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RESEARCH ARTICLES

New Results on Applications of Nevanlinna Methods to Value Sharing Problems*

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Abstract—Let $\mathbb{K}$ be a complete algebraically closed p-adic field of characteristic zero. We give a new Nevanlinna-type theorem that lets us obtain results of uniqueness for two meromrphic functions inside a disk, sharing 4 bounded functions CM. Let P be a polynomial of uniqueness for meromorphic functions in $\mathbb{K}$ or in an open disk, let $f,\ g$ be two transcendental meromorphic functions in the whole field $\mathbb{K}$ or meromorphic functions in an open disk of $\mathbb{K}$ that are not quotients of bounded analytic functions and let α be a small meromorphic function with respect to f and g. We apply results in algebraic geometry and a new Nevanlinna theorem for p-adic meromorphic functions in order to prove a result of uniqueness for functions: we show that if $f'P'(f)$ and $g'P'(g)$ share α counting multiplicity, then $f = g$, provided that the multiplicity order of zeros of P' satisfy certain inequalities. A breakthrough in this paper consists of replacing inequalities $n \geq k+2$ or $n \geq k+3$ used in previous papers by a new Hypothesis (G). Another consists of using the new Nevanlinna-type Theorem.

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1. INTRODUCTION

Notations and definitions: Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, complete with respect to an ultrametric absolute value $|\ .\ |$. We denote by $\mathcal{A}(\mathbb{K})$ the $\mathbb{K}$-algebra of entire functions in $\mathbb{K}$, by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in $\mathbb{K}$, i.e. the field of fractions of $\mathcal{A}(\mathbb{K})$ and by $\mathbb{K}(x)$ the field of rational functions. Throughout the paper, a is a point in $\mathbb{K}$ and R is a strictly positive number and we denote by $d(a,R)$ the disk $\{x \in \mathbb{K} \mid |x-a| \leq R\}$ and by $d(a,R^-)$ the "open" disk $\{x \in \mathbb{K} : |x-a| < R\}$, by $\mathcal{A}(d(a,R^-))$ the $\mathbb{K}$-algebra of analytic functions in $d(a,R^-)$ i.e. the $\mathbb{K}$-algebra of power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converging in $d(a,R^-)$ and we denote by $\mathcal{M}(d(a,R^-))$ the field of meromorphic functions inside $d(a,R^-)$, i.e. the field of fractions of $\mathcal{A}(d(a,R^-))$. Moreover, we denote by $\mathcal{A}_b(d(a,R^-))$ the $\mathbb{K}$-subalgebra of $\mathcal{A}(d(a,R^-))$ consisting of the bounded analytic functions in $d(a,R^-)$, i.e. which satisfy $\sup_{n\in\mathbb{N}} |a_n|R^n < +\infty$. And we denote by $\mathcal{M}_b(d(a,R^-))$ the field of fractions of $\mathcal{A}_b(d(a,R^-))$. Finally, we denote by $\mathcal{A}_u(d(a,R^-))$ the set of unbounded analytic functions in $d(a,R^-)$, i.e. $\mathcal{A}(d(a,R^-)) \setminus \mathcal{A}_b(d(a,R^-))$. Similarly, we set $\mathcal{M}_u(d(a,R^-)) = \mathcal{M}(d(a,R^-)) \setminus \mathcal{M}_b(d(a,R^-))$.

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Let $f \in \mathcal{M}(d(a,R^-))$, and let $r \in]0,R[$. By classical results [7] we know that $|f(x)|$ has a limit when $|x|$ tends to r, while being different from r. We set $|f|(r) = \lim_{|x-a|\to r, |x|\neq r} |f(x)|$.

Let $f,\ g,\ \alpha \in \mathcal{M}(\mathbb{K})$ $\big($resp. let $f, g, \alpha \in \mathcal{M}(d(a,R^-))\big)$. We say that f and g *share the function* α *C.M.*, if $f-\alpha$ and $g-\alpha$ have the same zeros with the same multiplicities in $\mathbb{K}$ $\big($resp. in $d(a,R^-)\big)$ and we say that f and g *share the function* α *I.M.*, if $f-\alpha$ and $g-\alpha$ have the same zeros without considering multiplicities in $\mathbb{K}$ $\big($resp. in $d(a,R^-)\big)$. In particular, those definitions apply to constants as small functions.

Throughout the paper, the symbol $\forall$ means *for all*.

The paper aims at showing a new Nevanlinna-type theorem for meromorphic functions both in the whole field and inside a disk $d(a,R^-)$, which is not a direct consequence of the classical p-adic Second Main Theorem. Concerning functions inside the disk, our reasoning lets us obtain a kind of "Second Main Theorem on n small functions" provided small functions are bounded inside the disk. Indeed, in the general situation, Yamanoi's Theorem proven in [17] in the complex context has no equivalent in the field $\mathbb{K}$.

Let us recall the definition of the Nevanlinna Functions for meromorphic functions in $\mathbb{K}$. Let log be a real logarithm function of base $b>1$ and let $f \in \mathcal{M}(\mathbb{K})$ $\big($resp. $f \in \mathcal{M}(d(0,R^-))\big)$ having no zero and no pole at 0. Let $r \in]0,+\infty[$ $\big($resp. $r \in]0,R[\big)$ and let $\gamma \in d(0,r)$. If f has a zero of order n at γ, we put $\omega_\gamma(f) = n$. If f has a pole of order n at γ, we put $\omega_\gamma(f) = -n$ and finally, if $f(\gamma) \neq 0, \infty$, we set $\omega_\gamma(f) = 0$.

We denote by $Z(r,f)$ the *counting function of zeros of* f in $d(0,r)$, counting multiplicity, defined as follows:

if f has no zero at 0, we set

$$Z(r,f) = \sum_{\omega_\gamma(f)>0,\ |\gamma|\leq r} \omega_\gamma(f)(\log r - \log|\gamma|),$$

and if f has a zero of order q at 0, we set

$$Z(r,f) = q\log r + \sum_{\omega_\gamma(f)>0,\ |\gamma|\leq r} \omega_\gamma(f)(\log r - \log|\gamma|),$$

Similarly, we denote by $\overline{Z}(r,f)$ the counting function of zeros of f in $d(0,r)$, ignoring multiplicity: if f has no zero at 0, we set

$$Z(r,f) = \sum_{\omega_\gamma(f)>0,\ |\gamma|\leq r} (\log r - \log|\gamma|),$$

and if f has a zero of order at 0, we set

$$Z(r,f) = \log r + \sum_{\omega_\gamma(f)>0,\ |\gamma|\leq r} (\log r - \log|\gamma|),$$

In the same way, we set $N(r,f) = Z\Big(r, \dfrac{1}{f}\Big)$ $\Big($resp. $\overline{N}(r,f) = \overline{Z}\Big(r, \dfrac{1}{f}\Big)\Big)$ to denote the *counting function of poles of* f in $d(0,r)$, counting multiplicity (resp. ignoring multiplicity).

For $f \in \mathcal{M}(d(0,R^-))$ the *characteristic Nevanlinna function* is defined by

$$T(r,f) = \max\{Z(r,f), N(r,f)\}$$

.

Remark: There exist other definitions of the Nevanlinna functions, involving for instance $|f(0)|$ when the function f has no zero and no pole at 0. Actually, all definitions are equivalent through inequalities, up to an additive constant.

As usual, given a function $f \in \mathcal{M}(\mathbb{K})$ (resp. $\mathcal{M}(d(0, R^-))$), we denote by $S_f(r)$ a function of r defined in $]0, +\infty[$ (resp. in $]0, R[$) such that $\lim_{r \to +\infty} \frac{S_f(r)}{T(r, f)} = 0$ (resp. $\lim_{r \to R} \frac{S_f(r)}{T(r, f)} = 0$).

Let us first recall the well known p-adic Nevanlinna Theorems:

Theorem N1. [6] *Let $a_1, ..., a_n \in \mathbb{K}$ with $n \geq 2, n \in \mathbb{N}$, and let $f \in \mathcal{M}(\mathbb{K})$ (resp. let $f \in \mathcal{M}(d(0, R^-))$). Let $S = \{a_1, ..., a_n\}$. Then, for $r > 0$ we have*

$$(n-1)T(r, f) \leq \sum_{j=1}^{n} \overline{Z}(r, f - a_j) + \overline{N}(r, f) - \log r + O(1),$$

(resp.

$$(n-1)T(r, f) \leq \sum_{j=1}^{n} \overline{Z}(r, f - a_j) + \overline{N}(r, f) + O(1)).$$

Now, we must recall the definition of a *small function* with respect to a meromorphic function and some pertinent properties.

Definition. Let $f \in \mathcal{M}(\mathbb{K})$ (resp. let $f \in \mathcal{M}(d(0, R^-))$) such that $f(0) \neq 0, \infty$. A function $\alpha \in \mathcal{M}(\mathbb{K})$ (resp. $\alpha \in \mathcal{M}(d(0, R^-))$) is called *a small function with respect to f*, if it satisfies $\lim_{r \to +\infty} \frac{T(r, \alpha)}{T(r, f)} = 0$ $\Big($resp. $\lim_{r \to R^-} \frac{T(r, \alpha)}{T(r, f)} = 0\Big)$.

We denote by $\mathcal{M}_f(\mathbb{K})$ $\big($resp. $\mathcal{M}_f(d(0, R^-))\big)$ the set of small meromorphic functions with respect to f in $\mathbb{K}$ $\big($resp. in $d(0, R^-)\big)$ and similarly we denote by $\mathcal{A}_f(\mathbb{K})$ $\big($resp. $\mathcal{A}_f(d(0, R^-))\big)$ the set of small analytic functions with respect to f in $\mathbb{K}$ $\big($resp. in $d(0, R^-)\big)$.

Remark: Thanks to classical properties of the Nevanlinna function $T(r, f)$ [9] with respect to the operations in a field of meromorphic functions, such as $T(r, f + g) \leq T(r, f) + T(r, g) + O(1)$ and $T(r, fg) \leq T(r, f) + T(r, g) + O(1)$, for $f, g \in \mathcal{M}(\mathbb{K})$ and $r > 0$, it is easily proven that $\mathcal{M}_f(\mathbb{K})$ $\big($resp. $\mathcal{M}_f(d(0, R^-))\big)$ is a subfield of $\mathcal{M}(\mathbb{K})$ $\big($resp. $\mathcal{M}(d(0, R^-))\big)$ and that $\mathcal{M}(\mathbb{K})$ $\big($resp. $\mathcal{M}(d(0, R^-))\big)$ is a transcendental extension of $\mathcal{M}_f(\mathbb{K})$ $\big($resp. of $\mathcal{M}_f(d(0, R^-))\big)$.

Theorem N2: [9, 11] *Let $f \in \mathcal{A}(\mathbb{K})$ (resp. let $f \in \mathcal{A}(d(0, R^-))$) and let $u \in f \in \mathcal{A}_f(\mathbb{K})$ (resp. let $u \in \mathcal{A}_f(d(0, R^-))$). Then $T(r, f) \leq \overline{Z}(r, f) + \overline{Z}(r, f - u) + S_f(r)$.*

A) A new Nevanlinna-type Theorem

2. RESULTS

Now, we can give here a new theorem which will be useful to obtain results in Part B comparatively to results of [4] and first we can obtain new results of uniqueness for functions inside a disk.

Theorem A1: *Let $f \in \mathcal{M}(\mathbb{K})$ and let $a_1, ..., a_q \in \mathbb{K}$ be distinct. Then*

$$(q-1)T(r, f) \leq \max_{1 \leq k \leq q} \Big(\sum_{j=1, j \neq k}^{q} Z(r, f - a_j) \Big) + O(1).$$

Corollary A1.1: *Let $f \in \mathcal{M}(\mathbb{K})$ and let $a_1, ..., a_q \in \mathbb{K}$ be distinct. Then*

$$(q-1)T(r,f) \leq \sum_{j=1}^{q} Z(r, f-a_j) + O(1).$$

Theorem A2: *Let $f \in \mathcal{M}(d(0,R^-)))$ and let $\theta_1, ..., \theta_q \in \mathcal{M}_b(d(0,R^-))$ be distinct. Then*

$$(q-1)T(r,f) \leq \max_{1 \leq k \leq q} \Big(\sum_{j=1, j \neq k}^{q} Z(r, f-\theta_j) \Big) + O(1).$$

Corollary A2.1: *Let $f \in \mathcal{M}(d(0,R^-)))$ and let $\theta_1, ..., \theta_q \in \mathcal{M}_b(d(0,R^-))$ be distinct. Then*

$$(q-1)T(r,f) \leq \sum_{j=1}^{q} Z(r, f-\theta_j) + O(1).$$

Remark: Corollary A1.1 does not hold in complex analysis. Indeed, let f be a meromorphic function in $\mathbb{C}$ omitting two values a and b, such as $f(x) = \dfrac{e^x}{e^x - 1}$. Then $Z(r, f-a) + Z(r, f-b) = 0$.

Concerning unbounded functions inside a disk, Corollary A2.1 may in certain sense, replace the Nevanlinna Theorem on n small functions proven by Yamanoi in $\mathbb{C}$ [17]: this theorem does not hold for meromorphic functions defined on the whole field $\mathbb{K}$.

Thanks to Corollaries A1.1 and A2.1 we can obtain a new result on functions sharing 4 bounded functions inside a disk. Let us first recall results already known on value sharing IM for p-adic functions [9]:

Definition: Two functions $f,\ g \in \mathcal{M}(K)$ or $\mathcal{M}(d(a,R^-))$ are said *to share I.M. a value $\alpha \in \mathbb{K}$* or *a function α defined in the same domain,* if $f-\alpha$ and $g-\alpha$ have the same distinct zeros, ignoring multiplicity, in their domain of defintion. And $f,\ g$ are said *to share C.M. a value $\theta \in \mathbb{K}$* or *a function α defined in the same domain,* if $f-\alpha$ and $g-\alpha$ have the same distinct zeros, counting multiplicity.

Theorem AC: *Let $f,\ g \in \mathcal{M}(\mathbb{K})$ (resp. $f,\ g \in \mathcal{M}_u(d(a,R^-)))$ share I.M. 4 (resp.5) distinct points $a_1, a_2, a_3, a_4 \in \mathbb{K}$ (resp. $a_1, a_2, a_3, a_4, a_5 \in \mathbb{K}$). Then $f = g$.*

Theorem AD: *Let $f,\ g \in \mathcal{A}(\mathbb{K})$ (resp. $f,\ g \in \mathcal{A}_u(d(a,R^-)))$ share I.M. 2 (resp.3) distinct points $a_1, a_2 \in \mathbb{K}$ (resp. $a_1, a_2, a_3 \in \mathbb{K}$). Then $f = g$.*

Now, thanks to Corollary A2.1 we can obtain a new result concerning value sharing bounded functions CM inside a disk:

Theorem A3: *Let $f,\ g \in \mathcal{M}(\mathbb{K})$ (resp. $f,\ g \in \mathcal{M}_u(d(a,R^-)))$ share C.M. 4 distinct points $a_1, a_2, a_3, a_4 \in \mathbb{K}$. Then $f = g$.*

Theorem A4: *Let $f,\ g \in \mathcal{M}_u(d(a,R^-))$ share C.M. 4 distinct functions $\theta_1, \theta_2, \theta_3, \theta_4 \in \mathcal{M}_b(d(a,R^-))$. Then $f = g$.*

In order to complete results known on this topic, we can notice Theorem A5 which does not need our new Nevanlinna theorems:

Theorem A5: *Let $f,\ g \in \mathcal{A}_u(d(a,R^-))$ share C.M. 2 distinct functions $\theta_1, \theta_2 \in \mathcal{A}_b(d(a,R^-))$. Then $f = g$.*

3. PROOFS OF PART A

First, we must recall Lemmas AL1 and AL2 that are classical.

Lemma AL1 [7] : *For every $r \in]0, R[$, the mapping $| \,.\, |(r)$ is an ultrametric multiplicative norm on $\mathcal{M}(d(0, R^-))$.*

The following Lemma AL2 is the p-adic Schwarz formula:

Lemma AL2 [9]: *Let $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}(d(0, R^-))$) and let $r',\ r'' \in]0, +\infty[$ (resp. let $r',\ r'' \in]0, R[$) satisfy $r' < r''$. Then $\log(|f|(r'')) - \log(|f|(r')) = Z(r'', f) - Z(r', f)$. If f has no zero and no pole at 0, then $\log(|f|(r)) - \log(|f(0)|) = Z(r, f)$.*

By Lemma AL2, we can derive Lemma AL3 which is also classical:

Lemma AL3 [9]: *Let $f,\ g \in \mathcal{A}(d(0, R^-))$ (resp. $f,\ g \in \mathcal{A}(\mathbb{K})$). The Nevanlinna functions T and Z satisfy $T(r, f) = Z(r, f),\ T(r, f + g) \le \max(T(r, f), T(r, g)) + O(1), r \in]0, R[$. Suppose $f,\ g \in \mathcal{A}(d(0, R^-))$ have no zero at the origin and let S be a subset of $]0, R[$ (resp. of $]0, +\infty[$) such that $Z(r, f) + \log|f(0)|) > Z(r, g) + \log|g(0)|)\ \forall r \in S$. Then $Z(r, f + g) = Z(r, f)\ \forall r \in S$.*

Lemma AL4 is essential and directly leads to the theorems:

Lemma AL4: *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(d(0, R^-))$). Suppose that there exists $\theta \in \mathbb{K}$ (resp. $\theta \in \mathcal{M}_b(d(0, R^-))$) and a sequence of intervals $I_n = [u_n, v_n]$ such that*
$u_n < v_n < u_{n+1}$, $\lim_{n\to+\infty} u_n = +\infty$ (resp. $\lim_{n\to+\infty} u_n = R$) and
$\lim\limits_{n\to+\infty}\Big(\inf\limits_{r\in I_n} T(r, f) - Z(r, f - \theta)\Big) = +\infty$ (resp. $\lim\limits_{n\to+\infty}\Big(\inf\limits_{r\in I_n} T(r, f) - Z(r, f - \theta)\Big) = +\infty$) Let $\tau \in \mathbb{K}$ (resp. let $\tau \in \mathcal{M}_b(d(0, R^-))$) , $\tau \ne \theta$. Then $Z(r, f - \tau) = T(r, f) + O(1)\ \forall r \in I_n$ when n is big enough.

Proof: We know that the Nevanlinna functions of a meromorphic function f are the same in $\mathbb{K}$ and in an algebraically closed complete extension of $\mathbb{K}$ whose absolute value extends that of $\mathbb{K}$. Consequently, without loss of generality, we can suppose that $\mathbb{K}$ is spherically complete because we know that such a field does admit a spherically complete algebraically closed extension whose absolute value expands that of $\mathbb{K}$. If f belongs to $\mathcal{M}(\mathbb{K})$, we can obviously set it in the form $\dfrac{g}{h}$ where $g,\ h$ belong to $\mathcal{A}(\mathbb{K})$ and have no common zero. Next, since $\mathbb{K}$ is supposed to be spherically complete, if f belongs to $\mathcal{M}(d(0, R^-))$ we can also set it in the form $\dfrac{g}{h}$ where $g,\ h$ belong to $\mathcal{A}(d(0, R^-))$ and have no common zero [9]. Consequently, we have $T(r, f) = \max(Z(r, g), Z(r, h))$.

When θ is a constant we can obviously suppose that $\theta = 0$. Suppose now $\theta \in \mathcal{M}_b(d(0, R^-))$. Then $f - \theta$ belongs to $\mathcal{M}_u(d(0, R^-))$ like f and $\tau - \theta$ belongs to $\mathcal{M}_b(d(0, R^-))$. Consequently, in both cases, we can assume $\theta = 0$ to prove the claim. Next, up to a change of origin, we can also assume that none of the functions we consider have a pole or a zero at the origin.

Now, we have $\lim\limits_{n\to+\infty}\Big(\inf\limits_{r\in I_n} T(r, f) - Z(r, f)\Big) = +\infty$, i.e.

$$\lim_{n\to+\infty}\Big(\inf_{r\in I_n}(Z(r, h) - Z(r, g))\Big) = +\infty.$$

Particularly, by Lemma AL3 we notice that $T(r, f) = Z(r, h) + O(1)$ whenever $r \in I_n$ when n is big enough.

Consider now $Z(r, f - \tau) = Z(r, g - \tau h)$. Then $Z(r, \tau h) = Z(r, h)$, hence by Lemma AL3, $Z(r, g - \tau h) = Z(r, h) + O(1)$, whenever $r \in I_n$ when n is big enough. Therefore $Z(r, f - \tau) = Z(r, h) + O(1) = T(r, f) + O(1),\ r \in I_n$ when n is big enough. So the claim is proven when τ is a constant.

Suppose now that $f \in \mathcal{M}(d(0,R^-))$ and $\tau \in \mathcal{M}_b(d(0,R^-))$. We can write τ in the form $\dfrac{\phi}{\psi}$ where $\phi, \psi \in \mathcal{A}_b((d(0,R^-))$ have no common zero. Consider
$Z(r, f-\tau) = Z(r, \dfrac{\psi g - \phi h}{\psi h})$. Since g and h have no common zero and since both ϕ, ψ are bounded, we have $Z(r, \dfrac{\psi g - \phi h}{\psi h}) = Z(r, \psi g - \phi h) + O(1)$. Now, since the norm $|\ .\ |(r)$ is multiplicative and increasing in r, by Lemma AL3 in I_n we have $|\psi g|(r) < |\phi h|(r)$ when n is big enough. Consequently, by Lemma AL1, $|\psi g - \phi h|(r) = |\phi h|(r)$ in I_n when n is big enough. Therefore, by Lemma AL3, $Z(r, \psi g - \phi h) = Z(r, \phi h) = Z(r, h) + O(1)$ in I_n when n is big enough and consequently we $Z(r, f-\tau) = Z(r, h) + O(1) = T(r, h) + O(1) = T(r, f) + O(1)$. That finishes proving Lemma AL4.

Problems of value sharing constants or functions, counting multiplicity or ignoring multiplicity, have been the focus of a lot of papers [4, 6, 12, 13, 15, 18]. Here we will apply Corollaries A1.1 and A2.1 to functions $f,\ g \in \mathcal{M}_u(d(a,R^-))$ sharing C.M. four constants or four functions $\theta_j \in \mathcal{M}_b(d(a,R^-))$.

Proof of Theorems A1 and A2: Suppose Theorem A1 (resp. Theorem A2) is wrong. In order to make a unique proof for the two theorems, in Theorem A1 we set $\theta_j = a_j$. Thus, there exists $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(d(0,R^-)))$ and $\theta_1, ..., \theta_q \in \mathbb{K}$ (resp. $\theta_1, ..., \theta_q \in \mathcal{M}_b(d(0,R^-)))$ such that $(q-1)T(r,f) - \max_{1\le k\le q}\left(\sum_{j=1, j\neq k}^{q} Z(r, f-\theta_j)\right)$ admits no superior bound in $]0,+\infty[$. So, there exists a sequence of intervals $J_s = [w_s, y_s]$ such that $w_s < y_s < w_{s+1}$, $\lim_{s\to+\infty} w_s = +\infty$ (resp. $\lim_{s\to+\infty} w_s = R$) and two distinct indices m and t such that

$$\lim_{s\to+\infty} \inf_{r\in J_s} \big(T(r,f) - Z(r, f-\theta_m)\big) = +\infty$$

and

$$\lim_{s\to+\infty} \inf_{r\in J_s} \big(T(r,f) - Z(r, f-\theta_t)\big) = +\infty.$$

But by Lemma AL4, this is impossible. This ends the proof of Theorems A1 and A2.

Proof of Theorems A3 and A4: In Theorem A3 we put $\theta_j = a_j,\ j = 1,2,3,4$. In Theorem A4 we can obviously assume $a = 0$. Suppose that f and g are not identical. We have

$$\sum_{j=1}^{4} Z(r, f-\theta_j) \le Z(r, f-g) \le T(r, f-g) \le T(r,f) + T(r,g).$$

On the other hand by Corollary A1.1 (resp. A2.1), we have $\sum_{j=1}^{4} Z(r, f-\theta_j) \ge 3T(r,f) + O(1)$. Consequently, $3T(r,f) \le T(r,f) + T(r,g)$. Similarly, $3T(r,g) \le T(r,f) + T(r,g)$, hence $3(T(r,f) + T(r,g)) \le 2(T(r,f) + T(r,g))$, a contradiction.

Remark: When $f,\ g$ belong to $\mathcal{M}(\mathbb{K})$, it is possible to prove the statement of Theorem A3 by using the classical p-adic Second Main Theorem. But when $f,\ g$ belong to $\mathcal{M}_u(d(0,R^-))$, the p-adic Second Main Theorem does not let us prove that statement.

Proof of Theorem A5: Suppose that f and g are not identical. By Theorem 2.4.15 [9] we have

$$\sum_{j=1}^{2} Z(r, f-\theta_j) \le Z(r, f-g) \le T(r, f-g) \le \max(T(r,f), T(r,g)).$$

On the other hand, since θ_j is bounded, so is $T(r,\theta_j)$ and therefore $T(r, f-\theta_j) = T(r,f) + O(1)$ and similarly, $T(r, g-\theta_j) = T(r,g) + O(1)$. Now, by definition, $T(r,f) = Z(r,f) + O(1), T(r,g) = Z(r,g) + O(1)$. Consequently, $T(r,f) + T(r,g) \le \max(T(r,f), T(r,g)) + O(1)$, a contradiction.

B) New results on p-adic meromorphic functions $f'P'(f),\ g'P'(g)$ sharing a small function

4. RESULTS

Throughout the paper we will denote by $P(X)$ a polynomial in $\mathbb{K}[X]$ such that $P'(X)$ is of the form $\prod_{i=1}^{l}(X-a_i)^{k_i}$ with $l \geq 2$ and $k_1 \geq 2$. The polynomial P will be said *to satisfy Hypothesis (G)* if $P(a_i)+P(a_j) \neq 0 \ \forall i \neq j$.

We will improve the main theorems obtained in [4] and [5] with the help of the new hypothesis Hypothesis (G) and by thorougly examining the situation in order to avoid a lot of exclusions.

Notation: Let L be an algebraically closed field and let $P \in L[x] \setminus L$ and let $\Xi(P)$ be the set of zeros c of P' such that $P(c) \neq P(d)$ for every zero d of P' other than c. We denote by $\Phi(P)$ its cardinal.

Definitions. Let $f,\ g,\ \alpha \in \mathcal{M}(\mathbb{K})$ (resp. let $f,\ g, \alpha \in \mathcal{M}(d(a,R^-))$). We say that f and g *share the function* α *C.M.*, if $f-\alpha$ and $g-\alpha$ have the same zeros with the same multiplicities in $\mathbb{K}$ (resp. in $d(0,R^-)$).

Recall that a polynomial $P \in \mathbb{K}[x]$ is called *a polynomial of uniqueness* for a family of functions $\mathcal{F}$ if for any two functions $f,\ g \in \mathcal{F}$ the property $P(f)=P(g)$ implies $f=g$.

The definition of polynomials of uniqueness was introduced in by H. Fujimoto [10] and was used in many papers, explicitly or implicitly, [2, 4, 9, 10, 12, 19] for complex functions and [1–3, 9, 16] for p-adic functions.

Let us recall general results on polynomials of uniqueness:

Theorem BU1 [8]: *Let $P(X) \in \mathbb{K}[X]$. If $\Phi(P) \geq 2$ then P is a polynomial of uniqueness for $\mathcal{A}(\mathbb{K})$. If $\Phi(P) \geq 3$ then P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ and for $\mathcal{A}_u(d(a,R^-))$. If $\Phi(P) \geq 4$ then P is a polynomial of uniqueness for $\mathcal{M}_u(d(a,R^-))$.*

Concerning polynomials such that P' has exactly two distinct zeros, we know other results:

Theorem BU2 [1, 8]: *Let $P \in \mathbb{K}[x]$ be such that P' has exactly two distinct zeros γ_1 of order c_1 and γ_2 of order c_2. If $\min\{c_1,c_2\} \geq 2$, then P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$. Moreover, if $c_1=1,\ c_2 \geq 2$, then P is a polynomial of uniqueness for $\mathcal{A}(\mathbb{K})$ and for $\mathcal{A}(d(a,R^-))$.*

Theorem BU3 [15]: *Let $P \in \mathbb{K}[x]$ be of degree $n \geq 6$, such that P' only has two distinct zeros, one of them being of order 2. Then P is a polynomial of uniqueness for $\mathcal{M}_u(d(a,R^-))$.*

In the present paper, thanks to the new Hypothesis (G) introduced above, we mean to avoid the hypothesis $k_1 \geq k+2$ for $\mathcal{M}(\mathbb{K})$ and $k_1 \geq k+3$ for $\mathcal{M}(d(a,R^-))$. On the other hand, here we will use a new Nevanlinna-type theorem.

Among the first results obtained in that domain, we must cite the work by W. Lin and H. Yi [13]. Here we first have a new theorem for p-adic analytic functions:

Theorem B1: *Let $P(X) \in \mathbb{K}[X]$ and let $P'(X) = b\prod_{i=1}^{l}(X-a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $f,\ g \in \mathcal{A}(\mathbb{K})$ (resp. $f,\ g \in \mathcal{A}(d(0,R^-))$) be a polynomial of uniqueness for $\mathcal{A}(\mathbb{K})$ (resp. for $\mathcal{A}(d(0,R^-))$) and be such that $f'P'(f)$ and $g'P'(g)$ share a function $\alpha \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$ C.M. (resp. $\alpha \in \mathcal{A}_f(d(0,R^-)) \cap \mathcal{A}_g(d(0,R^-))$ CM). If $\sum_{j=1}^{l} k_j \geq 2l+3$, then $f=g$. Moreover, if $f,\ g \in \mathcal{A}(\mathbb{K})$ and if $\alpha \in \mathbb{K}\setminus\{0\}$ and if $\deg(P) \geq 2l+2$, then $f=g$.*

Corollary B1.1 *Let $P(x) \in \mathbb{K}[x]$ be such that $\Phi(P) \geq 2$ and let $P'(x) = \prod_{i=1}^{l}(x - a_i)^{k_i}$ and let $f, g \in \mathcal{A}(\mathbb{K})$ be transcendental such that $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$. If $\sum_{i=1}^{l} k_i \geq 2l + 3$ then $f = g$. Moreover, if α is a constant and if $\deg(P) \geq 2l + 2$, then $f = g$..*

Corollary B1.2 *Let $P(x) \in \mathbb{K}[x]$ be such that $\Phi(P) \geq 3$ and let $P'(x) = \prod_{i=1}^{l}(x - a_i)^{k_i}$ and let $f,\ g \in \mathcal{A}_u(d(0, R^-))$ be such that $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha \in \mathcal{A}_f(d(0, R^-)) \cap \mathcal{A}_g(d(0, R^-))$. If $\deg(P) \geq 2l + 3$, then $f = g$..*

Example: Let $P(x) = \frac{x^9}{9} - \frac{3x^7}{7} + \frac{3x^5}{5} - \frac{x^3}{3}$. We can check that $P'(x) = x^2(x^2 - 1)^3$ hence $l = 3$. Next, we have $P(0) = 0,\ P(1) \neq 0,\ P(-1) = -P(1)$. Consequently, $\Phi(P) = 3$ and $\deg(P) = 2l + 3$. Then, given $f,\ g \in \mathcal{A}(\mathbb{K})$ (resp. $f,\ g \in \mathcal{A}_u(d(0, R^-))$) such that $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$ (rersp. $\alpha \in \mathcal{A}_f(d(0, R^-)) \cap \mathcal{A}_g(d(0, R^-))$) then $f = g$.

By Theorems BU2 and BU3 we can also derive Corollaries B1.3 and B1.4:

Corollary B1.3 *Let $f,\ g \in \mathcal{A}(\mathbb{K})$, let $\alpha \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$ and let $a,\ b \in \mathbb{K}(a \neq b)$. If $(f - a)^n(f - b)^k f'$ and $(g - a)^n(g - b)^k g'$ share the function α C.M. with $\max(n, k) \geq 2$, then $f = g$.*

Corollary B1.4: *Let $f,\ g \in \mathcal{A}_u(d(0, R^-))$, let $\alpha \in \mathcal{A}_f(d(0, R^-)) \cap \mathcal{A}_g(d(0, R^-))$ and let $a \in \mathbb{K} \setminus \{0\}$. Suppose $(f - a)^n(f - b)^k f'$ and $(g - a)^n(g - b)^k g'$ share the function α C.M. If $k = 1$, and $n \geq 2$ or if $k = 2$ and $n \geq 3$ then $f = g$.*

In order to improve results of [4] on p-adic meromorphic functions, we have to state Propositions BP derived from results of [3].

Notation and definition: Henceforth, we assume that $a_1 = P(a_1) = 0$ and that $P'(X)$ is of the form $b\prod_{i=1}^{l}(X - a_i)^{k_i}$ with $n \geq 2$.

Proposition BP: *Let $P \in \mathbb{K}[X]$ satisfy Hypothesis (G) and $\deg(P) \geq 3$ (resp. $\deg(P) \geq 4$). If meromorphic functions $f,\ g \in \mathcal{M}(\mathbb{K})$ (resp. $f,\ g \in \mathcal{M}(d(a, R^-))$) satisfy $P(f(x)) = P(g(x)) + C$ $(C \in \mathbb{K}^*)$, $\forall x \in \mathbb{K}$ (resp. $\forall x \in d(a, R^-)$) then both f and g are constant (resp. f and g belong to $\mathcal{M}_b(d(a, R^-))$).*

From [4] and thanks to Propositions BP we can now derive the following Theorems B2, B3, B4:

Theorem B2: *Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ (resp. for $\mathcal{M}(d(0, R^-))$) with $l \geq 2$, let $P'(X) = b\prod_{i=1}^{l}(X - a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $k_i \geq k_{i+1},\ 2 \leq i \leq l - 1$, let $k = \sum_{i=2}^{l} k_i$. For each $m \in \mathbb{N},\ m \geq 5$, let u_m be the biggest of the i such that $k_i > 4$, let $s_5 = \max(0, u_5 - 3)$ and for each $m \in \mathbb{N},\ m \geq 6$, let $s_m = \max(0, u_m - 2)$. Suppose P satisfies the following conditions:*

$$k_1 \geq 10 + \max(0, 5 - k_2) + \sum_{i=3}^{l} \max(0, 4 - k_i) - \min(\sum_{m=5}^{\infty} s_m, 2l)$$

either $k_1 \geq k+2$ *(resp.* $k_1 \geq k+3$*) or* P *satisfies Hypothesis (G),*

if $l = 2$*, then* $k_1 \neq k+1,\ 2k,\ 2k+1,\ 3k+1$*,*

if $l = 3$*, then* $k_1 \neq \frac{k}{2},\ k_1 \neq k+1,\ 2k+1,\ 3k_i - k\ \forall i = 2,\ 3$.

If $l \geq 4$*, then* $k_1 \neq k+1$

Let $f, g \in \mathcal{M}(\mathbb{K})$ *(resp.* $f, g \in \mathcal{M}_u((d(0, R^-))$ *be transcendental and let* $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ *(resp.* $\alpha \in \mathcal{M}_f(d(0, R^-)) \cap \mathcal{M}_g(d(0, R^-))$*) be non-identically zero. If* $f'P'(f)$ *and* $g'P'(g)$ *share* α *C.M., then* $f = g$*.*

Remark: The sum $\sum_{m=5}^{\infty} s_m$ is obviously finite.

Corollary B2.1 *Let* $P \in \mathbb{K}[x]$ *satisfy* $\Phi(P) \geq 3$ *and hypothesis (G), let* $P' = b\prod_{i=1}^{l}(X - a_i)^{k_i}$ *with* $b \in \mathbb{K}^*$*,* $l \geq 3$*,* $k_i \geq k_{i+1}$*,* $2 \leq i \leq l-1$*, let* $k = \sum_{i=2}^{l} k_i$*, and for each* $m \in \mathbb{N}$*,* $m \geq 5$*, let* u_m *be the biggest of the* i *such that* $k_i > 4$*, let* $s_5 = \max(0, u_5 - 3)$ *and for every* $m \geq 6$*, let* $s_m = \max(0, u_m - 2)$ *. Suppose* P *satisfies the following conditions:*

$$k_1 \geq 10 + \max(0, 5 - k_2) + \sum_{i=3}^{l} \max(0, 4 - k_i) - \min(\sum_{m=5}^{\infty} s_m, 2l)$$

if $l = 3$*, then* $k_1 \neq \dfrac{k}{2},\ k+1,\ 2k+1,\ 3k_i - k\ \forall i = 2, 3$*,*

if $l \geq 4$*, then* $n \neq k+1$*. Let* $f,\ g \in \mathcal{M}(\mathbb{K})$ *be transcendental and let* $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ *be non-identically zero. If* $f'P'(f)$ *and* $g'P'(g)$ *share* α *C.M., then* $f = g$*.*

Example: Let

$$P(X) = \frac{X^{20}}{20} - \frac{X^{19}}{19} - \frac{4X^{18}}{18} + \frac{4X^{17}}{17}$$

$$+\frac{6X^{16}}{16} - \frac{6X^{15}}{15} - \frac{4X^{14}}{14} + \frac{4X^{13}}{13} + \frac{X^{12}}{12} - \frac{X^{11}}{11}.$$

We can check that $P'(X) = X^{10}(X-1)^5(X+1)^4$ and

$$P(0) = 0,\ P(1) = \sum_{j=0}^{4} C_4^j (-1)^j \Big(\frac{1}{12+2j} - \frac{1}{11+2j}\Big),$$

$$P(-1) = -\sum_{j=0}^{4} C_4^j \Big(\frac{1}{12+2j} + \frac{1}{11+2j}\Big).$$

Consequently, we have $\Phi(P) = 3$ and we check that Hypothesis (G) is satisfied. Now, let $f,\ g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Remark: In that example, we have $k_1 = 10,\ k = 9$. Applying our previous work, a conclusion would have required $k_1 \geq k+2 = 11$.

Example: Let

$$P(X) = \frac{X^{24}}{24} - \frac{10X^{23}}{23} + \frac{36X^{22}}{22} - \frac{40X^{21}}{21} - \frac{74X^{20}}{20} + \frac{226X^{19}}{19}$$

$$-\frac{84X^{18}}{18} - \frac{312X^{17}}{17} + \frac{321X^{16}}{16} + \frac{88X^{15}}{15}$$

$$-\frac{280X^{14}}{14}+\frac{48X^{13}}{13}+\frac{80X^{12}}{12}-\frac{32X^{11}}{11}.$$

We can check that $P'(X)=X^{10}(X-2)^5(X+1)^4(X-1)^4$. Next, we have $P(2)<-134378$, $P(1)\in]-2,11;-2,10[$, $P(-1)\in]2,18;2,19[$. Therefore, $P(0),\ P(1),\ P(-1),\ P(2)$ are all distinct, hence $\Phi(P)=4$. Moreover, Hypothesis (G) is satisfied.

Now, let $f,\ g\in\mathcal{M}(\mathbb{K})$ (resp. let $f,\ g\in\mathcal{M}_u(d(0,R^-))$) and let $\alpha\in\mathcal{M}_f(\mathbb{K})\cap\mathcal{M}_g(\mathbb{K})$ (resp. let $\alpha\in\mathcal{M}_f(d(0,R^-))\cap\mathcal{M}_g(d(0,R^-))$) be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f=g$.

Remark: In that example, we have $k_1=10,\ k=13$. Applying our previous work, a conclusion would have required $k_1\geq k+2=15$ if $f,\ g$ belong to $\mathcal{M}(\mathbb{K})$ and $k_1\geq k+3=16$ if $f,\ g$ belong to $\mathcal{M}_u(d(0,R^-))$.

As noticed in [4], if $f,\ g$ belong to $\mathcal{M}(\mathbb{K})$ and if α is a constant or a Moebius function, we can get a more accurate statement:

Theorem B3: *Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, let $P'=b\prod_{i=1}^{l}(x-a_i)^{k_i}$ with $b\in\mathbb{K}^*$, $l\geq 2$, $k_i\geq k_{i+1}$, $2\leq i\leq l-1$, let $k=\sum_{i=2}^{l}k_i$. For each $m\in\mathbb{N},\ m\geq 5$, let u_m be the biggest of the i such that $k_i>4$, let $s_5=\max(0,u_5-3)$ and for each $m\in\mathbb{N},\ m\geq 6$, let $s_m=\max(0,u_m-2)$.*

Suppose P satisfies the following conditions:

$$k_1\geq 9+\max(0,5-k_2)+\sum_{i=3}^{l}\max(0,4-k_i)-\min(\sum_{m=5}^{\infty}s_m,2l-1)$$

either $k_1\geq k+2$ or P satisfies (G)

if $l=2$, then $k_1\neq k+1,\ 2k,\ 2k+1,\ 3k+1$,

if $l=3$, then $k_1\neq\frac{k}{2},\ k_1\neq k+1,\ 2k+1,\ 3k_i-k\ \forall i=2,3$.

Let $f,\ g\in\mathcal{M}(\mathbb{K})$ be transcendental and let α be a Moebius function. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f=g$.

By Theorem BU1, we have Corollary B3.1.

Corollary B3.1 *Let $P\in\mathbb{K}[x]$ satisfy $\Phi(P)\geq 3$, let $P'=b\prod_{i=1}^{l}(x-a_i)^{k_i}$ with $b\in\mathbb{K}^*$, $l\geq 3$, $k_i\geq k_{i+1}$, $2\leq i\leq l-1$, let $k=\sum_{i=2}^{l}k_i$. For each $m\in\mathbb{N},\ m\geq 5$, let u_m be the biggest of the i such that $k_i>4$, let $s_5=\max(0,u_5-3)$ and for each $m\in\mathbb{N},\ m\geq 6$, let $s_m=\max(0,u_m-2)$.*

Suppose P satisfies the following conditions:

$$k_1\geq 9+\max(0,5-k_2)+\sum_{i=3}^{l}\max(0,4-k_i)-\min(\sum_{m=5}^{\infty}s_m,2l-1)$$

either $k_1\geq k+2$ or P satisfies (G),

if $l=3$, then $k_1\neq\frac{k}{2},\ k_1\neq k+1,\ 2k+1,\ 3k_i-k\ \forall i=2,3$.

Let $f,g\in\mathcal{M}(\mathbb{K})$ be transcendental and let α be a Moebius function. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f=g$.

And by Theorem BU2 we have Corollary B3.2.

Corollary B3.2 *Let $P\in\mathbb{K}[x]$ be such that P' is of the form $b(x-a_1)^n(x-a_2)^k$ with $k\leq n$, $\min(k,n)\geq 2$ and with $b\in\mathbb{K}^*$. Suppose P satisfies the following conditions:*

$n\geq 9+\max(0,5-k)$,

either $n\geq k+2$ or P satisfies (G),

$n \neq k+1, 2k,\ 2k+1,\ 3k+1,$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a Moebius function. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Theorem B4: *Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, let $P' = b\prod_{i=1}^{l}(x-a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 2$, $k_i \geq k_{i+1},\ 2 \leq i \leq l-1$, let $k = \sum_{i=2}^{l} k_i$, and for each $m \in \mathbb{N},\ m \geq 5$, let u_5 be the biggest of the i such that $k_i > 4$, let $s_5 = \max(0, u_5 - 3)$ and for every $m \geq 6$ let $s_m = \max(0, u_m - 2)$. Suppose P satisfies the following conditions:*

either $k_1 \geq k+2$ or P satisfies (G)

$k_1 \geq 9 + \max(0, 5-k_2) + \sum_{i=3}^{l}\max(0, 4-k_i) - \min(\sum_{m=5}^{\infty} s_m, 2l),$

$k_1 \neq k+1.$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Theorem BU1, we have Corollary B4.1

Corollary B4.1 *Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geq 3$, let $P' = b\prod_{i=1}^{l}(x-a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 3$, $k_i \geq k_{i+1},\ 2 \leq i \leq l-1$, let $k = \sum_{i=2}^{l} k_i$.*

For each $m \in \mathbb{N},\ m \geq 5$, let u_m be the biggest of the i such that $k_i > 4$, let $s_5 = \max(0, u_5 - 3)$ and for every $m \geq 6$ let $s_m = \max(0, u_m - 2)$. Suppose P satisfies the following conditions:

$k_1 \geq k+2$ or P satisfies Hypothesis (G),

$k_1 \geq 9 + \max(0, 5-k_2) + \sum_{i=3}^{l}\max(0, 4-k_i) - \min(\sum_{m=5}^{\infty} s_m, 2l),$

$k_1 \neq k+1.$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

And by Theorem BU2, we have Corollary B4.2

Corollary B4.2 *Let $P \in \mathbb{K}[x]$ be such that P' is of the form $b(x-a_1)^n(x-a_2)^k$ with $\min(k, n) \geq 2$ and with $b \in \mathbb{K}^*$. Suppose P satisfies the following conditions:*

$k_1 \geq 9 + \max(0, 5-k),$

either $n \geq k+2$ or P satisfies (G),

$k_1 \neq k+1.$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Example: Let

$$P(X) = \frac{X^{15}}{15} + \frac{5X^{14}}{14} + \frac{10X^{13}}{13} + \frac{10X^{12}}{12} + \frac{5X^{11}}{11} + \frac{X^{10}}{10}.$$

Then $P'(X) = X^9(X+1)^5$. We can apply Corollary B4.2: given $f,\ g \in \mathcal{A}(\mathbb{K})$ transcendental such that $f'P'(f)$ and $g'P'(g)$ share a constant $\alpha \in \mathcal{M}(\mathbb{K})$ C.M., we have $f = g$.

Theorem B5: *Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ such that P' is of the form $b(x-a_1)^n\prod_{i=2}^{l}(x-a_i)$ with $l\geq 3$, $b\in\mathbb{K}^*$, satisfying:*
$n\geq l+10$,
if $l=3$, then $n\neq 2l-1$.
Let $f,\ g\in\mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha\in\mathcal{M}_f(\mathbb{K})\cap\mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f=g$.

By Theorem BU1, we have Corollary B5.1:

Corollary B5.1 *Let $P\in\mathbb{K}[x]$ satisfy $\Phi(P)\geq 3$ and be such that P' is of the form $b(x-a_1)^n\prod_{i=2}^{l}(x-a_i)$ with $l\geq 3$, $b\in\mathbb{K}^*$ satisfying:*
$n\geq l+10$,
if $l=3$, then $n\neq 2l-1$.
Let $f,g\in\mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha\in\mathcal{M}_f(\mathbb{K})\cap\mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f=g$.

Theorem B6: *Let $a\in\mathbb{K}$ and $R>0$. Let P be a polynomial of uniqueness for $\mathcal{M}_u(d(0,R^-))$ such that P' is of the form $P'=b(x-a_1)^n\prod_{i=2}^{l}(x-a_i)$ with $l\geq 3$, $b\in\mathbb{K}^*$ satisfying:*
$n\geq l+10$,
if $l=3$, then $n\neq 2l-1$.
Let $f,g\in\mathcal{M}_u(d(0,R^-))$ and let $\alpha\in\mathcal{M}_f(d(0,R^-))\cap\mathcal{M}_g(d(0,R^-))$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f=g$.

By Theorem BU1, we have Corollary B6.1:

Corollary B6.1 *Let $a\in\mathbb{K}$ and $R>0$. Let $P\in\mathbb{K}[x]$ satisfy $\Phi(P)\geq 4$ and be such that P' is of the form $P'=b(x-a_1)^n\prod_{i=2}^{l}(x-a_i)$ with $l\geq 4$, $b\in\mathbb{K}^*$ and $n\geq l+10$.*

Let $f,g\in\mathcal{M}_u(d(0,R^-))$ and let $\alpha\in\mathcal{M}_f(d(0,R^-))\cap\mathcal{M}_g(d(0,R^-))$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f=g$.

Example: Let $P(x)=\frac{x^{18}}{18}-\frac{2x^{17}}{17}-\frac{x^{16}}{16}+\frac{2x^{15}}{15}$. Then $P'(x)=x^{17}-2x^{16}-x^{15}+2x^{14}=x^{14}(x-1)(x+1)(x-2)$. We check that:
$P(0)=0$,
$P(1)=\frac{1}{18}-\frac{2}{17}-\frac{1}{16}+\frac{2}{15}$,
$P(-1)=\frac{1}{18}+\frac{2}{17}-\frac{1}{16}-\frac{2}{15}\neq 0,P(1)$, and $P(2)=\frac{2^{18}}{18}-\frac{2^{18}}{17}-\frac{2^{16}}{16}+\frac{2^{16}}{15}\neq 0,P(1),P(-1)$. Then $\Upsilon(P)=4$. So, P is a polynomial of uniqueness for both $\mathcal{M}(\mathbb{K})$ and $\mathcal{M}(d(0,R^-))$.

Given $f,\ g\in\mathcal{M}(\mathbb{K})$ transcendental or $f,\ g\in\mathcal{M}_u(d(0,R^-))$ such that $f'P'(f)$ and $g'P'(g)$ share C.M. a small function α, we have $f=g$.

Theorem B7: *Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ such that P' is of the form $P'=b(x-a_1)^n\prod_{i=2}^{l}(x-a_i)$ with $l\geq 3$, $b\in\mathbb{K}^*$ satisfying*

$n \geq l+9$,

if $l=3$, then $n \neq 2l-1$.

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a Moebius function or a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f=g$.

Example: Let $P(x)=x^q-ax^{q-2}+b$ with $a \in \mathbb{K}^*$, $b \in \mathbb{K}$, with $q \geq 5$ an odd integer. Then q and $q-2$ are relatively prime and hence by Theorem 3.21 in [11] P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ and P' admits 0 as a zero of order $n=q-3$ and two other zeros of order 1.

Let $f,\ g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}(\mathbb{K})$ be a small function such that $f,\ g$ share α C.M.

Suppose first $q \geq 17$. By Theorem B6 we have $f=g$. Now suppose $q \geq 15$ and suppose α is a Moebius function or a non-zero constant. Then by Theorems B7 we have $f=g$.

Theorem B8: *Let $f,\ g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. Let $a \in \mathbb{K} \setminus \{0\}$. If $f'f^n(f-a)$ and $g'g^n(g-a)$ share the function α C.M. and if $n \geq 12$, then either $f=g$ or there exists $h \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ such that $f=\dfrac{a(n+2)}{n+1}\Big(\dfrac{h^{n+1}-1}{h^{n+2}-1}\Big)h$ and $g=\dfrac{a(n+2)}{n+1}\Big(\dfrac{h^{n+1}-1}{h^{n+2}-1}\Big)$. Moreover, if α is a constant or a Moebius function, then the conclusion holds whenever $n \geq 11$.*

Inside an open disk, we have a version similar to the general case in the whole field.

Theorem B9: *Let $f,\ g \in \mathcal{M}_u(d(0,R^-))$, and let $\alpha \in \mathcal{M}_f(d(0,R^-)) \cap \mathcal{M}_g(d(0,R^-))$ be non-identically zero. Let $a \in \mathbb{K} \setminus \{0\}$. If $f'f^n(f-a)$ and $g'g^n(g-a)$ share the function α C.M. and $n \geq 12$, then either $f=g$ or there exists $h \in \mathcal{M}_u(d(0,R^-))$ such that $f=\dfrac{a(n+2)}{n+1}\Big(\dfrac{h^{n+1}-1}{h^{n+2}-1}\Big)h$ and $g=\dfrac{a(n+2)}{n+1}\Big(\dfrac{h^{n+1}-1}{h^{n+2}-1}\Big)$.*

Remark: In Theorems B8 and B9, the second conclusion does occur. Indeed, let $h \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ (resp. let $h \in \mathcal{M}_u(d(0,R^-))$). Now, let us precisely define f and g as: $g=(\dfrac{n+2}{n+1})\Big(\dfrac{h^{n+1)}-1}{h^{n+2}-1}\Big)$ and $f=hg$. Then, both $f,\ g$ are transcendental (resp. both $f,\ g$ belong to $\mathcal{M}_u(d(0,R^-))$) and then we can check that the polynomial $P(y)=\dfrac{1}{n+2}y^{n+2}-\dfrac{1}{n+1}y^{n+1}$ satisfies $P(f)=P(g)$, hence $f'P'(f)=g'P'(g)$, therefore $f'P'(f)$ and $g'P'(g)$ trivially share any function.

5. PROOFS OF PART B:

Notation: As usual, given a function $f \in \mathcal{M}(\mathbb{K})$ (resp. $\mathcal{M}(d(0,R^-))$), we denote by $S_f(r)$ a function of r defined in $]0,+\infty[$ (resp. in $]0,R[$) such that $\lim\limits_{r\to+\infty}\dfrac{S_f(r)}{T(r,f)}=0$ (resp. $\lim\limits_{r\to R}\dfrac{S_f(r)}{T(r,f)}=0$)

In the proof of Theorems B2, B3, B4 we will need the following Lemmas [11]:

Lemma BL1: *Let $Q \in \mathbb{K}[x]$ be of degree n and let $f \in \mathcal{M}(\mathbb{K})$, (resp. $f \in \mathcal{M}(d(0,R^-))$) be transcendental. Then $N(r,f')=N(r,f)+\overline{N}(r,f)$, $Z(r,f') \leq Z(r,f)+\overline{N}(r,f)+O(1)$, $nT(r,f) \leq T(r,f'Q(f)) \leq (n+2)T(r,f)-\log r+O(1)$ (resp. $nT(r,f) \leq T(r,f'Q(f)) \leq (n+2)T(r,f)+O(1)$). Particularly, if $f \in \mathcal{A}(\mathbb{K})$, (resp. $f \in \mathcal{A}(d(0,R^-))$), then $nT(r,f) \leq T(r,f'Q(f)) \leq (n+1)T(r,f)-\log r+O(1)$ (resp. $nT(r,f) \leq T(r,f'Q(f)) \leq (n+1)T(r,f)+O(1)$).*

Let $P \in \mathcal{M}_b(d(0,R^-))[X]$ be of degree n and let $f \in \mathcal{M}_u(d(0,R^-))$. Then $T(r,P(f))=nT(r,f)+O(1)$.

Lemma BL2 : *Let $f \in \mathcal{M}(d(0,R^-))$. Then, $Z(r,f') - N(f',r) \leq Z(r,f) - N(r,f) - \log r + O(1)$. Moreover, $T(r,f) - Z(r,f) \leq T(r,f') - Z(r,f') + O(1)$. Further, given $\alpha \in \mathcal{M}(d(0,R^-))$, we have $T(r,\alpha f) - Z(r,\alpha f) \leq T(r,f) - Z(r,f) + T(r,\alpha)$.*

The following lemma is given in [4], for p-adic meromorphic functions. The same applies for complex meromorphic functions [5].

Lemma BL3: *Let $Q(x) = (x-a_1)^n \prod_{i=2}^{l}(x-a_i)^{k_i} \in \mathbb{K}[x]$ $(a_i \neq a_j,\ \forall i \neq j)$ with $l \geq 2$ and $n \geq \max\{k_2, .., k_l\}$ and let $k = \sum_{i=2}^{l} k_i$. Let $f,\ g \in \mathcal{M}(\mathbb{K})$ be transcendental (resp. $f,\ g \in \mathcal{M}_u(d(0,R^-))$) such that $\theta = Q(f)f'Q(g)g'$ is a small function with respect to f and g. We have the following :*

If $l = 2$ then n belongs to $\{k,\ k+1,\ 2k,\ 2k+1,\ 3k+1\}$.

If $l = 3$ then n belongs to $\{\frac{k}{2}, k+1, 2k+1, 3k_2 - k, .., 3k_l - k\}$.

If $l \geq 4$ then $n = k+1$.

If θ is a constant and $f,\ g \in \mathcal{M}(\mathbb{K})$ then $n = k+1$.

Lemma BL4: *Let $P \in \mathbb{K}[x] \setminus \mathbb{K}$ with $\deg(P) > 1$ and let $f,\ g \in \mathcal{A}(\mathbb{K}) \setminus \mathbb{K}$ (resp. $f,\ g \in \mathcal{A}_u(d(a,R^-))$) be such that $P(f) = P(g) + c,\ c \in \mathbb{K}$ (resp. $P(f) = P(g) + h,\ h \in \mathcal{M}_b(d(a,R^-))$). Then $c = 0$ (resp. $h = 0$).*

Proof: Let $P(x) = \sum_{k=0}^{n} a_k x^k$ with $a_n \neq 0$. For each $k = 1, ..., n-1$, let $Q_k(x,y) = a_k \sum_{j=0}^{k} x^j y^{k-j}$. Then $P(x) - P(y) = (x-y)(\sum_{k=1}^{n-1} Q_k(x,y))$. Suppose first $f,\ g \in \mathcal{A}(\mathbb{K})$ and suppose $c \neq 0$. Since $(f-g)(\sum_{k=1}^{n-1} Q_k(f,g))$ is a constant, both $f-g$ and $\sum_{k=1}^{n-1} Q_k(f,g)$ are constants different from 0 because the semi-norm $|\ .\ |(r)$ is multiplicative on $\mathcal{A}(\mathbb{K})$ (resp. on $\mathcal{A}_u(d(0,R^-))$) and is an increasing function in r. Thus we have $g = f + b$ with $b \in \mathbb{K}$. Let $G(x) = \sum_{k=1}^{n-1} Q_k(x, x+b))$. Since $\mathbb{K}$ has characteristic 0, we can check that G is a polynomial of degree $n-1$. And since $G(f)$ is a constant, we have $n - 1 = 0$, a contradiction. Consequently, $c = 0$.

Similarly, suppose now $f,\ g \in \mathcal{A}_u(d(a,R^-))$. Since $P(f) - P(g)$ belongs to $\mathcal{A}_b(d(a,R^-))$, both $f - g$ and $\sum_{k=1}^{n-1} Q_k(f,g)$ are bounded and not identically 0, so we have $g = f + h$, with $h \in \mathcal{A}_b(d(a,R^-))$. Suppose that h is not identically zero. Consider the polynomial
$B(x) = \sum_{k=1}^{n-1} Q_k(x, x+h) \in \mathcal{M}_b(d(a,R^-))[x]$. Clearly, $B(x)$ is a polynomial with coefficients in $\mathcal{M}_b(d(a,R^-))$ and $\deg(B))$ is $n-1$, hence we have $T(r,B(f)) = (n-1)T(r,f) + o(T(r,f))$. But since $B(f)$ is bounded, it belongs to $\mathcal{M}_b(d(a,R^-))[x]$, hence $T(r,B(f))$ is bounded and so is $(n-1)T(r,f)$, which leads to $n = 1$, a contradiction again.

Proof of Theorem B1. Put $F = f'b\prod_{j=1}^{l}(f-a_j)^{k_j}$ and $G = g'b\prod_{j=1}^{l}(g-a_j)^{k_j}$. Since $f,\ g \in \mathcal{A}(\mathbb{K})$ (resp. $f, g \in \mathcal{A}_u(d(0,R^-))$) and since F and G share α C.M., then $\dfrac{F-\alpha}{G-\alpha}$ is a meromorphic function having no zero and no pole in $\mathbb{K}$ (resp. in $d(0,R^-)$), hence it is a constant w in $\mathbb{K} \setminus \{0\}$ (resp. it is an invertible function $w \in \mathcal{A}_b(d(0,R^-))$).

Suppose $w \neq 1$. Then, $F = wG + \alpha(1-w)$.

Let $r > 0$. Since $\alpha(1-w) \in \mathcal{A}_f(\mathbb{K})$ (resp. $\alpha(1-w) \in \mathcal{A}_f(d(0,R^-))$), $\alpha(1-w)$ obviously belongs to $\mathcal{A}_F(\mathbb{K})$ (resp. to $\mathcal{A}_F(d(0,R^-))$). So, applying Theorem N1 to F, we obtain

$$T(r,F) \leq \overline{Z}(r,F) + \overline{Z}(r, F - \alpha(1-w)) + S_F(r) = \overline{Z}(r,F) + \overline{Z}(G) + S_F(r)$$

$$= \sum_{j=1}^{l} \overline{Z}(r,(f-a_j)^k) + \overline{Z}(r,f') + \sum_{j=1}^{l} \overline{Z}(r,(g-a_j)^k) + \overline{Z}(r,g') + S_f(r)$$

$$\leq l(T(r,f)+T(r,g)) + Z(r,f') + Z(r,g') + S_f(r).$$

We also notice that if $f,\ g \in \mathcal{A}(\mathbb{K})$ and if $\alpha \in \mathbb{K}\setminus\{0\}$, we have $T(r,F) \leq \overline{Z}(r,F) + \overline{Z}(r,F-\alpha(1-w)) - \log r + O(1)$ and therefore we obtain

$$T(r,F) \leq l(T(r,f)+T(r,g)) + Z(r,f') + Z(r,g') - \log r + O(1).$$

Now, let us go back to the general case. Since f is entire (resp. f belongs to $\mathcal{M}_u(d(0,R^-))$), by Lemma BL1 we have $T(r,F) = (\sum_{j=1}^{l} k_j)T(r,f) + Z(r,f') + O(1)$. Consequently, $(\sum_{j=1}^{l} k_j)T(r,f) \leq l(T(r,f)+T(r,g)) + Z(r,g') + S_f(r)$.

Similarly, $(\sum_{j=1}^{l} k_j)T(r,g) \leq l(T(r,f)+T(r,g)) + Z(r,f') + S_f(r)$. Therefore

$$(\sum_{j=1}^{l} k_j)(T(r,f)+T(r,g)) \leq 2l(T(r,f)+T(r,g)) + Z(r,f') + Z(r,g') + S_f(r)$$

$$\leq (2l+1)(T(r,f)+T(r,g)) + S_f(r).$$

So, $\sum_{j=1}^{l} k_j \leq 2l+1$. Thus, since $\sum_{j=1}^{l} k_j > 2l+1$ we have $w=1$.

And if $\alpha \in \mathbb{K}\setminus\{0\}$ and if $f,\ g$ belong to $\mathcal{A}(\mathbb{K})$, by applying Theorem N1 we obtain

$$\sum_{j=1}^{l} k_j(T(r,f)+T(r,g)) \leq 2l(T(r,f)+T(r,g)) + Z(r,f') + Z(r,g') - 2\log r + O(1)$$

$$\leq (2l+1)(T(r,f)+T(r,g)) - 4\log r + O(1)$$

because $T(r,f') \leq T(r,f) - \log r + O(1)$, hence $\sum_{j=1}^{l} k_j \leq 2l$ which also contradicts the hypothesis $w \neq 1$ whenever $\sum_{j=1}^{l} k_j > 2l$.

Consequently, in the general case, whenever $\sum_{j=1}^{l} k_j > 2l+1$, we have $w=1$ and therefore $f'P'(f) = g'P'(g)$ hence $P(f)-P(g)$ is a constant c. And by Lemma BL4 we have $c=0$. But since P is a polynomial of uniqueness for $\mathcal{A}(\mathbb{K})$ (resp. for $\mathcal{A}(d(0,R^-))$), that yields $f=g$.

And similarly, if $\alpha \in \mathbb{K}$ and $f,\ g \in \mathcal{A}(\mathbb{K})$, whenever $\sum_{j=1}^{l} k_j > 2l$, we have $w=1$ and therefore we can conclude in the same way.

From results of [3] we can extract this:

Theorem BF: *Let $P,\ Q \in \mathbb{K}[x]$ of respective degree m and n with $m \leq n$ and P monic and let $P'(x) = m\prod_{i=1}^{h}(x-a_i)^{k_I},\ Q'(x) = nb\prod_{i=1}^{l}(x-b_i)^{q_I}$, where $a_1,\dots,a_h$ are distinct and $b_1,\dots,b_l$ are distinct.*

Let $H = \{i \quad 1 \le i \le h,\ P(a_i) \neq Q(b_j)\ \forall j = 1, ..., l\}$ *and let* $L = \{j \quad 1 \le j \le l,\ Q(b_j) \neq P(a_i)$ $\forall i = 1, ..., h\}$.

Suppose that one of the following two statement holds:

$\sum_{a_i \in H} k_i \ge n - m + 2$ *(resp.* $\sum_{a_i \in H} k_i \ge n - m + 3$*),*

$\sum_{b_j \in L} q_j \ge 2$ *(resp.* $\sum_{b_i \in L} q_j \ge 3$*).*

If two meromorphic functions $f,\ g \in \mathcal{M}(\mathbb{K})$ *(resp.* $f,\ g \in \mathcal{M}(d(0, R^-))$*)) satisfy* $P(f(x)) = Q(g(x)),\ \forall x \in \mathbb{K}$*, (resp.* $\forall x \in d(0, R^-)$*) then both* f *and* g *are constant (resp. belong to* $\mathcal{M}_b(d(0, R^-))$*)).*

Proof of Proposition BP: Let $n = \deg(P)$. Suppose that two functions $f,\ g \in \mathcal{M}(\mathbb{K})$ (resp. $f,\ g \in \mathcal{M}(d(0, R^-))$) satisfy $P(f(x)) = P(g(x)) + C$ $(C \in \mathbb{K}^*)$, $\forall x \in \mathbb{K}$ (resp. $\forall x \in d(0, R^-)$). We can apply Theorem BF by putting $Q(X) = P(X) + C$ and next keeping the same notations. So, here we have $h = l,\ m = n$ and $b_i = a_i,\ i = 1, ..., l$. Let Γ be the curve of equation $P(X) - P(Y) = C$. By hypothesis we have $n \ge 3$, so Γ is of degree ≥ 3. Therefore, if Γ has no singular point, it is of genus ≥ 1 and hence, by Picard-Berkovich Theorem, the conclusion is immediate. Consequently, we can assume that Γ has a singular point (α, β). But then $P'(\alpha) = P'(\beta) = 0$ and hence (α, β) is of the form (a_h, a_k). Consequently, $C = P(a_h) - P(a_k)$ and since $C \neq 0$, we have $h \neq k$. We will prove that either $a_1 \in H$, or $a_1 \in L$.

Suppose first that $a_1 \notin H \cup L$. Since $a_1 \notin H$, there exists $i \in \{2, ..., l\}$ such that $P(a_1) = P(a_i) + C$. Now since $1 \notin L$, there exists $j \in \{2, ..., l\}$ such that $P(a_1) + C = P(a_i)$. But since $C = -P(a_i)$, we have $P(a_j) = -P(a_i)$, therefore $P(a_i) + P(a_j) = 0$. Since P satisfies (G), we have $i = j$, hence $P(a_i) = 0$. But then $C = 0$, a contradiction. Therefore, we have proven that $a_1 \in F' \cup F''$. Now, by Theorem BF, f and g are constant (resp. f and g belong to $\mathcal{M}_b(d(0, R^-))$).

The following basic lemma applies to both complex and meromorphic functions. A proof is given in [4].

Lemma BL5: *Let* $f \in \mathcal{M}(\mathbb{K})$*, (resp.* $f \in \mathcal{M}(d(0, R^-))$*). Then*

$$T(r, f) - Z(r, f) \le T(r, f') - Z(r, f') + O(1).$$

Notation: Given two meromorphic functions $f,\ g \in \mathcal{M}(K)$ (resp. $f,\ g \in \mathcal{M}(d(0, R^-))$), we will denote by $\Psi_{f,g}$ the function

$$\frac{f''}{f'} - \frac{2f'}{f-1} - \frac{g''}{g'} + \frac{2g'}{g-1}.$$

We denote by $Z_{[2]}(r, f)$ the counting function of zeros of f in $\mathbb{K}$ (resp. in $d(0, R^-)$) where zeros of order > 2 are only counted with multiplicity order 2. Similarly, we denote by $N_{[2]}(r, f)$ the counting function of poles of f in $\mathbb{K}$ (resp. in $d(0, R^-)$) where poles of order > 2 are only counted with multiplicity order 2.

Now, we can extract the following Lemma BL6 from a result that is proven in several papers and particularly in Lemma 11 [4].

Lemma BL6: *Let* $f,\ g \in \mathcal{M}(\mathbb{K})$ *(resp.* $f,\ g \in \mathcal{M}(d(0, R^-))$*) share the value* 1 *CM. If* $\Psi_{f,g}$ *is not identically zero, then,* $\max(T(r, f), T(r, g)) \le N_{[2]}(r, f) + Z_{[2]}(r, f) + N_{[2]}(r, g) + Z_{[2]}(r, g) - 3 \log r$.

We will need the following Lemma BL7:

Lemma BL7: *Let $f,\ g \in \mathcal{M}(\mathbb{K})$ be transcendental (resp. $f,\ g \in \mathcal{M}_u(d(0, R^-))$). Let $P(x) = x^{n+1}Q(x)$ be a polynomial such that $n \geq deg(Q) + 2$ (resp. $n \geq deg(Q) + 3$). If $f'P'(f) = g'P'(g)$ then $P(f) = P(g)$.*

The following lemma holds in the same way in p-adic analysis and in complex analysis. It is proven in [4]:

By Lemma 8 in [4], we have the following Lemma BL8

Lemma BL8: *Let $F, G \in \mathcal{M}(\mathbb{K})$ (resp. Let $F, G \in \mathcal{M}(d(0, R^-))$) be non-constant, having no zero and no pole at 0 and sharing the value 1 C.M.*

If $\Theta_{F,G} = 0$ and if

$$\limsup_{r\to+\infty}\Big(T(r,F) - [\overline{Z}(r,F) + \overline{N}(r,F) + \overline{Z}(r,G) + \overline{N}(r,G)]\Big) = +\infty$$

(resp.

$$\limsup_{r\to R^-}\Big(T(r,F) - [\overline{Z}(r,F) + \overline{N}(r,F) + \overline{Z}(r,G) + \overline{N}(r,G)]\Big) = +\infty)$$

then either $F = G$ or $FG = 1$.

Proofs of Theorems. Theorems B5, B6, B7, B8, B9 were proven in [4]. Consequently, our work only consists of proving Theorem B2, B3 and B4.

For simplicity, now we set $n = k_1$. Set $F = \dfrac{f'P'(f)}{\alpha}$, $G = \dfrac{g'P'(g)}{\alpha}$ and $\widehat{F} = P(f)$, $\widehat{G} = P(g)$. Suppose $F \neq G$. We notice that $P(x)$ is of the form $x^{n+1}Q(x)$ with $Q \in \mathbb{K}[x]$ of degree k. Now, with help of Lemma BL5, we can check that we have Since $(\widehat{F})' = \alpha F$, by Lemma BL2 we have

$$T(r,\widehat{F}) \leq T(r,F) + Z(r,\widehat{F}) - Z(r,F) + T(r,\alpha) + O(1), \tag{1}$$

hence, by (1), we obtain

$$T(r,\widehat{F}) \leq T(r,F) + (n+1)Z(r,f) + Z\big(r,Q(f)\big) - nZ(r,f)$$
$$- \sum_{i=2}^{l} k_i Z(r, f - a_i) - Z(r,f') + T(r,\alpha) + O(1),$$

i.e.

$$T(r,\widehat{F}) \leq T(r,F) + Z(r,f) + Z\big(r,Q(f)\big) - \sum_{i=2}^{l} k_i Z(r,f-a_i) - Z(r,f') + T(r,\alpha) + O(1), \tag{2}$$

and similarly,

$$T(r,\widehat{G}) \leq T(r,G) + Z(r,g) + Z\big(r,Q(g)\big) - \sum_{i=2}^{l} k_i Z(r,g-a_i) - Z(r,g') + T(r,\alpha) + O(1). \tag{3}$$

Now, it follows from the definition of F and G that

$$Z_{[2]}(r,F) + N_{[2]}(r,F) \leq 2Z(r,f) + 2\sum_{i=2}^{l} Z(r,f-a_i) + Z(r,f') + 2\overline{N}(r,f) + T(r,\alpha) + O(1) \tag{4}$$

and similarly

$$Z_{[2]}(r,G)+N_{[2]}(r,G)\leq 2Z(r,g)+2\sum_{i=2}^{l}Z(r,g-a_i)+Z(r,g')+2\overline{N}(r,g)+T(r,\alpha)+O(1). \quad (5)$$

And particularly, if $k_i=1,\ \forall i\in\{2,..,l\}$, then

$$Z_{[2]}(r,F)+N_{[2]}(r,F)\leq 2Z(r,f)+\sum_{i=2}^{l}Z(r,f-a_i)+Z(r,f')+2\overline{N}(r,f)+T(r,\alpha)+O(1) \quad (6)$$

and similarly

$$Z_{[2]}(r,G)+N_{[2]}(r,G)\leq 2Z(r,g)+\sum_{i=2}^{l}Z(r,g-a_i)+Z(r,g')+2\overline{N}(r,g)+T(r,\alpha)+O(1). \quad (7)$$

We will now prove that $\Psi_{F,G}$ is identically zero. Indeed, suppose now that $\Psi_{F,G}$ is not identically zero.

By Lemma BL6, we have

$$T(r,F)\leq Z_{[2]}(r,F)+N_{[2]}(r,F)+Z_{[2]}(r,G)+N_{[2]}(r,G)-3\log r$$

hence by (2), we obtain

$$T(r,\widehat{F})\leq Z_{[2]}(r,F)+N_{[2]}(r,F)+Z_{[2]}(r,G)+N_{[2]}(r,G)+Z(r,f)+Z(r,Q(f))$$

$$-\sum_{i=2}^{l}k_iZ(r,f-a_i)-Z(r,f')+T(r,\alpha)-3\log r+O(1)$$

and hence by (4) and (5):

$$T(r,\widehat{F})\leq 2Z(r,f)+2\sum_{i=2}^{l}Z(r,f-a_i)+Z(r,f')+2\overline{N}(r,f)+2Z(r,g)+2\sum_{i=2}^{l}Z(r,g-a_i)$$

$$+Z(r,g')+2\overline{N}(r,g)+Z(r,f)+Z(r,Q(f))$$

$$-\sum_{i=2}^{l}k_iZ(r,f-a_i)-Z(r,f')+T(r,\alpha)-3\log r+O(1) \quad (8)$$

and similarly,

$$T(r,\widehat{G})\leq 2Z(r,g)+2\sum_{i=2}^{l}Z(r,g-a_i)+Z(r,g')+2\overline{N}(r,g)+2Z(r,f)+2\sum_{i=2}^{l}Z(r,f-a_i)+Z(r,f')$$

$$+2\overline{N}(r,f)+Z(r,g)+Z(r,Q(g))-\sum_{i=2}^{l}k_iZ(r,g-a_i)-Z(r,g')+T(r,\alpha)-3\log r+O(1). \quad (9)$$

Consequently,

$$T(r,\widehat{F})+T(r,\widehat{G})\leq 5(Z(r,f)+Z(r,g))+\sum_{i=2}^{l}(4-k_i)(Z(r,f-a_i)+Z(r,g-a_i))+(Z(r,f')$$

$$+Z(r,g')) + 4(\overline{N}(r,f) + \overline{N}(r,g)) + (Z(r,Q(f)) + Z(r,Q(g))) + 6T(r,\alpha) - 6\log r + O(1). \quad (10)$$

By Lemma BL1 we can write $Z(r,f') + Z(r,g') \le Z(r,f-a_2) + Z(r,g-a_2) + \overline{N}(r,f) + \overline{N}(r,g) - 2\log r$. Hence, in general, by (10) we obtain

$$T(r,\widehat{F}) + T(r,\widehat{G}) \le 5(Z(r,f) + Z(r,g))$$

$$+\sum_{i=3}^{l}(4-k_i)\big((Z(r,f-a_i) + Z(r,g-a_i))\big) + (5-k_2)\big((Z(r,f-a_2) + Z(r,g-a_2))$$

$$+5(\overline{N}(r,f) + \overline{N}(r,g)) + (Z(r,Q(f)) + Z(r,Q(g))) + 6T(r,\alpha) - 8\log r + O(1)$$

and hence, since $T(r,Q(f)) = kT(r,f) + O(1)$ and $T(r,Q(g)) = kT(r,g) + O(1)$,

$$T(r,\widehat{F}) + T(r,\widehat{G}) \le 5(T(r,f) + T(r,g))$$

$$+\sum_{i=3}^{l}(4-k_i)\big((Z(r,f-a_i) + Z(r,g-a_i))\big) + (5-k_2)\big((Z(r,f-a_2) + Z(r,g-a_2))$$

$$+5(\overline{N}(r,f) + \overline{N}(r,g)) + k(T(r,f) + T(r,g)) + 6T(r,\alpha) - 8\log r + O(1). \quad (12)$$

Now, since $\widehat{F}$ is a polynomial in f of degree $n+k+1$, we have $T(r,\widehat{F}) = (n+k+1)T(r,f) + O(1)$ and similarly, $T(r,\widehat{G}) = (n+k+1)T(r,g) + O(1)$, hence by (12) we can derive

$$(n+k+1)(T(r,f) + T(r,g)) \le 5(T(r,f) + T(r,g))$$

$$+(5-k_2)(Z(r,f-a_2) + Z(r,g-a_2)) + \sum_{i=3}^{l}(4-k_i)\big((Z(r,f-a_i) + Z(r,g-a_i))\big)$$

$$+5(\overline{N}(r,f) + \overline{N}(r,g)) + k(T(r,f) + T(r,g)) + 6T(r,\alpha) - 8\log r + O(1). \quad (15)$$

Hence

$$(n+k+1)(T(r,f) + T(r,g)) \le 10(T(r,f) + T(r,g))$$

$$+\sum_{i=3}^{l}(4-k_i)\big((Z(r,f-a_i) + Z(r,g-a_i))\big) + (5-k_2)\big((Z(r,f-a_2) + Z(r,g-a_2))$$

$$+k(T(r,f) + T(r,g)) + 6T(r,\alpha) - 8\log r + O(1)),$$

and hence

$$n(Tr,f) + T(r,g)) \le 9(T(r,f) + T(r,g)) + (5-k_2)\big((Z(r,f-a_2) + Z(r,g-a_2))$$

$$+\sum_{i=3}^{l}(4-k_i)\big((Z(r,f-a_i) + Z(r,g-a_i))\big) + 6T(r,\alpha) - 8\log r + O(1)). \quad (16)$$

Then $(5-k_2)(Z(r,f-a_2) + Z(r,g-a_2)) \le \max(0,5-k_2)(T(r,f) + T(r,g)) + O(1)$ and at least, for each $i = 3,..,l$ we have $(4-k_i)(Z(r,f-a_i) + Z(r,g-a_i)) \le \max(0,4-k_i)(T(r,f) + T(r,g)) + O(1)$.

Now suppose $s_5 > 0$. That means that $k_i \geq 5\ \forall i = 3, ..., u_5$ with $l \geq 5$. We notice that the number of indices i superior or equal to 2 such that $k_i \geq 5$ is $u_5 - 2$. Similarly, for each $m > 5$, the number of indices superior or equal to 1 such that $k_i \geq m$ is $u_m - 1$.

Then we can apply Theorem A1 and we obtain
$\sum_{i=3}^{u_5} Z(r, f - a_i) \geq (u_5 - 3)T(r, f) + O(1)$
and for each $m \geq 6$,
$\sum_{i=3}^{u_m} Z(r, f - a_i) \geq (u_m - 2)T(r, f) + O(1)$, i.e.

$\sum_{i=3}^{u_5} Z(r, f - a_i) \geq s_5 T(r, f) + O(1)$
and for each $m \geq 6$,
$\sum_{i=3}^{u_m} Z(r, f - a_i) \geq s_m T(r, f) + O(1)$,
and similarly for g.

Consequently, by (16) we obtain

$$n(Tr, f) + T(r, g)) \leq 9(T(r, f) + T(r, g)) + \max(0, 5 - k_2)(Z(r, f - a_2) + Z(r, g - a_2))$$

$$+ \sum_{i=3}^{l} \max(0, 4 - k_i)\big(Z(r, f - a_i) + Z(r, g - a_i)\big)$$

$$- \sum_{m=5}^{\infty} s_m(T(r, f) + T(r, g)) + 4T(r, \alpha) - 8 \log r + O(1)), \tag{17}$$

therefore

$$n \leq 9 + \max(5 - k_2) + \sum_{i=3}^{l} \max(0, 4 - k_i) - \sum_{j=5}^{\infty} s_j, \tag{18}$$

a contradiction to the hypotheses of Theorem B2.

Consider now the situation in Theorems B3 and B4. Here we have $T(r, \alpha) \leq \log r + O(1)$. Consequently, Relation (16) now implies

$$n(Tr, f) + T(r, g)) \leq 9(T(r, f) + T(r, g)) + \max(0, 5 - k_2)(Z(r, f - a_2) + Z(r, g - a_2))$$

$$+ \sum_{i=3}^{l} \max(0, 4 - k_i)\big(Z(r, f - a_i) + Z(r, g - a_i)\big) - \sum_{m=5}^{\infty} s_m(T(r, f) + T(r, g)) - 2 \log r + O(1)),$$

therefore

$$n < 9 + \max(0, 5 - k_2) + \sum_{i=3}^{l} \max(0, 4 - k_i) - \sum_{m=5}^{\infty} s_m,$$

but this is uncompatible with the hypotheses

$n \geq 9 + \max(5 - k_2) + \sum_{i=3}^{l} \max(0, 4 - k_i) - \min(\sum_{j=5}^{\infty} s_j, 2l - 1)$ in Theorem B3 and

$n \geq 9 + \max(5 - k_2) + \sum_{i=3}^{l} \max(0, 4 - k_i) - \min(\sum_{j=5}^{\infty} s_j, 2l)$ in Theorem B4.

Thus, in the hypotheses of Theorems B2, B3 and B4 we have proven that $\Psi_{F,G}$ is identically zero. Henceforth, we can assume that $\Psi_{F,G} = 0$ in all theorems. Note that we can write $\Psi_{F,G} = \dfrac{\phi'}{\phi}$

with $\phi = \Big(\dfrac{F'}{(F-1)^2}\Big)\Big(\dfrac{(G-1)^2}{G'}\Big)$. Since $\Psi_{F,G} = 0$, there exist $A, B \in \mathbb{K}$ such that

$$\frac{1}{G-1} = \frac{A}{F-1} + B \tag{19}$$

and $A \neq 0$. We notice that
$\overline{Z}(r,f) \leq T(r,f)$, $\overline{N}(r,f) \leq T(r,f)$, $\overline{Z}(r,f-a_i) \leq T(r,f-a_i) \leq T(r,f) + O(1)$, $i = 2,...,l$ and $\overline{Z}(r,f') \leq T(r,f') \leq 2T(r,f) + O(1)$. Similarly for g and g'. Moreover, by Lemma BL1 we have

$$T(r,F) \geq (n+k)T(r,f). \tag{20}$$

We will show that $F = G$ in each theorem. We first notice that hypotheses of Theorems B2 and B3 imply

$$n + k \geq 2l + 7, \tag{21}$$

and that in Theorem B4 we have

$$n + k \geq 2l + 6. \tag{22}$$

Indeed, set $t = \sum_{i=5}^{\infty} s_m$, $s = \min(t, 2l)$ and $s' = \min(t, 2l-1)$. In theorem B2 we have

$$n + k \geq 10 + k + \max(0, 5-k_2) + \sum_{i=3}^{\infty} \max(0, 4-k_i) - s$$

$$= 10 + [k_2 + \max(0, 5-k_2)] + \sum_{i=3}^{\infty}[k_i + \max(0, 4-k_i)] - s$$

$$= 10 + \max(k_2, 5) + \sum_{i=3}^{\infty}[\max(k_i, 4)] - s \geq 10 + 5 + 4(l-2) - 2l = 2l + 7.$$

And in Theorem B3 we have

$$n + k \geq 9 + k + \max(0, 5-k_2) + \sum_{i=3}^{\infty} \max(0, 4-k_i) - s'$$

$$= 9 + [k_2 + \max(0, 5-k_2)] + \sum_{i=3}^{\infty}[k_i + \max(0, 4-k_i)] - s'$$

$$= 9 + \max(k_2, 5) + \sum_{i=3}^{\infty}[\max(k_i, 4)] - s' \geq 9 + 5 + 4(l-2) - 2l = 2l + 7.$$

That finishes proving (21) in Theorems B2 and B3.

Now, in Theorem B4 we have

$$n + k \geq 9 + k + \max(0, 5-k_2) + \sum_{i=3}^{\infty} \max(0, 4-k_i) - s'$$

$$= 9 + [k_2 + \max(0, 5-k_2)] + \sum_{i=3}^{\infty}[k_i + \max(0, 4-k_i)] - s'$$

$$= 9 + \max(k_2, 5) + \sum_{i=3}^{\infty}[\max(k_i, 4)] - s \geq 9 + 5 + 4(l-2) - 2l = 2l + 6.$$

We will consider the following two cases: $B = 0$ and $B \neq 0$.

Case 1: $B = 0$.

Suppose $A \neq 1$. Then, by (19), we have $F = \frac{1}{A}G + \Big(1 - \frac{1}{A}\Big)$. Applying Theorem N1 to F, we obtain

$$T(r,F) \leq \overline{Z}(r,F) + \overline{Z}\Big(r, F - \Big(1 - \frac{1}{A}\Big)\Big) + \overline{N}(r,F) - \log r + O(1) \leq \overline{Z}(r,f) + \sum_{i=2}^{l} \overline{Z}(r, f - a_i)$$

$$+\overline{Z}(r,f') + \overline{Z}(r,g) + \sum_{i=2}^{l} \overline{Z}(r, g - a_i) + \overline{Z}(r,g') + \overline{N}(r,f) + 3T(r,\alpha) - \log r + O(1).$$

But $\overline{Z}(r,f) \leq T(r,f)$, $\overline{N}(r,f) \leq T(r,f)$, $\overline{Z}(r,f-1) \leq T(r,f-1) \leq T(r,f) + O(1)$ and $\overline{Z}(r,f') \leq T(r,f') \leq 2T(r,f) + O(1)$. Moreover, by Lemma BL1, we have
$T(r,F) \geq (n+k)T(r,f) - T(r,\alpha)$. Then, considering all the previous inequalities in (12), we can deduce that

$$(n+k)T(r,f) \leq (l+3)T(r,f) + (l+2)T(r,g) + 4T(r,\alpha) - \log r + O(1). \tag{23}$$

And similarly,

$$(n+k)T(r,g) \leq (l+3)T(r,g) + (l+2)T(r,f) + 4T(r,\alpha) - \log r + O(1). \tag{24}$$

Hence, adding (23) and (24), we have

$$(n+k)\big[T(r,f) + T(r,g)\big] \leq (2l+5)\big[T(r,f) + T(r,g)\big] + 4T(r,\alpha) - 2\log r + O(1), \tag{25}$$

which shows that $n + k|eq 2l + 5$ and hence leads to a contradiction whenever $n + k \geq (2l + 6)$. Thus, by (21), this leads to a contradiction in Theorems B2 and B3.

In the same way, in Theorem B4, we have $T(r,\alpha) = 0$, hence Relation (25) shows that $n + k < 2l + 5$, a contradiction to (22).

Thus, we have $A = 1$ and this implies that $F = G$. Now, $\alpha F = \alpha G$, i.e. $(\widehat{F})' = (\widehat{G})'$. We assume $n \geq k + 2$ in Theorem B2 when f, g belong to $\mathcal{M}(\mathbb{K})$ and in Theorems B3 and B4. And we assume $n \geq k + 3$ in Theorem B2 when f, g belong to $\mathcal{M}(d(0, R^-))$.

Consequently, by Proposition BP and by Lemma BL4, we have $\widehat{F} = \widehat{G}$, i.e. $P(f) = P(g)$. But in Theorems B2, B3, B4, B5, B6, B7, P is a polynomial of uniqueness for the family of meromorphic functions we consider, hence we have $f = g$. And in Theorems B8 and B9, the conclusion was given in [4]. That finishes Case 1: $B = 0$.

Case 2: $B \neq 0$.

We have $\overline{Z}(r,F) \leq \overline{Z}(r,f) + \sum_{i=2}^{l} \overline{Z}(r, f - a_i) + \overline{Z}(r,f') + T(r,\alpha)$ and $\overline{N}(r,F) \leq \overline{N}(r,f) + T(r,\alpha) + O(1)$ and similarly for G, so we can derive

$$\overline{Z}(r,F) + \overline{Z}(r,G) + \overline{N}(r,F) + \overline{N}(r,G) \leq \overline{Z}(r,f) + \sum_{i=2}^{l} \overline{Z}(r, f - a_i) + \overline{Z}(r,f') + \overline{Z}(r,g)$$

$$+\sum_{i=2}^{l} \overline{Z}(r, g - a_i) + \overline{Z}(r,g') + \overline{N}(r,f) + \overline{N}(r,g) + 4T(r,\alpha) + O(1)$$

$$\leq (l+3)\big[T(r,f) + T(r,g)\big] + 4T(r,\alpha) + O(1). \tag{26}$$

Moreover, by (19), $T(r,F) = T(r,G) + O(1)$ and, by Lemma BL1, we have

$$T(r,f) \leq \frac{1}{n+k}(T(r,F)+T(r,\alpha))+O(1)$$

and $T(r,g) \leq \frac{1}{n+k}(T(r,G)+T(r,\alpha))+O(1)$. Consequently,

$$T(r,f)+T(r,g) \leq 2\Big[\frac{1}{n+k}(T(r,F)+T(r,\alpha))\Big]+O(1).$$

Thus, (26) is equivalent to

$$\overline{Z}(r,F)+\overline{Z}(r,G)+\overline{N}(r,F)+\overline{N}(r,G) \leq \frac{2(l+3)}{n+k}T(r,F)+(\frac{10}{n+k}+4)T(r,\alpha)+O(1).$$

Hence in Theorems 2 and 3, by (21) we have

$$\limsup_{r\to+\infty}\left(T(r,F)-(\overline{Z}(r,F)+\overline{Z}(r,G)+\overline{N}(r,F)+\overline{N}(r,G))\right)=+\infty$$

(resp.

$$\limsup_{r\to R^-}\left(T(r,F)-(\overline{Z}(r,F)+\overline{Z}(r,G)+\overline{N}(r,F)+\overline{N}(r,G))\right)=+\infty.$$

Next, in Theorem B4, we have $\overline{Z}(r,F) \leq \overline{Z}(r,f)+\sum_{i=2}^{l}\overline{Z}(r,f-a_i)+\overline{Z}(r,f')$ and $\overline{N}(r,F) \leq \overline{N}(r,f)+O(1)$ and similarly for G, so we can derive

$$\overline{Z}(r,F)+\overline{Z}(r,G)+\overline{N}(r,F)+\overline{N}(r,G) \leq \overline{Z}(r,f)+\sum_{i=2}^{l}\overline{Z}(r,f-a_i)+\overline{Z}(r,f')+\overline{Z}(r,g)$$

$$+\sum_{i=2}^{l}\overline{Z}(r,g-a_i)+\overline{Z}(r,g')+\overline{N}(r,f)+\overline{N}(r,g)+O(1)$$

$$\leq (l+3)\big[T(r,f)+T(r,g)\big]-2\log r+O(1),$$

therefore

$$\overline{Z}(r,F)+\overline{Z}(r,G)+\overline{N}(r,F)+\overline{N}(r,G) \leq \frac{2l+6}{n+k}T(r,F)-2\log r+O(1).$$

Consequently, by (22) we have again

$$\limsup_{r\to+\infty}\left(T(r,F)-(\overline{Z}(r,F)+\overline{Z}(r,G)+\overline{N}(r,F)+\overline{N}(r,G))\right)=+\infty.$$

Thus, in each theorem, the hypotheses of Lemma BL8 are satisfied and hence, either $F=G$, or $FG=1$.

If $FG=1$, then $f'P'(f)g'P'(g)=\alpha^2$. In Theorems B2, B3, B4 we have assumed that
if $l=2$, then $k_1 \neq k+1,\ 2k,\ 2k+1,\ 3k+1$,
if $l=3$, then $k_1 \neq \frac{k}{2},\ k_1 \neq k+1,\ 2k+1,\ 3k_i-k\ \forall i=2,\ 3$.
If $l \geq 4$, then $k_1 \neq k+1$.

And these hypotheses are automatically satisfied in the other theorems. Consequently, by Lemma BL3, $FG=1$ is impossible. Consequently, $F=G$, hence $(\widehat{F})'=(\widehat{G})'$ and therefore we can conclude as in the case $B=0$.

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REFERENCES

1. T. T. H. An, J. T. Y. Wang and P. M. Wong, "Unique range sets and uniqueness polynomials in positive characteristic II.," Acta Arithm. 115–143 (2005).
2. T. T. H. An, J. T. Y. Wang and P. M. Wong, "Strong uniqueness polynomials: the complex case," Complex Var. **49** (1), 25–54 (2004).
3. T. T. H. An and A. Escassut, "Meromorphic solutions of equations over non-Archimedean fields," Ramanujan J. **15** (3), 415–433 (2008).
4. K. Boussaf, A. Escassut and J. Ojeda, "p-Adic meromorphic functions $f'P'(f)$, $g'P'(g)$ sharing a small function," Bull. Scien. Mathématiques **136** (2), 172–200 (2012).
5. K. Boussaf, A. Escassut and J. Ojeda, "Complex meromorphic functions $f'P'(f)$, $g'P'(g)$ sharing a small function," Indagationes (N.S.) **24** (1), 15–41 (2013).
6. A. Boutabaa, "Théorie de Nevanlinna p-adique," Manuscripta Math. **67**, 251–269 (1990).
7. A. Escassut, *Analytic Elements in p-Adic Analysis* (World Scientific Publ. Co. Pte. Ltd. Singapore, 1995).
8. A. Escassut, "Meromorphic functions of uniqueness," Bull. Scien. Mathématiques **131** (3), 219–241 (2007).
9. A. Escassut, "p-Adic value distribution," *Some Topics on Value Distribution and Differentability in Complex and p-Adic Analysis* 42–138, Mathematics Monograph, Series **11** (Science Press, Beijing, 2008).
10. H. Fujimoto, "On uniqueness of meromorphic functions sharing finite sets," Amer. J. Math. **122** (6), 1175–1203 (2000).
11. P. C. Hu and C. C. Yang, *Meromorphic Functions over non-Archimedean Fields* (Kluwer Academic Publ., 2000).
12. X. Hua and C. C. Yang, "Uniqueness and value-sharing of meromorphic functions," Ann. Acad. Sci. Fenn. Math. **22**, 395–406 (1997).
13. W. Lin and H. Yi, "Uniqueness theorems for meromorphic functions concerning fixed-points," Complex Var. Theory Appl. **49** (11), 793–806 (2004).
14. R. Nevanlinna, *Le théoreme de Picard-Borel et la théorie des fonctions méromorphes* (Gauthiers-Villars, Paris, 1929).
15. J. Ojeda, "Uniqueness for ultrametric analytic functions," Bull. Math. Sci. Mathématiques de Roumanie **54** (102, 2), 153–165 (2011).
16. J. T. Y. Wang, "Uniqueness polynomials and bi-unique range sets," Acta Arithm. **104**, 183–200 (2002).
17. K. Yamanoi "The second main theorem for small functions and related problems," Acta Math. **192**, 225–294 (2004).
18. C. C. Yang and P. Li, "Some further results on the functional equation $P(f) = Q(g)$," Series Advances in Complex Analysis and Applications, Value Distribution Theory and Related Topics, pp. 219–231 (Kluwer Academic Publ., Dordrecht-Boston-London, 2004).
19. C. C. Yang and X. Hua, "Unique polynomials of entire and meromorphic functions," Matemat. Fizika Anal. Geom. **4** (3), 391–398 (1997).