{2.6}$$

Below, by means of induction, we prove that

$$a) \;\; f_m(x) \;\; \uparrow \;\; \text{in} \;\; m \tag{2.7}$$

$$b) \;\; f_m(x) \leq 1, \;\; m = 0, 1, 2, \ldots, \;\; x \in \mathbb{R}^+ \tag{2.8}$$

$$c) \;\; f_m(x) \;\; \uparrow \;\; \text{in} \;\; x \;\; \text{on} \;\; \mathbb{R}^+. \tag{2.9}$$

First we prove the statements $a)$ and $b)$. From (2.6) taking into account (1.3), (1.4) we have

$$cos(n\, arccos f_1(x)) \geq 0 = cos\left(n\, arccos cos\frac{\pi}{2n}\right), \tag{2.10}$$

$$\begin{aligned} cos(n\, arccos f_1(x)) &\leq \int_0^\infty [K(x-t) - K_0(x+t)] dt \leq \int_0^\infty K(x-t) dt \\ &\leq \int_{-\infty}^{+\infty} K(\tau) d\tau = 1 = cos(n\, arccos 1). \end{aligned} \tag{2.11}$$

Due to continuity and monotonicity $P_n(z)$ on interval $\left[cos\frac{\pi}{2n}, 1\right]$ from (2.10),
it follows that

$$cos\frac{\pi}{2n} \leq f_1(x) \leq 1. \tag{2.12}$$

We assume that $f_m(x) \geq f_{m-1}(x)$ and $f_m(x) \leq 1, \quad x \in \mathbb{R}^+$, for some $m \in \mathbb{N}$. Consequently, since $P_n(z) \;\; \uparrow \;\;$ in z on $\left[cos\frac{\pi}{2n}, 1\right]$ and $P_n \in C\left[cos\frac{\pi}{2n}, 1\right]$, then from inequalities

$$\begin{aligned} cos(n\, arccos f_{m+1}(x)) &\geq \int_0^\infty [K(x-t) - K_0(x+t)] f_{m-1}(t) dt \\ &= cos(n\, arccos f_m(x)) \end{aligned} \tag{2.13}$$

and

$$cos(n\, arccos f_{m+1}(x)) \leq \int_0^\infty [K(x-t) - K_0(x+t)] dt \leq 1 = cos(n\, arccos 1) \tag{2.14}$$

it follows

$$f_{m+1}(x) \geq f_m(x) \;\; \text{and} \;\; f_m(x) \leq 1, \;\; x \in \mathbb{R}^+. \tag{2.15}$$

Thus we proved that sequence of functions $\{f_m(x)\}_{m=0}^\infty$ possesses the properties $a)$ and $b)$. Therefore, sequence of functions $\{f_m(x)\}_{m=0}^\infty$ has pointwise limit when $m \to +\infty$.

$$\lim_{m \to \infty} f_m(x) = f(x) \tag{2.16}$$

and in addition for the limit function two-sided estimates take place

$$cos\frac{\pi}{2n} \leq f(x) \leq 1, \;\; x \in \mathbb{R}^+. \tag{2.17}$$

Due to continuity $P_n(z)$ and taking into consideration B. Levi theorem from (2.6) we conclude that $f(x)$ is the solution of equation (1.1).

Now let us prove that

$$\lim_{x\to+\infty} f(x) = 1. \tag{2.18}$$

First, we prove that for each $m \in \mathbb{N} \cup \{0\}$

$$f_m(x) \;\; \uparrow \;\; \text{in} \;\; x \;\; \text{on} \;\; \mathbb{R}^+. \tag{2.19}$$

Let $x_1, x_2 \in \mathbb{R}^+, \;\; x_1 > x_2$ be arbitrary numbers. In the case when $m = 0$ the statement (2.19) is obvious. Assume that

$$f_m(x_1) \geq f_m(x_2) \;\; \text{for some} \;\; m \in \mathbb{N}. \tag{2.20}$$

The iteration (2.6) can be written as:

$$\begin{gathered} cos(n\, arccos f_{m+1}(x)) = \int\limits_{-\infty}^{x} K(\tau) f_m(x-\tau)d\tau - \int\limits_{0}^{\infty} K_0(x+t) f_m(t)dt, \\ f_0(x) = cos\frac{\pi}{2n}, \;\; m = 0,1,2,\ldots, \;\; n > 1, \;\; n \in \mathbb{N}. \end{gathered} \tag{2.21}$$

We consider the difference

$$\begin{aligned} & cos(n\, arccos f_{m+1}(x_1)) - cos(n\, arccos f_{m+1}(x_2)) = \int\limits_{-\infty}^{x_1} K(\tau) f_m(x_1-\tau)d\tau \\ & - \int\limits_{-\infty}^{x_2} K(\tau) f_m(x_2-\tau)d\tau - \int\limits_{0}^{\infty} K_0(x_1+t) f_m(t)dt + \int\limits_{0}^{\infty} K_0(x_2+t) f_m(t)dt \\ & \geq \int\limits_{-\infty}^{x_2} K(\tau)[f_m(x_1-\tau) - f_m(x_2-\tau)]d\tau + \int\limits_{0}^{\infty} [K_0(x_2+t) - K_0(x_1+t)] f_m(t)dt \geq 0. \end{aligned}$$

Hence

$$cos(n\, arccos f_{m+1}(x_1)) \geq cos(n\, arccos f_{m+1}(x_2)).$$

Since $P_n(z) \;\; \uparrow \;\; \text{in} \;\; z \;\; \text{on} \;\; \left[cos\frac{\pi}{2n}, 1\right]$, then taking into consideration statements $a)$ and $b)$ we conclude that

$$f_{m+1}(x_1) \geq f_{m+1}(x_2).$$

Thus the statement (2.19) is proved.

In (2.20) tending $m \to +\infty$ we obtain that $f(x_1) \geq f(x_2)$. Therefore $f(x) \;\; \uparrow \;\; \text{in} \;\; x \;\; \text{on} \;\; \mathbb{R}^+$.

Taking into account the above mentioned we can state that there exists

$$\lim_{x\to\infty} f(x) = c. \tag{2.22}$$

We verify that $c = 1$. First from(2.17) it follows that

$$cos\frac{\pi}{2n} \leq c \leq 1. \tag{2.23}$$

From the chain of inequalities

$$0 \leq \int_0^\infty K_0(x+t)f(t)dt \leq \int_x^\infty K_0(y)dy$$

it follows that

$$\lim_{x\to\infty} \int_0^\infty K_0(x+t)f(t)dt = 0. \tag{2.24}$$

Substituting the constructed solution $f(x)$ into the equation (1.1), taking into account (2.24) and tending $x \to +\infty$ due to continuity $P_n(z)$ we obtain

$$cos(n\, arccos c) = \lim_{x\to\infty} \int_0^\infty K(x-t)f(t)dt. \tag{2.25}$$

Since $f(x) \uparrow c$ when $x \to +\infty$ and kernel $K(x)$ possesses property (1.3), due to well known limit relation for convolution operation (see [9, 10])

$$\lim_{x\to\infty} \int_0^\infty K(x-t)f(t)dt = \int_{-\infty}^{+\infty} K(x)dx \cdot \lim_{x\to\infty} f(x) = c$$

from (2.25)we have

$$cos(n\, arccos c) = c. \tag{2.26}$$

On the other hand we have proved that equation (2.3) on interval $\left[cos\frac{\pi}{2n}, 1\right]$ has unique solution $z = 1$, therefore $c = 1$.

It is known that convolution of bounded and integrable functions presents continuous function (see, [4]). Due to $P_n(z) \in C\left[cos\frac{\pi}{2n}, 1\right]$ and $P_n(z) \uparrow$ in z on $\left[cos\frac{\pi}{2n}, 1\right]$ it follows that constructed solution is continuous function on $[0, +\infty)$.

Thus the following theorem holds.

Theorem 2.1. *Let kernels $K(x)$ and $K_0(x)$ satisfy conditions* (1.3)*,* (1.4) *and* $n > 1,\ n \in \mathbb{N}$. *Then equation* (1.1) *has positive continuous monotonically increasing and bounded solution f(x). Moreover,*

- $\lim_{x\to\infty} f(x) = 1$
- $cos\frac{\pi}{2n} \leq f(x) \leq 1,\ x \in \mathbb{R}^+.$

Now let us prove uniqueness of a solution of equation (1.1) in the certain class of functions. We prove by means of contradicting assumption. We assume that equation (1.1) has two different measurable solutions, $\varphi_1(x)$ and $\varphi_2(x)$,

$$\varphi_1, \varphi_2 \in \mathfrak{P} = \{\varphi(x) :\ cos\frac{\pi}{2n} \leq \varphi(x) \leq 1, \quad x \in \mathbb{R}\}.$$

We denote

$$arccos\varphi_1(x) = u_1(x) \in \left[0, \frac{\pi}{2n}\right], \quad arccos\varphi_2(x) = u_2(x) \in \left[0, \frac{\pi}{2n}\right]. \tag{2.27}$$

It is easy to check that

$$\frac{\triangle u}{2} = \frac{u_1 - u_2}{2} \in \left[-\frac{\pi}{4n}, \frac{\pi}{4n}\right], \quad \frac{u_1 + u_2}{2} \in \left[0, \frac{\pi}{2n}\right], \ n \geq 2, \ n \in \mathbb{N}. \tag{2.28}$$

From (1.1) and (2.27) we have

$$\sin \frac{n(u_1(x) - u_2(x))}{2} \sin \frac{n(u_1(x) + u_2(x))}{2}$$
$$= \int_0^\infty [K(x-t) - K_0(x+t)] \sin \frac{u_1(t) - u_2(t)}{2} \sin \frac{u_1(t) + u_2(t)}{2} dt.$$

Since $\frac{u_1 + u_2}{2} \in \left[0, \frac{\pi}{2n}\right]$ then $\sin \frac{u_1 + u_2}{2} \geq 0$ and $\sin \frac{n(u_1 + u_2)}{2} \geq 0$,
hence

$$| \sin \frac{n(u_1(x) - u_2(x))}{2} | \sin \frac{n(u_1(x) + u_2(x))}{2}$$
$$\leq \sup_{t \geq 0} \left[\sin \frac{u_1(t) + u_2(t)}{2} \mid \sin \frac{n(u_1(t) - u_2(t))}{2} \mid \right] \int_0^\infty [K(x-t) - K_0(x+t)] dt$$
$$\leq \sup_{t \geq 0} \left[\sin \frac{u_1(t) + u_2(t)}{2} \mid \sin \frac{(u_1(t) - u_2(t))}{2} \mid \right].$$

We show that

$$| \, sin \frac{n\Delta u}{2} \, | \geq | \, sin \Delta u \, |, \ \ \Delta u \in \left[-\frac{\pi}{2n}, \frac{\pi}{2n}\right]. \tag{2.29}$$

Let $\Delta u \in \left[0, \frac{\pi}{2n}\right]$, $\frac{n\Delta u}{2} \in \left[0, \frac{\pi}{4}\right]$ and since $\frac{n\Delta u}{2} \geq \Delta u, y = sint \ \uparrow$ in t on $\left[0, \frac{\pi}{2}\right]$ hence $sin\frac{n\Delta u}{2} \geq 0, \ \ sin\Delta u \geq 0.$
If $\Delta u \in \left[-\frac{\pi}{2n}, 0\right]$, $-\Delta u \in \left[0, \frac{\pi}{2n}\right]$ then we have:

$$| \, sin\frac{n\Delta u}{2} \, | = | \, -sin\frac{n(-\Delta u)}{2} \, | = | \, sin\frac{n(-\Delta u)}{2} \, | \geq | \, sin(-\Delta u) \, | = | \, sin\Delta u \, | \, .$$

So inequality (2.29) holds.
Given (2.29), we obtain

$$2 \mid sin\frac{\Delta u(x)}{2} \mid \cdot \mid cos\frac{\Delta u(x)}{2} \mid sin\frac{n(u_1(x) + u_2(x))}{2}$$
$$\leq \sup_{t \geq 0} \left[\sin \frac{u_1(t) + u_2(t)}{2} \mid \sin \frac{\Delta u(t)}{2} \mid \right]. \tag{2.30}$$

It is easy to see that

$$cos\frac{\Delta u(x)}{2} \geq cos\frac{\pi}{4n}, \ \ sin\frac{n(u_1(x) + u_2(x))}{2} \geq sin\frac{u_1(x) + u_2(x)}{2}.$$

Hence

$$2cos\frac{\pi}{4n} \mid sin\frac{\Delta u(x)}{2} \mid sin\frac{u_1(x) + u_2(x)}{2}$$
$$\leq \sup_{t \geq 0} \left[\sin \frac{u_1(t) + u_2(t)}{2} \mid \sin \frac{\Delta u(t)}{2} \mid \right]. \tag{2.31}$$

From (2.31) it follows that

$$\left[2cos\frac{\pi}{4n}-1\right]\sup_{t\geq 0} sin\frac{u_1(t)+u_2(t)}{2} \mid \sin\frac{\Delta u(t)}{2} \mid \leq 0. \tag{2.32}$$

Since $0<\frac{\pi}{4n}<\frac{\pi}{3}$ and $cos x \ \downarrow$ on $\left[0,\frac{\pi}{2}\right]$ then $2cos\frac{\pi}{4n}-1>0$.

Hence from (2.32) it follows that $sin\frac{u_1(t)+u_2(t)}{2} \mid \sin\frac{\Delta u(t)}{2} \mid = 0$.

Note that $sin\frac{u_1(t)+u_2(t)}{2}\neq 0$. Indeed if $sin\frac{u_1(t)+u_2(t)}{2}=0$, then $u_1+u_2=2\pi k,\ \ k\in\mathbb{N}\cup\{0\}$. Since $u_1,u_2\in\left[0,\frac{\pi}{2n}\right],\ \ u_1=u_2=0$. From (2.27) it follows $\varphi_1=\varphi_2=1$. This is impossible, because φ_1 and φ_2 are the solutions of initial equation (1.1), in addition, it is easy to check that $\varphi(x)\equiv 1$ does not satisfy equation (1.1).

Thus $sin\frac{\Delta u(x)}{2}=0,\ \ \Delta u=2\pi m,\ \ m\in\mathbb{N}\cup\{0\}$. On the other hand, $\Delta u\in\left[-\frac{\pi}{2n},\frac{\pi}{2n}\right]$, then $m=0$, i.e. $u_1(x)=u_2(x)$ on $\mathbb{R}^+$. Due to $arccos x \ \downarrow$ on $\left[\cos\frac{\pi}{2n},1\right]$ from (2.27) we obtain $\varphi_1(x)=\varphi_2(x)$.

Thus the following theorem is true:

Theorem 2.2. *Let the conditions* (1.3) *and* (1.4) *are fulfilled. Then if* $n\geq 2$ *the equation* (1.1) *has a unique solution on* $\mathbb{R}$ *in the following class of measurable functions*

$$\mathfrak{P}=\{\varphi(x):\ \ cos\frac{\pi}{2n}\leq\varphi(x)\leq 1,\quad x\in\mathbb{R}^+\}.$$

The result of the Theorem 2.1 can be applied for the following more general equation with "double" nonlinearity

$$cos(n\, arccos f(x))=\int_0^\infty[K(x-t)-K_0(x+t)]G(f(t))dt,\quad x\in\mathbb{R}^+, \tag{2.33}$$

where

$$G\in C\left[cos\frac{\pi}{2n},1\right],\ \ G\uparrow\ \text{ on }\ \left[cos\frac{\pi}{2n},1\right],\ \ G(z)\geq 0,\ \ z\in\left[cos\frac{\pi}{2n},1\right],\ \ G(1)=1. \tag{2.34}$$

The following theorem is true.

Theorem 2.3. *Let the conditions* (1.3), (1.4), (2.34) *are fulfilled. Then equation* (2.33) *in case* $n>1$, *possesses positive, monotonically increasing and bounded solution* $f(x)$ *and,in addition,*

$$cos\frac{\pi}{2n}\leq f(x)\leq 1,\ \ x\in\mathbb{R}^+.$$

Moreover, if unity is the first positive root of the equation

$$G(z)=cos(n\, arccos z), \tag{2.35}$$

then limit of the solution of equation (2.33) *at infinity is equal to 1, i.e.*

$$\lim_{x\to\infty}f(x)=1.$$

The proof of the Theorem 2.3 is done similarly to the proof of Theorem 2.1, simply we have to take into account that since $f_m\ \downarrow$ in $m,\ \ f_m(x)\geq cos\frac{\pi}{2n}$, therefore first iteration should start from the unit, i.e. $f_0=1$.

Remark 1. It is easy to see that if the order of the polynomial is enough large then the range of the constructed solution is sufficiently narrow.

Remark 2. It should be noted if $n = 2k + 1$, $k \in \mathbb{N}$ and $G(0) = 0$, then equation (2.33) in addition to trivial solution ($f(x) \equiv 0$), has also nontrivial solution.

Below are particular examples of functions $G(x)$:

- $G(x) = x^p$
- $G(x) = x \pm \frac{1}{\pi} sin^2 \pi x$
- $G(x) = e^{x-1}$
- $G(x) = \sqrt{xcos(n\, arccosx)}$.

It is easy to check that all examples of the function $G(x)$ satisfy conditions (2.34). In the case of last example the solution of equation (2.33) has also unit limit at infinity.

3. APPENDIX

As a kernels $K(x)$ and $K_0(x)$ we take

$$K_0(x) = K(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}. \tag{3.1}$$

The kernel of the type (3.1) arises in p-adic string theory and dynamic theory of universe (see [6–8]) and satisfies the conditions (1.3),(1.4).

From (1.1) we have

if $n = 2$

$$2f^2(x) - 1 = \frac{1}{\sqrt{\pi}} \int_0^\infty \left[e^{-(x-t)^2} - e^{-(x+t)^2}\right] f(t)dt, \;\; x > 0, \;\; n = 2 \tag{3.2}$$

if $n = 3$

$$4f^3(x) - 3f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \left[e^{-(x-t)^2} - e^{-(x+t)^2}\right] f(t)dt, \;\; x > 0, \;\; n = 3 \tag{3.3}$$

if $n = 4$

$$8f^4(x) - 8f^2(x) + 1 = \frac{1}{\sqrt{\pi}} \int_0^\infty \left[e^{-(x-t)^2} - e^{-(x+t)^2}\right] f(t)dt, \;\; x > 0, \;\; n = 4. \tag{3.4}$$

Instead of equations (3.2)-(3.4) we consider the following auxiliary equation:

$$F(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \left[e^{-(x-t)^2} - e^{-(x+t)^2}\right] cos\left(\frac{1}{n} arccos F(t)\right) dt, \;\; n = 2, 3, 4 \tag{3.5}$$

where

$$F(x) = cos(arccos f(x))$$

$$f(x) = cos\left(\frac{1}{n} arccos F(x)\right).$$

The equation (3.5) can be solved by simple iteration

$$F_{m+1}(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \left[e^{-(x-t)^2} - e^{-(x+t)^2}\right] cos\left(\frac{1}{n} arccos F_m(t)\right) dt, \\ F_0 = 0, \;\; m = 0, 1, 2, \dots, \;\; n = 2, 3, 4. \tag{3.6}$$

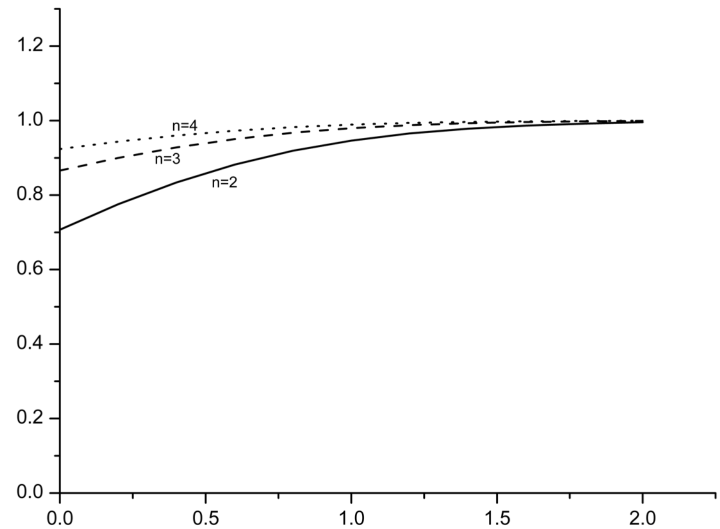

Fig. 1. Solutions of equation (1.1) at different values n .

We have already proved the convergence of iteration in the previous section.
In fig.1 the graphs of the unique solution $f(x)$ of equation (1.1) at $n = 2, 3, 4$ are presented .

We also consider the integral equation (2.33) with "double" nonlinearity in case when

$$G(x) = e^{x-1}, \quad G(x) = \sqrt{xcos(n\, arccos x)}$$

at $n = 2$.

We have

$$2f^2(x) - 1 = \frac{1}{e\sqrt{\pi}} \int_0^\infty \left[e^{-(x-t)^2} - e^{-(x+t)^2}\right] e^{f(t)} dt, \tag{3.7}$$

$$2f^2(x) - 1 = \frac{1}{\sqrt{\pi}} \int_0^\infty \left[e^{-(x-t)^2} - e^{-(x+t)^2}\right] \sqrt{f(t)cos(n\, arccos f(t))} dt. \tag{3.8}$$

Instead of the equations (3.7), (3.8) we consider the following iterations:

$$F_{m+1}(x) = \frac{1}{e\sqrt{\pi}} \int_0^\infty \left[e^{-(x-t)^2} - e^{-(x+t)^2}\right] e^{\sqrt{\frac{F_m(t)+1}{2}}} dt, \qquad F_0 = 1, \quad m = 0, 1, 2, \ldots, \tag{3.9}$$

$$F_{m+1}(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \left[e^{-(x-t)^2} - e^{-(x+t)^2}\right] \sqrt{\sqrt{\frac{F_m(t)+1}{2}} F_m(t)} dt, \qquad F_0 = 1, \quad m = 0, 1, 2, \ldots \tag{3.10}$$

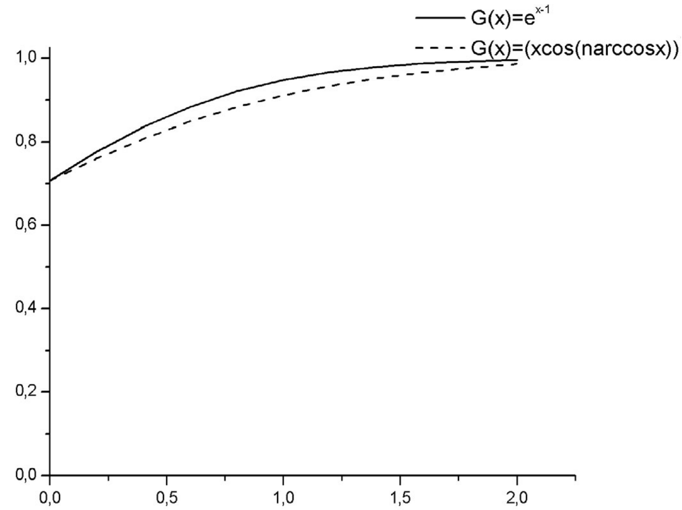

Fig. 2. Solutions of equation (2.33) .

In fig. 2 the solutions $f(x)=\sqrt{\dfrac{F(x)+1}{2}}$ of equation (2.33) with "double" nonlinearity are plotted, which correspond to $n=2,\ G(x)=e^{x-1},\ G(x)=\sqrt{x\,cos(n\,arccosx)}$.

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