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SHORT COMMUNICATIONS

A Theorem on Weighted Means in Non-Archimedean Fields*

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Abstract—K denotes a complete, non-trivially valued, non-archimedean field. Infinite matrices, sequences and series have entries in K. In this paper, we prove an interesting result, which gives an equivalent formulation of summability by weighted mean methods. Incidentally this result includes the non-archimedean analogue of a theorem proved by Móricz and Rhoades (see [2], Theorem MR, p.188).

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1. INTRODUCTION

Throughout this short note, K denotes a complete, non-trivially valued, non-archimedean field. Infinite matrices, sequences and series have entries in K.

Given an infinite matrix $A = (a_{nk})$, $n, k = 0, 1, 2, \dots$ and a sequence $x = \{x_k\}$, $k = 0, 1, 2, \dots$, by the A-transform of $x = \{x_k\}$, we mean the sequence $Ax = \{(Ax)_n\}$,

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \quad n = 0, 1, 2, \dots,$$

where we suppose that the series on the right converge. If $\lim_{n\to\infty} (Ax)_n = \ell$, we say that $x = \{x_k\}$ is A-summable or summable A to ℓ. If $\lim_{n\to\infty} (Ax)_n = \ell$ whenever $\lim_{k\to\infty} x_k = \ell$, we say that A is regular. The following result, which gives necessary and sufficient conditions for an infinite matrix $A = (a_{nk})$ to be regular in terms of its entries, is well-known (see [1]).

Theorem 1. *$A = (a_{nk})$ is regular if and only if*

(i) $\sup_{n,k} |a_{nk}| < \infty$;

(ii) $\lim_{n\to\infty} a_{nk} = 0$, $k = 0, 1, 2, \dots$; *and*

(iii) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1.$

An infinite series $\sum_{k=0}^{\infty} x_k$ is said to be A-summable to ℓ if $\{s_n\}$ is A-summable to ℓ, where $s_n = \sum_{k=0}^{n} x_k$, $n = 0, 1, 2, \dots$.

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2. WEIGHTED MEANS AND THE MAIN RESULT

In developing summability methods in non-archimedean fields, Srinivasan [4] defined the analogue of the classical weighted means $(\bar{N}, p_n)$ under the assumption that the sequence $\{p_n\}$ of weights satisfies the conditions: $|p_0| < |p_1| < |p_2| < \cdots < |p_n| < \ldots$; and $\lim_{n\to\infty} |p_n| = \infty$. However, it turned out that these weighted means were equivalent to convergence. Natarajan remedied the situation with the following definition (see [3]).

Definition 6. The weighted mean method in K, denoted by $(\bar{N}, p_n)$, is defined by the infinite matrix (a_{nk}), where

$$a_{nk} = \begin{cases} \frac{p_k}{P_n}, & k \le n; \\ 0, & k > n, \end{cases}$$

$p_n \neq 0, n = 0, 1, 2, \ldots$ and $|p_i| \le |P_j|, i = 0, 1, 2, \ldots, j; j = 0, 1, 2, \ldots, P_n = \sum_{k=0}^{n} p_k, n = 0, 1, 2, \ldots.$

An example of such an $(\bar{N}, p_n)$ method corresponds to

$$p_n = \begin{cases} p^n, & \text{if } n \text{ is odd;} \\ \frac{1}{p^n}, & \text{if } n \text{ is even,} \end{cases}$$

where $K = Q_p$, the p-adic field for a prime p. We need, in the sequel, the following result.

Theorem 2. *([3], Theorem 1, p. 194) $(\bar{N}, p_n)$ is regular if and only if $\lim_{n\to\infty} |P_n| = \infty$.*

We now prove the main result which gives an equivalent formulation of summability by weighted mean methods. Incidentally we note that this result includes the non-archimedean analogue of a theorem proved by Móricz and Rhoades (see [2], Theorem MR, p. 188).

Theorem 3. *Let $(\bar{N}, p_n)$, $(\bar{N}, q_n)$ be two regular weighted mean methods. Let $\sum_{n=0}^{\infty} b_n$ converge to ℓ, where*

$$b_n = q_n \sum_{k=n}^{\infty} \frac{x_k}{Q_k}, n = 0, 1, 2, \ldots.$$

Then $\sum_{n=0}^{\infty} x_n$ is $(\bar{N}, p_n)$ summable to ℓ if and only if

$$\sup_{n,k} \left| \frac{p_k Q_{k+1}}{P_n q_{k+1}} \right| < \infty.$$

Proof. Let $B_n = \sum_{k=0}^{n} b_k \to \ell, n \to \infty$. Now,

$$\frac{b_n}{q_n} - \frac{b_{n+1}}{q_{n+1}} = \sum_{k=n}^{\infty} \frac{x_k}{Q_k} - \sum_{k=n+1}^{\infty} \frac{x_k}{Q_k} = \frac{x_n}{Q_n},$$

so that

$$x_n = Q_n \left(\frac{b_n}{q_n} - \frac{b_{n+1}}{q_{n+1}} \right), \quad n = 0, 1, 2, \ldots.$$

Consequently

$$\begin{aligned} s_m &= \sum_{k=0}^{m} x_k = \sum_{k=0}^{m} Q_k \left(\frac{b_k}{q_k} - \frac{b_{k+1}}{q_{k+1}}\right) = \sum_{k=0}^{m} Q_k \cdot \frac{b_k}{q_k} - \sum_{k=1}^{m+1} Q_{k-1} \cdot \frac{b_k}{q_k} \\ &= Q_0 \cdot \frac{b_0}{q_0} + \sum_{k=1}^{m} (Q_k - Q_{k-1}) \frac{b_k}{q_k} - Q_m \cdot \frac{b_{m+1}}{q_{m+1}} = b_0 + \sum_{k=1}^{m} q_k \cdot \frac{b_k}{q_k} - Q_m \cdot \frac{b_{m+1}}{q_{m+1}} \\ &= b_0 + \sum_{k=1}^{m} b_k - Q_m \cdot \frac{b_{m+1}}{q_{m+1}} = \sum_{k=0}^{m} b_k - Q_m \cdot \frac{b_{m+1}}{q_{m+1}} = B_m - Q_m \cdot \frac{b_{m+1}}{q_{m+1}}. \end{aligned} \tag{2.1}$$

By hypothesis, $\displaystyle\sum_{k=0}^{\infty} \frac{x_k}{Q_k}$ converges so that $\frac{b_n}{q_n} = \sum_{k=n}^{\infty} \frac{x_k}{Q_k} \to 0, \quad n \to \infty$. Now,

$$\frac{s_m}{Q_m} = \frac{B_m}{Q_m} - \frac{b_{m+1}}{q_{m+1}}, \quad \text{using}\,(2.1).$$

Since $\{B_n\}$ converges, it is bounded so that for some $M > 0$, $|B_n| \leq M$, $n = 0, 1, 2, \ldots$. As $(\bar{N}, q_n)$ is regular, $|Q_n| \to \infty$, $n \to \infty$ so that

$$\left|\frac{B_m}{Q_m}\right| \leq \frac{M}{|Q_m|} \to 0, \quad m \to \infty.$$

Thus $\frac{s_m}{Q_m} \to 0, \quad m \to \infty$. For $n = 0, 1, 2, \ldots,$

$$\begin{aligned} b_n &= q_n \sum_{k=n}^{\infty} \frac{x_k}{Q_k} = q_n \lim_{m\to\infty} \sum_{k=n}^{m} \frac{s_k - s_{k-1}}{Q_k} \quad (\text{where } s_{-1} = 0) = q_n \lim_{m\to\infty} \left\{\sum_{k=n}^{m} \frac{s_k}{Q_k} - \sum_{k=n-1}^{m-1} \frac{s_k}{Q_{k+1}}\right\} \\ &= q_n \lim_{m\to\infty} \left\{\sum_{k=n}^{m-1} \frac{s_k}{Q_k} + \frac{s_m}{Q_m} - \sum_{k=n}^{m-1} \frac{s_k}{Q_{k+1}} - \frac{s_{n-1}}{Q_n}\right\} \\ &= q_n \lim_{m\to\infty} \left\{\sum_{k=n}^{m-1} \left(\frac{1}{Q_k} - \frac{1}{Q_{k+1}}\right) s_k + \frac{s_m}{Q_m} - \frac{s_{n-1}}{Q_n}\right\} \\ &= -q_n \frac{s_{n-1}}{Q_n} + q_n \sum_{k=n}^{\infty} \left(\frac{1}{Q_k} - \frac{1}{Q_{k+1}}\right) s_k, \quad \left(\text{since } \lim_{m\to\infty} \frac{s_m}{Q_m} = 0\right) \\ &= -q_n \frac{s_{n-1}}{Q_n} + q_n \sum_{k=n}^{\infty} u_k s_k, \quad \text{where } u_k = \frac{1}{Q_k} - \frac{1}{Q_{k+1}}. \end{aligned} \tag{2.2}$$

Now,

$$\begin{aligned} B_n &= \sum_{k=0}^{n-1} b_k + b_n = \sum_{k=0}^{n-1} \frac{b_k}{q_k} q_k + b_n = \sum_{k=0}^{n-1} q_k \left(\sum_{u=k}^{\infty} \frac{x_u}{Q_u}\right) + b_n \\ &= q_0 \sum_{u=0}^{\infty} \frac{x_u}{Q_u} + q_1 \sum_{u=1}^{\infty} \frac{x_u}{Q_u} + q_2 \sum_{u=2}^{\infty} \frac{x_u}{Q_u} + \cdots + q_{n-1} \sum_{u=n-1}^{\infty} \frac{x_u}{Q_u} + b_n \\ &= (q_0 + q_1 + \cdots + q_{n-1}) \sum_{u=n-1}^{\infty} \frac{x_u}{Q_u} + q_0 \sum_{u=0}^{n-2} \frac{x_u}{Q_u} + q_1 \sum_{u=1}^{n-2} \frac{x_u}{Q_u} + q_2 \sum_{u=2}^{n-2} \frac{x_u}{Q_u} + \cdots + q_{n-2} \frac{x_{n-2}}{Q_{n-2}} + b_n \\ &= Q_{n-1} \sum_{u=n-1}^{\infty} \frac{x_u}{Q_u} + b_n + \frac{x_{n-2}}{Q_{n-2}} Q_{n-2} + \frac{x_{n-3}}{Q_{n-3}} Q_{n-3} + \cdots + \frac{x_0}{Q_0} Q_0 \end{aligned}$$

$$
\begin{aligned}
&= Q_{n-1}\sum_{u=n-1}^{\infty}\frac{x_u}{Q_u} + b_n + \sum_{k=0}^{n-2} x_k = s_{n-2} + b_n + Q_{n-1}\sum_{u=n-1}^{\infty}\frac{x_u}{Q_u}\\
&= s_{n-2} + q_n\sum_{u=n}^{\infty}\frac{x_u}{Q_u} + Q_{n-1}\sum_{u=n-1}^{\infty}\frac{x_u}{Q_u}\\
&= s_{n-2} + (Q_n - Q_{n-1})\sum_{u=n}^{\infty}\frac{x_u}{Q_u} + Q_{n-1}\sum_{u=n-1}^{\infty}\frac{x_u}{Q_u}\\
&= s_{n-2} + Q_n\sum_{u=n}^{\infty}\frac{x_u}{Q_u} + Q_{n-1}\frac{x_{n-1}}{Q_{n-1}} = s_{n-1} + Q_n\sum_{u=n}^{\infty}\frac{x_u}{Q_u}\\
&= s_{n-1} + Q_n\frac{b_n}{q_n} = s_{n-1} + Q_n\left[-\frac{s_{n-1}}{Q_n} + \sum_{k=n}^{\infty} u_k s_k\right], \quad \text{using (2.2)}\\
&= Q_n\sum_{k=n}^{\infty} u_k s_k,
\end{aligned}
$$

so that $\frac{B_n}{Q_n} = \sum_{k=n}^{\infty} u_k s_k$. Consequently

$$
u_n s_n = \frac{B_n}{Q_n} - \frac{B_{n+1}}{Q_{n+1}}, \quad n = 0, 1, 2, \ldots. \tag{2.3}
$$

Let $\{T_n\}$ be the $(\bar{N}, p_n)$ transform of $\{s_k\}$ so that

$$
\begin{aligned}
T_n &= \frac{1}{P_n}\sum_{k=0}^{n} p_k s_k = \frac{1}{P_n}\sum_{k=0}^{n} p_k \frac{1}{u_k}\left\{\frac{B_k}{Q_k} - \frac{B_{k+1}}{Q_{k+1}}\right\}, \text{using (2.3)}\\
&= \frac{1}{P_n}\left[\frac{p_0}{u_0}\frac{B_0}{Q_0} + \sum_{k=1}^{n}\left\{\frac{p_k}{u_k} - \frac{p_{k-1}}{u_{k-1}}\right\}\frac{B_k}{Q_k} - \frac{p_n}{u_n}\frac{B_{n+1}}{Q_{n+1}}\right] = \sum_{k=0}^{\infty} a_{nk} B_k,
\end{aligned}
$$

where

$$
a_{nk} = \begin{cases}
\frac{1}{P_n}\frac{p_0}{u_0 Q_0}, & k = 0;\\
\frac{1}{P_n}\left\{\frac{p_k}{u_k} - \frac{p_{k-1}}{u_{k-1}}\right\}\frac{1}{Q_k}, & 1 \le k \le n;\\
-\frac{1}{P_n}\frac{p_n}{u_n Q_{n+1}}, & k = n+1;\\
0, & k \ge n+2.
\end{cases}
$$

It is clear that $\lim\limits_{n\to\infty} a_{nk} = 0$, $k = 0, 1, 2, \ldots$. Also,

$$
\begin{aligned}
\sum_{k=0}^{\infty} a_{nk} = \sum_{k=0}^{n+1} a_{nk} &= \frac{1}{P_n}\left[\frac{p_0}{u_0 Q_0} + \sum_{k=1}^{n}\left\{\frac{p_k}{u_k} - \frac{p_{k-1}}{u_{k-1}}\right\}\frac{1}{Q_k} - \frac{p_n}{u_n Q_{n+1}}\right]\\
&= \frac{1}{P_n}\left[\frac{p_0}{u_0 Q_0} + \left(\frac{p_1}{u_1} - \frac{p_0}{u_0}\right)\frac{1}{Q_1} + \left(\frac{p_2}{u_2} - \frac{p_1}{u_1}\right)\frac{1}{Q_2} + \cdots + \left(\frac{p_n}{u_n} - \frac{p_{n-1}}{u_{n-1}}\right)\frac{1}{Q_n} - \frac{p_n}{u_n Q_{n+1}}\right]\\
&= \frac{1}{P_n}\left[\frac{p_0}{u_0}\left(\frac{1}{Q_0} - \frac{1}{Q_1}\right) + \frac{p_1}{u_1}\left(\frac{1}{Q_1} - \frac{1}{Q_2}\right) + \frac{p_2}{u_2}\left(\frac{1}{Q_2} - \frac{1}{Q_3}\right) + \cdots + \frac{p_n}{u_n}\left(\frac{1}{Q_n} - \frac{1}{Q_{n+1}}\right)\right]\\
&= \frac{1}{P_n}\left[\frac{p_0}{u_0}u_0 + \frac{p_1}{u_1}u_1 + \cdots + \frac{p_n}{u_n}u_n\right] = \frac{1}{P_n}P_n = 1, \quad n = 0, 1, 2, \ldots,
\end{aligned}
$$

so that $\lim_{n\to\infty}\sum_{k=0}^{\infty} a_{nk} = 1$. By hypothesis, $B_k \to \ell$, $k \to \infty$. Using Theorem 1, $T_n \to \ell$, $n \to \infty$. i.e.,

$\sum_{n=0}^{\infty} x_n$ is $(\bar{N}, p_n)$ summable to ℓ if and only if

$$\sup_n \frac{1}{|P_n|}\left[\frac{1}{|Q_0|}\left|\frac{p_0}{u_0}\right|, \max_{1\le k\le n}\frac{1}{|Q_k|}\left\{\left|\frac{p_k}{u_k}-\frac{p_{k-1}}{u_{k-1}}\right|\right\}, \frac{1}{|Q_{n+1}|}\left|\frac{p_n}{u_n}\right|\right] < \infty,$$

i.e., if and only if

$$\sup_n \frac{1}{|P_n|}\left[\max_{1\le k\le n}\left|\frac{p_k Q_{k+1}}{q_{k+1}}-\frac{p_{k-1}Q_{k-1}}{q_k}\right|, \left|\frac{p_n Q_n}{q_{n+1}}\right|\right] < \infty,$$

which is equivalent to

$$\sup_n \frac{1}{|P_n|}\left[\max_{1\le k\le n}\left|\frac{p_k Q_{k+1}}{q_{k+1}}-\frac{p_{k-1}Q_{k-1}}{q_k}\right|\right] < \infty$$

(here we use the fact that $|P_n| \le |P_{n+1}|$, $|Q_n| \le |Q_{n+1}|$, $n = 0, 1, 2, \dots$).

This completes the proof of the theorem. □

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