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RESEARCH ARTICLES

On Spaces Extremal for the Gomory-Hu Inequality*

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Abstract—Let (X, d) be a finite ultrametric space. In 1961 E.C. Gomory and T.C. Hu proved the inequality $|\mathrm{Sp}(X)| \leqslant |X|$ where $\mathrm{Sp}(X) = \{d(x, y) \colon x, y \in X\}$. Using weighted Hamiltonian cycles and weighted Hamiltonian paths we give new necessary and sufficient conditions under which the Gomory-Hu inequality becomes an equality. We find the number of non-isometric (X, d) satisfying the equality $|\mathrm{Sp}(X)| = |X|$ for given $\mathrm{Sp}(X)$. Moreover it is shown that every finite semimetric space Z is an image under a composition of mappings $f \colon X \to Y$ and $g \colon Y \to Z$ such that X and Y are finite ultrametric spaces, X satisfies the above equality, f is an ε-isometry with an arbitrary $\varepsilon > 0$, and g is a ball-preserving map.

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1. INTRODUCTION

Recall some necessary definitions from the theory of metric spaces. An *ultrametric* on a set X is a function $d \colon X \times X \to \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$, such that for all $x, y, z \in X$:

(i) $d(x, y) = d(y, x)$,

(ii) $(d(x, y) = 0) \Leftrightarrow (x = y)$,

(iii) $d(x, y) \leq \max\{d(x, z), d(z, y)\}$.

Inequality (iii) is often called the *strong triangle inequality*. By studying the flows in networks, R. Gomory and T. Hu [1], deduced an inequality that can be formulated, in the language of ultrametric spaces, as follows: if (X, d) is a finite nonempty ultrametric space with the *spectrum*

$$\mathrm{Sp}(X) = \{d(x, y) \colon x, y \in X\},$$

then

$$|\mathrm{Sp}(X)| \leqslant |X|.$$

Definition 1.1. Define by $\mathfrak{U}$ the class of finite ultrametric spaces X with $|\mathrm{Sp}(X)| = |X|$.

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Two descriptions of $X \in \mathfrak{U}$ were obtained in terms of the representing trees and, respectively, so-called diametrical graphs of X (see [2] theorems 2.3 and 3.1.). Our paper is also a contribution to this lines of studies. We give a new criterium of $X \in \mathfrak{U}$ in terms of weighted Hamiltonian cycles and weighted Hamiltonian paths (see Theorem 2.5) and find the number of non-isometric $X \in \mathfrak{U}$ with given $\mathrm{Sp}(X)$ (see Proposition 3.2). It is also shown that every finite semimetric X is an image of a space $Y \in \mathfrak{U}$, $X = g(f(Y))$, where g is a ball-preserving map and f is an ε-isometry (see Theorem 4.5 and Theorem 4.6).

Recall that a *graph* is a pair (V, E) consisting of nonempty set V and (probably empty) set E elements of which are unordered pairs of different points from V. For the graph $G = (V, E)$, the set $V = V(G)$ and $E = E(G)$ are called *the set of vertices* and *the set of edges*, respectively. A graph G is empty if $E(G) = \varnothing$. A graph is complete if $\{x, y\} \in E(G)$ for all distinct $x, y \in V(G)$. Recall that a *path* is a nonempty graph $P = (V, E)$ of the form

$$V = \{x_0, x_1, ..., x_k\}, \quad E = \{\{x_0, x_1\}, ..., \{x_{k-1}, x_k\}\},$$

where x_i are all distinct. The number of edges of a path is the length. Note that the length of a path can be zero. A Hamiltonian path is a path in the graph that visits each vertex exactly once. A finite graph C is a *cycle* if $|V(C)| \geq 3$ and there exists an enumeration $(v_1, v_2, ..., v_n)$ of its vertices such that

$$(\{v_i, v_j\} \in E(C)) \Leftrightarrow (|i - j| = 1 \quad \text{or} \quad |i - j| = n - 1).$$

For the graph $G = (V, E)$ a *Hamiltonian cycle* is a cycle which is a subgraph of G that visits every vertex exactly once. A connected graph without cycles is called a tree. A tree T may have a distinguished vertex called the *root*; in this case T is called a *rooted tree*.

Generally we follow terminology used in [3]. A graph $G = (V, E)$ together with a function $w\colon E \to \mathbb{R}^+$, where $\mathbb{R}^+ = [0, +\infty)$, is called a *weighted* graph, and w is called a *weight* or a *weighting function*. The weighted graphs we denote by (G, w).

A nonempty graph G is called *complete k-partite* if its vertices can be divided into k disjoint nonempty subsets $X_1, ..., X_k$ so that there are no edges joining the vertices of the same subset X_i and any two vertices from different X_i, X_j, $1 \leqslant i, j \leqslant k$ are adjacent. In this case we write $G = G[X_1, ..., X_k]$.

2. CYCLES IN ULTRAMETRIC SPACES

In the following we identify a finite ultrametric space (X, d) with a complete weighted graph (G_X, w_d) such that $V(G_X) = X$ and

$$\forall\, x, y \in X,\ x \neq y\colon \quad w_d(\{x, y\}) = d(x, y). \tag{2.1}$$

The following lemma was proved in [4].

Lemma 2.1. *Let (X, d) be an ultrametric space with $|X| \geqslant 3$. Then for every cycle $C \subseteq G_X$ there exist at least two distinct edges $e_1, e_2 \in C$ such that*

$$w_d(e_1) = w_d(e_2) = \max_{e \in E(C)} w_d(e). \tag{2.2}$$

We shall say that a weighted cycle (C, w) is *characteristic* if the following conditions hold.

(i) There are exactly two distinct $e_1, e_2 \in E(C)$ such that (2.2) holds.

(ii) The restriction of w on the set $E(C) \setminus \{e_1, e_2\}$ is strictly positive and injective.

Remark 2.2. Let us explain the choice of a name for such a type of cycles. It was proved in [4] that for every characteristic weighted cycle (C, w) there is a unique ultrametric $d\colon V(C) \times V(C) \to \mathbb{R}^+$ such that

$$d(x, y) = w(\{x, y\})$$

for all $\{x, y\} \in E(C)$. In other words we can uniquely reconstruct whole the ultrametric space (X, d) by characteristic cycle $(C, w_d) \subseteq (G_X, w_d)$ if $|V(C)| = |X|$.

We need the following definition.

Definition 2.3 ([1]). Let (X,d) be a finite ultrametric space. Define the graph G^d_X as follows $V(G^d_X) = X$ and

$$(\{u,v\} \in E(G^d_X)) \Leftrightarrow (d(u,v) = \operatorname{diam} X).$$

We call G^d_X a *diametrical graph* of the space (X,d).

Lemma 2.4 ([1]). *Let (X,d) be a finite ultrametric space, $|X| \geqslant 2$. If $(X,d) \in \mathfrak{U}$, then G^d_X is a bipartite graph, $G^d_X = G^d_X[X_1, X_2]$ and $X_1 \in \mathfrak{U}$, $X_2 \in \mathfrak{U}$.*

We shall say that a weighted path (P,w) is *characteristic* if the weighting function $w\colon E(P) \to \mathbb{R}^+$ is injective and strictly positive.

The next theorem is the main result of this section.

Theorem 2.5. *Let (X,d) be a finite ultrametric space with $|X| \geqslant 3$. Then the following conditions are equivalent.*

(i) $(X,d) \in \mathfrak{U}$.

(ii) There exists a characteristic Hamiltonian path in G_X.

(iii) There exists a characteristic Hamiltonian cycle in G_X.

Proof. **(i)⇒(ii)**. We shall prove the implication (i)⇒(ii) by induction on $|X|$. Let $(X,d) \in \mathfrak{U}$. If $|X| = 3$, then the existence of a characteristic Hamiltonian path is evident. Suppose the implication (i)⇒(ii) holds for X with $|X| \leqslant n-1$. Let $|X| = n$. Let us prove that there exists a characteristic Hamiltonian path in G_X. According to Lemma 2.3 we have

$$G^d_X = G^d_X[X_1, X_2], \quad |X_1| \leqslant n-1, \quad |X_2| \leqslant n-1 \tag{2.3}$$

and $X_1 \in \mathfrak{U}$, $X_2 \in \mathfrak{U}$. By the induction supposition there exist characteristic Hamiltonian paths $P_1 \subseteq G_{X_1}$ and $P_2 \subseteq G_{X_2}$. Let $V(P_1) = \{x_1, ..., x_m\}$ and $V(P_2) = \{x_{m+1}, ..., x_n\}$, $1 \leqslant m \leqslant n-1$. Since $G^d_X = G^d_X[X_1, X_2]$, we have

$$\operatorname{diam} X \notin \operatorname{Sp}(X_1) \quad \text{and} \quad \operatorname{diam} X \notin \operatorname{Sp}(X_2).$$

Moreover, the equality

$$\operatorname{Sp}(X_1) \cap \operatorname{Sp}(X_2) = \{0\} \tag{2.4}$$

holds. Indeed, it is clear that

$$0 \in \operatorname{Sp}(X_1) \cap \operatorname{Sp}(X_2),$$

but if $|\operatorname{Sp}(X_1) \cap \operatorname{Sp}(X_2)| \geqslant 2$, then using the equality

$$\operatorname{Sp}(X) = \operatorname{Sp}(X_1) \cup \operatorname{Sp}(X_2) \cup \{\operatorname{diam} X\} \tag{2.5}$$

and the Gomory-Hu inequality we obtain

$$|\operatorname{Sp}(X)| \leqslant 1 + |X_1| + |X_2| - |X_1 \cap X_2| < |X_1| + |X_2| = |X|$$

contrary to $(X,d) \in \mathfrak{U}$. The equality $d(x_m, x_{m+1}) = \operatorname{diam} X$, (2.4) and (2.5) imply that the path P with $V(P) = \{x_1, .., x_m, x_{m+1}, ..., x_n\}$ is a characteristic Hamiltonian path in G_X.

(ii)⇒(iii). Let P be a characteristic Hamiltonian path in G_X with $V(P) = \{x_1, ..., x_n\}$. Consider the cycle $C = (x_1, ..., x_n)$. It is clear that C is Hamiltonian. According to Lemma 2.1 the equality

$$w_d(\{x_1, x_n\}) = \max_{e \in E(P)} w_d(e)$$

holds. This means that C is characteristic.

(iii)⇒(i). Let (X,d) be a finite ultrametric space and let C be a characteristic Hamiltonian cycle in G_X. Using Lemma 2.1 with this C we easily show that $|\operatorname{Sp}(X)| = |X|$. Condition (i) follows.

□

With every finite ultrametric space (X, d), we can associate (see [2]) a labeled rooted m-ary tree T_X by the following rule. If $X = \{x\}$ is a one-point set, then T_X is a tree consisting of one node x considered strictly binary by definition. Let $|X| \geqslant 2$ and $G_X^d = G_X^d[X_1, ..., X_k]$ be the diametrical graph of the space (X, d). In this case the root of the tree T_X is labeled by $\operatorname{diam} X$ and, moreover, T_X has k nodes $X_1, ..., X_k$ of the first level with the labels

$$l_i = \begin{cases} \operatorname{diam} X_i, & \text{if } |X_i| \geqslant 2, \\ x, & \text{if } X_i \text{ is a one-point set} \\ & \text{with the single element } x, \end{cases} \tag{2.6}$$

$i = 1, ..., k$. The nodes of the first level indicated by labels $x \in X$ are leaves, and those indicated by labels $\operatorname{diam} X_i$ are internal nodes of the tree T_X. If the first level has no internal nodes, then the tree T_X is constructed. Otherwise, by repeating the above-described procedure with $X_i \subset X$ corresponding to internal nodes of the first level, we obtain the nodes of the second level, etc. Since $|X|$ is finite, and the cardinal numbers $|Y|$, $Y \subseteq X$, decrease strictly at the motion along any path starting from the root, consequently all vertices on some level will be leaves, and the construction of T_X is completed. The above-constructed labeled tree T_X is called the *representing tree* of the space (X, d). We note that every element $x \in X$ is ascribed to some leaf, and all internal nodes are labeled as $r \in \operatorname{Sp}(X)$. In this case, different leaves correspond to different $x \in X$, but different internal nodes can have coinciding labels.

Recall that a rooted tree is *strictly binary* if every internal node has exactly two children. Note that the correspondence between trees and ultrametric spaces is well known [5–7].

Define by L_T the set of leaves of the tree T and by $l(v)$ the label of the vertex v.

The proof of the following two lemmas is immediate.

Lemma 2.6. *Let X be a finite ultrametric space having a strictly binary tree T_X. If v_0 and v_1 are internal nodes of T_X and v_1 is a direct successor of v_0 then the inequality $l(v_1) < l(v_0)$ holds.*

Lemma 2.7. *Let (X, d) be a finite ultrametric space with $|X| \geqslant 3$ and let $G_X^d = G_X^d[X_1, \ldots, X_k]$ be the diametrical graph of (X, d). Then a tree T_X is strictly binary if and only if $k = 2$ and T_{X_1} and T_{X_2} are strictly binary.*

Proposition 2.8. *Let (X, d) be a finite ultrametric space with $|X| \geqslant 3$. The following conditions are equivalent.*

(i) *T_X is strictly binary.*

(ii) *If $X_1 \subseteq X$ and $|X_1| \geqslant 3$, then there exists a Hamiltonian cycle $C \subseteq G_{X_1}$ with exactly two edges of maximal weight.*

(iii) *There is no equilateral triangle in (X, d).*

Proof. **(i)⇒(ii)**. Suppose T_X is strictly binary. Let X_1 be a subset of X, $|X_1| \geqslant 3$. According to construction of T_X all elements of X_1 are labels of leaves of T_X. Let v_0 be a smallest common predecessor for the leaves of T_X labeled by elements of X_1. Let v_0^1 and v_0^2 be the two offsprings of v_0 (direct successors) and let T_1 and T_2 be the subtrees of the tree T_X with the roots v_0^1 and v_0^2. Let $L_1 = L_{T_1} \cap X_1$ and $L_2 = L_{T_2} \cap X_1$ and let $P_1 = \{x_1, ..., x_m\}$ and $P_2 = \{x_{m+1}, ..., x_{|X_1|}\}$, $1 \leqslant m \leqslant |X_1| - 1$, be Hamiltonian paths in the spaces (L_1, d) and (L_2, d). By the property of representing trees of ultrametric spaces we have $d(x, y) = l(v_0)$ for all $x \in L_1$ and $y \in L_2$. Since $X_1 = L_1 \cup L_2$, we obtain that the Hamiltonian cycle $C = (x_1, ..., x_m, x_{m+1}, ..., x_{|X_1|})$ has exactly the two edges $\{x_1, x_{|X_1|}\}$ and $\{x_m, x_{m+1}\}$ of maximal weight.

(ii)⇒(iii). This implication is evident.

(iii)⇒(i). We will prove (i) by induction on $|X|$. The statement (i) evidently follows from (iii) if $|X| = 3$. Assume that (iii)⇒(i) is satisfied for all finite ultrametric spaces (X, d) with $3 \leqslant |X| \leqslant n$, $n \in \mathbb{N}$. Let

$G_X^d = G_X^d[X_1, \dots, X_k]$ be the diametrical graph of (X, d). Statement (i) holds if $k = 2$. Indeed, since the inequality $|X_i| < |X|$ holds, the induction assumption implies that for every $i = 1, \dots, k$, T_{X_i} is a strictly binary tree. Hence if $k = 2$, then T_X is a strictly binary tree by Lemma 2.7. To complete the proof it suffices to note that if $k \geqslant 3$ and $x_i \in X_i$ for $i = 1, 2, 3$, then the points x_1, x_2, x_3 form an equilateral triangle with $d(x_1, x_2) = d(x_2, x_3) = d(x_3, x_1) = \operatorname{diam} X$. ☐

3. THE NUMBER OF NON-ISOMETRIC $X \in \mathfrak{U}$ WITH GIVEN $\operatorname{Sp}(X)$

Let $n \in \mathbb{N}$ and $\mathfrak{U}_n$ denote the class of ultrametric spaces $X \in \mathfrak{U}$ such that $|X| = n$. In the present section we study the following question: how many non-isometric spaces having the same spectrum are in the class $\mathfrak{U}_n$? Let us denote this number by $\kappa(\mathfrak{U}_n)$.

Definition 3.1 ([8]). Let (X, d_X), (Y, d_Y) be metric spaces. A bijective mapping $\Phi\colon X \to Y$ is a *weak similarity* if there is a strictly increasing bijective function $f\colon \operatorname{Sp}(Y) \to \operatorname{Sp}(X)$ such that the equality

$$d_X(x, y) = f(d_Y(\Phi(x), \Phi(y))) \tag{3.1}$$

holds for all $x, y \in X$. Write $X \simeq Y$ if a weak similarity $\Phi : X \to Y$ exists.

It is clear that $\simeq$ is an equivalence relation. It was proved in [8] that if X and Y are compact ultrametric spaces with the same spectrum, then every week similarity $\Phi\colon X \to Y$ is an isometry. So, the main question of this section can be reformulated as follows. How many spaces are there in $\mathfrak{U}_n$ *up to weak similarity*?

Proposition 3.2. *Let* $\mathfrak{U}_n := \{X \in \mathfrak{U} : |X| = n\}$, $n \in \mathbb{N}$, *let* $\mathfrak{U}_n/\simeq$ *be the quotient set of* $\mathfrak{U}_n$ *by* $\simeq$ *and let*

$$\kappa(\mathfrak{U}_n) := \operatorname{card}(\mathfrak{U}_n/\simeq).$$

Then the equality

$$\kappa(\mathfrak{U}_n) = \sum_{k=2}^{n-1} C_{n-3}^{k-2} \kappa(\mathfrak{U}_k)\kappa(\mathfrak{U}_{n-k}) \tag{3.2}$$

holds for every integer $n \geqslant 3$ *with* $\kappa(\mathfrak{U}_1) = \kappa(\mathfrak{U}_2) = 1$ *and*

$$C_{n-3}^{k-2} = \frac{(n-3)!}{(k-2)!(n-k-1)!}.$$

Proof. Directly we can find the initial values

$$\kappa(\mathfrak{U}_1) = \kappa(\mathfrak{U}_2) = 1.$$

Let $n \geq 3$. The number $\kappa(\mathfrak{U}_n)$ coincides with the number of non-isometric $(X, d) \in \mathfrak{U}_n$ having the spectrum $\{0, 1, ..., n-1\}$. For every such $(X, d) \in \mathfrak{U}_n$ we write $G_X^d[X_1, X_2]$ for the diametrical graph of (X, d). The inequality $n \geqslant 3$ implies that $\operatorname{diam} X = n - 1 > 1$. Since

$$\operatorname{Sp}(X) = \{n-1\} \cup \operatorname{Sp}(X_1) \cup \operatorname{Sp}(X_2)$$

and

$$\operatorname{Sp}(X_1) \cap \operatorname{Sp}(X_2) = \{0\},$$

we may assume, without loss of generality, that

$$1 \in \operatorname{Sp}(X_1) \text{ and } 1 \notin \operatorname{Sp}(X_2).$$

Let $|X_1| = k$. It follows from $1 \in \operatorname{Sp}(X_1)$ that $k \geqslant 2$. Moreover the statement $X_2 \neq \varnothing$ implies that $k \leqslant n - 1$. As was noted in the second section of the paper we have

$$X_1 \in \mathfrak{U}_k \text{ and } X_2 \in \mathfrak{U}_{n-k}.$$

Let $\mathrm{Sp}(X_1) = \{0, 1, n_1, ..., n_{k-2}\}$ where $1 < n_1 < ... < n_k$ (if $k \geqslant 3$). The set $\{n_1, ..., n_{k-2}\}$ can be selected from the set $\{2, ..., n-2\}$ in C_{n-3}^{k-2} ways. It is clear that if (X, d), $(Y, \rho) \in \mathfrak{U}_n$ and

$$\mathrm{Sp}(X) = \mathrm{Sp}(Y) = \{0, 1, ..., n-1\}$$

and if for the diametrical graphs $G_X^d[X_1, X_2]$, $G_Y^\rho[Y_1, Y_2]$ we have

$$1 \in \mathrm{Sp}(X_1) \text{ and } 1 \in \mathrm{Sp}(Y_1),$$

then X and Y are isometric if and only if X_1 is isometric to Y_1 and X_2 is isometric to Y_2. Now using the multiplication principle and additional principle we obtain (3.2). □

Corollary 3.3. *The number $\kappa(\mathfrak{U}_n)$ of all non-isometric spaces $X \in \mathfrak{U}_n$ with given* $\mathrm{Sp}(X)$ *equals to*

$$\sum_{k=2}^{n-1} C_{n-3}^{k-2} \kappa(\mathfrak{U}_k)\kappa(\mathfrak{U}_{n-k}),$$

where $\kappa(\mathfrak{U}_1) = \kappa(\mathfrak{U}_2) = 1$.

Using formula (3.2) we can find $\kappa(\mathfrak{U}_3) = 1$, $\kappa(\mathfrak{U}_4) = 2$, $\kappa(\mathfrak{U}_5) = 5$, $\kappa(\mathfrak{U}_6) = 16$, $\kappa(\mathfrak{U}_7) = 61$ and so on.

Remark 3.4. As was shown in [2] there is an isomorphism between spaces from $\mathfrak{U}$ and strictly decreasing binary trees.

It is easy to see that there is also a bijection between the strictly decreasing binary trees and the ranked trees $\mathcal{R}_n$. The definition of the ranked trees $\mathcal{R}_n$ one can find in [9]. It was noted in [9] that numbers of $\mathcal{R}_n$ correspond to sequence A000111 from [10].

4. BALL-PRESERVING MAPPINGS, ε-ISOMETRIES AND SEMIMETRIC SPACES

Let X be a set. A *semimetric* on X is a function $d\colon X \times X \to \mathbb{R}^+$ such that $d(x, y) = d(y, x)$ and $(d(x, y) = 0) \Leftrightarrow (x = y)$ for all $x, y \in X$. A pair (X, d), where d is a semimetric on X, is called a *semimetric space* (see, for example, [11]).

A *directed graph* or *digraph* is a set of nodes connected by edges, where the edges have a direction associated with them. In formal terms a digraph is a pair $G = (V, A)$ of

- a set V, whose element are called vertices or nodes,
- a set A of ordered pairs of vertices, called arcs, directed edges, or arrows.

An arc $e = \langle x, y \rangle$ is considered to be directed from x to y; y is said to be a *direct successor* of x, and x is said to be a *direct predecessor* of y. If a path made up of one or more successive arcs leads from x to y, then y is said to be a *successor* of x, and x is said to be a *predecessor* of y.

A Hasse diagram for a partially ordered set $(X, \leqslant_X)$ is a digraph (X, A_X), where X is the set of vertices and $A_X \subseteq X \times X$ is the set of directed edges such that the pair $\langle v_1, v_2 \rangle$ belongs to A_X if and only if $v_1 \leqslant_X v_2$, $v_1 \neq v_2$, and implication

$$(v_1 \leqslant_X w \leqslant_X v_2) \Rightarrow (v_1 = w \vee v_2 = w)$$

holds for every $w \in X$.

Recall that a subset B of a semimetric space (X, d) is called a closed ball if it can be represented as follows:

$$B = B_r(t) = \{x \in X\colon d(x, t) \leqslant r\},$$

where $t \in X$ and $r \in [0, \infty)$. Denote by $\mathbf{B}_X$ the set of all distinct balls of semimetric space (X, d).

Definition 4.1. Let X and Y be semimetric spaces. A mapping $F\colon X \to Y$ is ball-preserving if

$$F(Z) \in \mathbf{B}_Y, \tag{4.1}$$

for every $Z \in \mathbf{B}_X$.

Definition 4.2. Let $G_1 = (V_1, A_1)$ and $G_2 = (V_2, A_2)$ be directed graphs. A map $F\colon V_1 \to V_2$ is a *graph homomorphism* if the implication

$$(\langle u, v\rangle \in A_1) \Rightarrow (\langle F(u), F(v)\rangle \in A_2)$$

holds for all $u, v \in V_1$. A homomorphism $F\colon V_1 \to V_2$ is an *isomorphism* if F bijective and the inverse map F^{-1} is also a homomorphism.

According to [12] we shall say that a graph homomorphism $F\colon V_1 \to V_2$ from $G_1 = (V_1, A_1)$ to $G_2 = (V_2, A_2)$ is a *surjective* homomorphism if $V_2 = F(V_1)$ and $A_2 = F(A_1)$ where

$$F(A_1) = \{\langle F(u), F(v)\rangle \colon \langle u, v\rangle \in A_1\}.$$

Remark 4.3. It is evident that every isomorphism is a surjective homomorphism.

It was shown in [13] that if X and Y are finite ultrametric spaces, then the following conditions are equivalent.

- There is a bijective ball-preserving mapping $F\colon X \to Y$ such that the inverse mapping $F^{-1}\colon Y \to X$ is also ball-preserving.
- The Hasse diagrams $(\mathbf{B}_X, A_{\mathbf{B}_X})$ and $(\mathbf{B}_Y, A_{\mathbf{B}_Y})$ of the posets $(\mathbf{B}_X, \subseteq)$ and $(\mathbf{B}_Y, \subseteq)$ are isomorphic as directed graphs.

Definition 4.4. Let (X, d) and (Y, ρ) be semimetric spaces and let $\varepsilon > 0$. A surjective mapping $F\colon X \to Y$ is an ε-isometry if the inequality

$$|d(x, y) - \rho(F(x), F(y))| \leqslant \varepsilon$$

holds for all $x, y \in X$.

The main result of the present section is the following two theorems.

Theorem 4.5. *Let X be a finite nonempty semimetric space. Then there is a finite ultrametric space Y and a surjective ball-preserving function $F\colon Y \to X$ such that the mapping*

$$\boldsymbol{B}_Y \ni B \mapsto F(B) \in \boldsymbol{B}_X$$

is a surjective homomorphism from the Hasse diagram $(\boldsymbol{B}_Y, A_Y)$ of $(\boldsymbol{B}_Y, \subseteq)$ to the Hasse diagram $(\boldsymbol{B}_X, A_X)$ of $(\boldsymbol{B}_X, \subseteq)$.

Theorem 4.6. *Let (Y, d) be a finite ultrametric space. Then for every $\varepsilon > 0$ there is a bijective ε-isometry $\Phi\colon W \to Y$ such that $W \in \mathfrak{U}$.*

Theorems 4.5 and 4.6 imply the following

Corollary 4.7. *For every finite nonempty semimetric space X and every $\varepsilon > 0$ there are mappings $F\colon Y \to X$ and $\Phi\colon Z \to Y$ such that Y is finite and ultrametric, $Z \in \mathfrak{U}$, F is ball-preserving, Φ is an ε-isometry and*

$$X = F(\Phi(Z)).$$

The next lemma will be used in the proof of Theorem 4.5.

Lemma 4.8. *Let X be a finite semimetric space. If $B \in \boldsymbol{B}_X$ and $|B| \geqslant 2$, then the following statements hold.*

(i) The ball B has at least two direct predecessors in the Hasse diagram $(\boldsymbol{B}_X, A_{\boldsymbol{B}_X})$.

(ii) The union of all direct predecessors of B coincides with B.

Proof. Let $B \in \mathbf{B}_X$ and $|B| \geqslant 2$. The set of all direct predecessors of B is simply the set of all maximal elements of the subset

$$\mathbf{S} = \{S \in \mathbf{B}_X \colon S \subseteq B \text{ and } S \neq B\} \tag{4.2}$$

of the poset $(\mathbf{B}_X, \subseteq)$. The inequality $|B| \geqslant 2$ implies

$$B \subseteq \bigcup S, \quad S \in \mathbf{S},$$

because $\{x\} \in \mathbf{S}$ for every $x \in B$. Since X is finite, $\mathbf{S}$ is also finite and consequently for every $x \in B$ there is a maximal element S of $\mathbf{S}$ such that $x \in S$. Statement (ii) follows. Now to finish the proof it suffices to note that if B contains a unique direct predecessor S, then $B = S$ contrary to (4.2). □

Proof of Theorem 4.5. Let $(\mathbf{B}_X, A_{\mathbf{B}_X})$ be a Hasse diagram of the poset $(\mathbf{B}_X, \subseteq)$. To this diagram we assign n-ary rooted labeled tree T by the following procedure. Let the root v_0 of T be labeled by X. Let $B_1, ..., B_k$ be direct predecessors of X in $(\mathbf{B}_X, A_{\mathbf{B}_X})$. Define $v_1, .., v_k$ to be the children (nodes of the first level) of v_0 with the labels $B_1, ..., B_k$ respectively. Let us look at the nodes of the first level of the tree T. Define the children of the nodes v_i, $i = 1, ..., k$, as follows: if there is no Y such that $\langle Y, B_i \rangle \in A_{\mathbf{B}_X}$ then v_i is a leaf of T; if $B_{i1}, B_{i2}, ..., B_{in}$ are direct predecessors of B_i in $(\mathbf{B}_X, A_{\mathbf{B}_X})$, then define $v_{i1}, v_{i2}, ..., v_{in}$ to be the children of v_i (nodes of the second level) with labels $B_{i1}, B_{i2}, ..., B_{in}$ respectively. Note that the nodes of the second level may have the identical labels in the case when B_{ij} is a direct predecessor both B_{k_1} and B_{k_2}. Do the same procedure with the nodes of the second level and so on. By Lemma 4.8 T is n-ary tree with $n \geqslant 2$. Note also that the leaves of T are labeled with the balls $\{x_i\}$, $x_i \in X$.

Let n be the number of leaves of T. We define a new names y_i, $i = 1, .., n$, for the leaves of T in any order but save the labels of these leaves. Let Y be an ultrametric space with representing tree isomorphic to T, $Y = \{y_1, ..., y_n\}$. Define $F \colon Y \to X$ by the rule

$$F(y_i) = x_i \text{ if the label of } y_i \text{ is } x_i.$$

We claim that F is ball-preserving. Indeed, by Lemma 4 in [13] for every $B \in \mathbf{B}_Y$ there exists a node $\tilde{v}$ of T such that $\Gamma_T(\tilde{v}) = B$, where $\Gamma_T(\tilde{v})$ is the set of all leaves of subtree with the root $\tilde{v}$. And let $\tilde{B}$ be the label of $\tilde{v}$. According to Lemma 4.8 and the construction of T the set $F(B)$ coincides with $\tilde{B}$. It suffices to note that $\tilde{B}$ is a ball in $\mathbf{B}_X$ because all the nodes in T are labeled by balls of semimetric space X. Furthermore, it is easily seen that the mapping

$$\mathbf{B}_X \ni B \mapsto F(B) \in \mathbf{B}_Y$$

is a surjective homomorphism from $(\mathbf{B}_Y, A_Y)$ to $(\mathbf{B}_X, A_X)$ as required. □

Definition 4.9. Let (Y, d_Y) and (W, d_W) be bounded metric spaces and let $\Delta > 0$. The Gromov-Hausdorff distance $d_{GH}(Y, W)$ is less than Δ if there exists a metric spaces (Z, d_Z) with subspaces Y' and W' such that

- Y and Y' are isometric;
- W and W' are isometric;
- We have the inclusions

$$Y' \subseteq \bigcup_{w \in W'} O_\Delta(w) \text{ and } W' \subseteq \bigcup_{y \in Y'} O_\Delta(y), \tag{4.3}$$

where for $t \in Z$, $O_\Delta(t) = \{z \in Z \colon d_Z(t, z) < \Delta\}$ is an open ball from (Z, d_Z) that has the radius Δ.

The next lemma is a reformulation of Proposition 4.1 from [2].

Lemma 4.10. *Let Y be a finite ultrametric space and let $\varepsilon > 0$. Then there is a finite ultrametric space $W \in \mathfrak{U}$ such that $|Y| = |W|$ and*

$$d_{GH}(Y, W) < \varepsilon.$$

Now we are ready to prove Theorem 4.6.

Proof of Theorem 4.6. The theorem is trivial if $|Y| \leqslant 2$. Let $|Y| \geqslant 3$, let $\varepsilon > 0$ and let

$$\delta = \min\{d_Y(x,y)\colon x, y \in Y, x \neq y\}.$$

Since $3 \leqslant |Y| < \infty$, we have $0 < \delta < \infty$. By Lemma 4.10 for every Δ from the interval $(0, \min(\frac{\delta}{2}, \frac{\varepsilon}{2}))$ there is $W \in \mathfrak{U}$ such that $d_{GH}(Y, W) < \Delta$. Let (Z, d_Z) be metric space which contains isometric copies Y' and W' of Y and W respectively such that inclusions (4.3) hold. We claim that for every $w \in W'$ there is a unique $y \in Y'$ such that $y \in O_\Delta(w)$. Suppose we can find $w \in W$ and two distinct $y_1, y_2 \in Y'$ which satisfy

$$y_1 \in O_\Delta(w) \text{ and } y_2 \in O_\Delta(w).$$

Then the triangle inequality and the definitions of δ and Δ imply

$$\delta \leqslant d_Z(y_1, y_2) \leqslant d_Z(y_1, w) + d_Z(w, y_2) \leqslant 2\Delta < \delta.$$

This contradiction shows that, for every $w \in W'$, the set

$$O_\Delta(w) \cap Y'$$

is either empty or contains a single point. Consequently, if there exists $w^* \in W'$ such that

$$O_\Delta(w^*) \cap Y' = \varnothing,$$

then from the first inclusion in (4.3) it follows that

$$|Y'| = \left| \bigcup_{w \in W'} O_\Delta(w) \cap Y' \right| = \sum_{\substack{w \in W' \\ w \neq w^*}} |O_\Delta(w) \cap Y'| \leqslant |W'| - 1,$$

contrary to $|Y'| = |Y| = |W| = |W'|$.

Let $\varphi\colon W \to W'$ and $\psi\colon Y \to Y'$ be isometries. We define a function $\Phi\colon W \to Y$ by setting

$$(\Phi(w) = y) \Leftrightarrow (\psi(y) \in O_\Delta(\varphi(w))) \tag{4.4}$$

for all $w \in W$ and $y \in Y$. The first part of the proof shows that this definition is correct and Φ is bijective. It remains to prove that Φ is an ε-isometry. For this purpose note that if $w_1, w_2 \in W$ and $y_1 = \Phi(w_1)$, $y_2 = \Phi(w_2)$, then

$$d_W(w_1, w_2) = d_Z(\varphi(w_1), \varphi(w_2)),$$

$$d_Y(\Phi(w_1), \Phi(w_2)) = d_Z(\psi(\Phi(w_1)), \psi(\Phi(w_2)))$$

and, by (4.4),

$$d_Z(\varphi(w_i), \psi(\Phi(w_i))) < \Delta$$

for $i = 1, 2$. Now using the triangle inequality and the inequality $\Delta < \frac{\varepsilon}{2}$ we obtain

$$\begin{aligned} &|d_W(w_1, w_2) - d_Y(\Phi(w_1), \Phi(w_2))| \\ &= |d_Z(\varphi(w_1), \varphi(w_2)) - d_Z(\psi(\Phi(w_1)), \psi(\Phi(w_2)))| \\ &\leqslant d_Z(\varphi(w_1), \psi(\Phi(w_1))) + d_Z(\varphi(w_2), \psi(\Phi(w_2))) < \varepsilon. \end{aligned}$$

Thus Φ is an ε-isometry as required. □

The class $\mathfrak{U}$ consisting of finite ultrametric spaces which are extremal for the Gomory-Hu inequality can be extended by the following way. If X is a compact ultrametric space, then we define $X \in \mathfrak{U}_C$ if $Y \in \mathfrak{U}$ for every finite $Y \subseteq X$. It was shown in [2] that $Y \in \mathfrak{U}$ if $Y \subset X$ and $X \in \mathfrak{U}$. Hence the class $\mathfrak{U}$ is a subclass of $\mathfrak{U}_C$. The following conjecture seems to be a natural generalization of theorems 4.5 and 4.6.

Conjecture 4.11. Let X be a compact nonempty semimetric space and let $\varepsilon > 0$. Then there are continuous mappings $F\colon Y \to X$ and $\Phi\colon W \to Y$ such that Y is compact ultrametric, $W \in \mathfrak{U}_C$, Φ is an ε-isometry and F is ball-preserving and

$$\mathbf{B}_Y \ni B \mapsto F(B) \in \mathbf{B}_X$$

is a surjective homomorphism from $(\mathbf{B}_Y, A_Y)$ to $(\mathbf{B}_X, A_X)$.

This statement can be considered as a variation of the following "universal" property of the Cantor set: "Any compact metric space is a continuous image of the Cantor set."

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