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RESEARCH ARTICLES

A Muckenhoupt's Weight Problem and Vector Valued Maximal Inequalities Over Local Fields*

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Abstract—In this paper, an important and interesting Muckenhoupt's problem over local fields was firstly solved. Weighted weak and strong type norm inequalities for the Fefferman-Stein vector-valued maximal operator were firstly established over local fields, too. These results are very useful. They could be applied in various mathematical areas, for instance, in theories of functions, of partial differential equations, in harmonic analysis.

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1. INTRODUCTION

In [5, 6, 12], it is noticed that many interesting relations and mutual influences between abstract, applied and computational harmonic analysis, deterministic as well as stochastic, are more and more widespread, especially, when we are interested not only in the cases of real, complex fields, but also in other cases such as, the cases of local fields, of other topological, algebraic areas (see [7—11, 20, 25, 29, 30, 32]). For instance, the surprising relation and influence between p-adic spectral analysis and wavelet analysis over $\mathbb{R}$-field was recently discovered by S.V. Kozyrev in [26], and investigated in detail in works [21—24].

Here, in order to solve the Muckenhoupt's problem, we have to prove a lot of necessary covering lemmas for local fields, to prove a celebrated duality inequality of C. Fefferman and E. M. Stein, for local fields, too. From the proof of the duality inequality, it is interesting to know that the norm of the Hardy-Littlewood maximal operator M from $L^1\left(\mathbb{K}^d\right)$ to weak-$L^1\left(\mathbb{K}^d\right)$ is not greater than 1, which is different from the Euclidean case. We exploit the Marcinkiewicz interpolation theorems and some special geometric properties of local fields, which do not belong to $\mathbb{R}$ and $\mathbb{C}$-fields, to solve completely the Muckenhoupt's problem [27] over such fields. Note that with the needed tools just mentioned above, it is also hard to choose a special key function for solving successfully this difficult problem over local fields. In the Euclidean case, this problem was already solved independently by Wo-Sang Young [33] and by Ángel E. Gatto, Cristian E. Gutiérrez [18]. In the last section of this paper, some famous weighted weak and strong inequalities for the Fefferman-Stein vector-valued maximal operator are proved. These maximal inequalities generalize the ones most recently established by us in [12].

The locally compact, non-discrete, complete fields have been completely characterized. They are either connected, which are the usual real and complex number fields, or totally disconnected. Let $\mathbb{K}$ be a *local field*, which is a totally disconnected, locally compact, non-discrete, complete field. Such fields have been completely classified (see [30]). If $\mathbb{K}$ is of characteristic zero, then $\mathbb{K}$ is either a $p-$adic field $\mathbb{Q}_p$ for some rational prime p, or a finite algebraic extension of such a field. If $\mathbb{K}$ has a finite characteristic, then $\mathbb{K}$ is isomorphic to a field of formal power series over a finite field. Let $\mathbb{K}$ be a fixed local field.

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Then there exist an integer $q = p^r$ and a norm $|\cdot|$ on $\mathbb{K}$ (where p is a prime number and r is a positive integer number), such that if $x \in \mathbb{K}^* = \mathbb{K} \setminus \{0\}$, then $|x| = q^k$ for some integer k. The norm is non-Archimedean, that is $|x+y| \le \max\{|x|, |y|\}$ for all $x, y \in \mathbb{K}$, and $|x+y| = \max\{|x|, |y|\}$ if $|x| \ne |y|$. Let dx be the Haar measure on $(\mathbb{K}, +)$ normalized so that $\int_{B_0} dx = 1$, where $B_0 = \{x : |x| \le 1\}$. There exists an additive character χ on the additive group $(\mathbb{K}, +)$ such that χ is trivial on B_0, but is not trivial on $B_1 = \{x : |x| \le q\}$.

Let $\mathbb{K}^d$ be the d-dimensional vector space over $\mathbb{K}$, $\mathbb{K}^d = \{x = (x_1, \ldots, x_d) : \; x_i \in \mathbb{K}, i = \overline{1, d}\,\}$. There is a norm $|\cdot|$, defined on $\mathbb{K}^d$ by $|x| = \max\{|x_1|, \ldots, |x_d|\}$, $x = (x_1, \ldots, x_d) \in \mathbb{K}^d$. It is easy to check that $|x+y| \le \max\{|x|, |y|\}$, for all $x, y \in \mathbb{K}^d$ and that $|x+y| = \max\{|x|, |y|\}$ if $|x| \ne |y|$. $\mathbb{K}^d$ is a locally compact, Abelian, topological group under vector addition and with the usual product topology (which coincides with the norm topology). A Haar measure is given by $dx = dx_1 \ldots dx_d$, where dx_i is the normalized additive Haar measure on that copy of $\mathbb{K}$, which is the i^{th} coordinate space of $\mathbb{K}^d$. For each $\alpha \in \mathbb{K}^*$, we have $d(\alpha x) = |\alpha|^d dx$. Let us define $\alpha \cdot x = (\alpha x_1, \ldots, \alpha x_d)$ for any $\alpha \in \mathbb{K}$, and $x = (x_1, \ldots, x_d) \in \mathbb{K}^d$.

Fix $1 \le \ell \le \infty$ and a non-negative measurable function u on $\mathbb{K}^d$. We denote by $L^\ell(u)$ the space of all measurable functions f from $\mathbb{K}^d$ to $\mathbb{C}$ so that the norm $\|f\|_{L^\ell(u)} = \left(\int\limits_{\mathbb{K}^d} |f(y)|^\ell u(y) dy\right)^{1/\ell} < +\infty$, with a usual modification made when $\ell = \infty$. Let A be a measurable subset of $\mathbb{K}^d$, u be a measurable function from A to $\mathbb{C}$, and $(\gamma, x) \in \mathbb{Z} \times \mathbb{K}^d$. It is convenient to introduce the following

$$x + B_\gamma = \{y \in \mathbb{K}^d : \; |y - x| \le q^\gamma\}\,, \; B_\gamma = 0 + B_\gamma,$$

$$x + S_\gamma = \{y \in \mathbb{K}^d : \; |y - x| = q^\gamma\}\,, \; S_k = 0 + S_\gamma,$$

$$|A| = \int_A dx\,, \; u(A) = \int_A u(x) dx.$$

Thus, $x + B_\gamma$ is the ball centered at x, with radius q^γ and $x + S_\gamma$ is its boundary. It is easy to see that

$$|x + B_\gamma| = |B_\gamma| = q^{d\gamma}\,, \; |x + S_\gamma| = |S_\gamma| = q^{d\gamma}\Big(1 - \frac{1}{q^d}\Big).$$

We shall need the facts that

$$\int\limits_{\mathbb{K}^d} f dx = \sum_{\gamma=-\infty}^{+\infty} \int\limits_{S_\gamma} f dx, \tag{1.1}$$

and

$$\int\limits_{\mathbb{K}^d} f(\alpha x) dx = \frac{1}{|\alpha|^d} \int\limits_{\mathbb{K}^d} f dx\,, \qquad (\forall \alpha \in \mathbb{K}^*)\,, \tag{1.2}$$

for any measurable function f from $\mathbb{K}^d$ to $\mathbb{C}$. For other facts, see [30] and [32].

The *Hardy-Littlewood maximal operator* M is defined for a locally integrable function f on $\mathbb{K}^d$ by

$$Mf(x) = \sup_{\gamma \in \mathbb{Z}} \frac{1}{q^{\gamma d}} \int\limits_{x + B_\gamma} |f(y)| dy \qquad (x \in \mathbb{K}^d), \tag{1.3}$$

the supremum being taken over all integer numbers γ. It is interesting to note that, over local fields, the centered maximal function and uncentered maximal function are equivalent. The maximal operator M and its variants on local fields have been studied by many authors, in particular, the following inequalities

$$\int\limits_{\mathbb{K}^d} |Mf(x)|^\ell dx \le C_\ell \int\limits_{\mathbb{K}^d} |f(x)|^\ell dx \; (1 < \ell < \infty), \tag{1.4}$$

$$\left|\{x \in \mathbb{K}^d : \ Mf(x) > \alpha\}\right| \leq \frac{C}{\alpha} \int\limits_{\mathbb{K}^d} |f(x)|dx. \tag{1.5}$$

It was proved by us in [12] that, the constant C of inequality (1.5) can be chosen as $C = q^d$.

Let T be an operator defined on a linear space of complex-valued measurable functions on a measure space (X, μ) and taking values in the set of all complex-valued finite almost everywhere measurable functions on a measure space (Y, ν). T is called *sublinear* if for all f, g and $\lambda \in \mathbb{C}$ we have

$$|T(f+g)(x)| \leq |Tf(x)| + |Tg(x)| \quad \text{and} \quad |T(\lambda f)(x)| = |\lambda| \cdot |T(f)(x)| \qquad (\forall x \in X).$$

For $1 \leq s \leq \infty$, the space *weak*-$L^s(X, \mu)$ is defined as the set of all μ-measurable functions f such that

$$||f||_{L^{s,\infty}} := \inf \left\{ C > 0 : \ \mu\{x : \ |f(x)| > \alpha\} \leq \frac{C^s}{\alpha^s} \text{ for all } \alpha > 0 \right\}$$

is finite, with a usual modification made when $s = \infty$. The *weak*-$L^s(X, \mu)$ spaces are denoted by $L^{s,\infty}(X, \mu)$. Operators that map L^r to L^s are said to be of strong type of (r, s) and operators that map L^r to $L^{s,\infty}$ is called of weak type (r, s). Inequalities (1.4), (1.5) show that M is of weak type $(1, 1)$ and of strong type (ℓ, ℓ) for $1 < \ell < \infty$.

Theorem 1. (Marcinkiewicz Theorem) *Let (X, μ) and (Y, ν) be measure spaces and let T be a sublinear operator defined in both $L^{s_0}(X, \mu)$ and $L^{s_1}(X, \mu)$ for some pair $1 \leq s_0 < s_1 \leq \infty$ and take values in the space of all ν-measurable functions on Y. Assume that there exist two positive constants A_0, A_1 such that*

$$||Tf||_{L^{s_0,\infty}(Y,\nu)} \leq A_0 ||f||_{L^{s_0}(X,\mu)} \qquad \textit{for all } f \in L^{s_0}(X, \mu),$$

$$||Tf||_{L^{s_1,\infty}(Y,\nu)} \leq A_1 ||f||_{L^{s_1}(X,\mu)} \qquad \textit{for all } f \in L^{s_1}(X, \mu).$$

Then for all $s_0 < s < s_1$ and for all f in $L^s(X, \mu)$ we have the estimate

$$||Tf||_{L^s(Y,\nu)} \leq A ||f||_{L^s(X,\mu)}.$$

See [16], pp. 31-34 for a proof of Theorem 1.

Let ℓ^r $(1 \leq r < \infty)$ be the space of all complex sequences $x = \{x_k\}_{k=1}^{\infty}$ such that

$$|x|_r := \left(\sum_{k=1}^{\infty} |x_k|^r \right)^{1/r} < \infty.$$

Let $\mathcal{S}$ be the linear space of measurable functions $f : \mathbb{K}^d \to \mathbb{C}$ which take only a finite number of values. Let $\mathcal{S}(\ell^r)$ be the linear space of sequences of functions $f = \{f_k\}$ so that $f_k \in \mathcal{S}$ and $f_k(x) \equiv 0$ for all sufficiently large k. Then $\mathcal{S}$ is dense in $L^t(\mathbb{K}^d)$, $1 \leq t < \infty$. Furthermore, from [3] and [13], if $\omega \geq 0$ is a locally integrable function on $\mathbb{K}^d$, then $S(\ell^r)$ is dense in $L^t_\omega(\ell^r)$ for $1 \leq t, r < \infty$, where $L^t_\omega(\ell^r)$ is the space of sequences $f = \{f_k\}$ with norm

$$||f||_{L^t_\omega(\ell^r)} := \left(\int_{\mathbb{K}^d} |f(x)|_r^t \omega(x) dx \right)^{1/t} < \infty.$$

We shall employ a vector-valued version of the Marcinkiewicz interpolation Theorem. It is essentially contained in [2], Lemma 1. The details of the proof are standard, so they are omitted (see [2, 16, 34]).

Theorem 2. *Let $\omega(x) \geq 0$ be locally integrable on $\mathbb{K}^d$, $1 < r < \infty$, $1 \leq \ell_1 < \ell_2 < \infty$. Suppose that T is a sublinear operator defined on $\mathcal{S}(\ell^r)$ with values in $\mathcal{M}(\mathbb{K}^d)$, where $\mathcal{M}(\mathbb{K}^d)$ is the set of all sequences of measurable functions $g = \{g_k\}$ on $\mathbb{K}^d$. Let $\overrightarrow{T} f = \{Tf_k\}$ such that*

$$\omega\left(\left\{x \in \mathbb{K}^d : \ |\overrightarrow{T} f|_r > \alpha \right\}\right) \leq C_i^{\ell_i} \alpha^{-\ell_i} \int\limits_{\mathbb{K}^d} |f(x)|_r^{\ell_i} \omega(x) dx$$

for $i = 1, 2$ *and* $f \in \mathcal{S}(\ell^r)$.

Then for $\ell_1 < s < \ell_2$, $\overrightarrow{T}$ *extends uniquely to a sublinear operator on* $L^s_\omega(\ell^r)$ *and there is a constant* $C = C_s$ *so that*

$$\int_{\mathbb{K}^d} |\overrightarrow{T} f(x)|^s_r \, \omega(x)dx \le C_s \int_{\mathbb{K}^d} |f(x)|^s_r \, \omega(x)dx$$

for any $f \in L^s_\omega(\ell^r)$.

A function ω on $\mathbb{K}^d$ is called a *weight*, if it is a non-negative, measurable, and locally integrable function. A weight ω is a $\mathcal{A}_\ell$ weight (or in $\mathcal{A}_\ell$ class), for $1 < \ell < \infty$, if there exists a constant $C > 0$ such that

$$\left(\int_{x+B_\gamma} \omega(y)dy \right) \cdot \left(\int_{x+B_\gamma} \omega(y)^{-\frac{1}{\ell-1}} \right)^{\ell-1} \le C \cdot q^{d\gamma} < +\infty, \tag{1.6}$$

for any $(\gamma, x) \in \mathbb{Z} \times \mathbb{K}^d$. A weight ω is said to be in $\mathcal{A}_1$ if there exist $C > 0$ such that for all $x \in \mathbb{K}^d$ and $\gamma \in \mathbb{Z}$,

$$\frac{1}{|B_\gamma|} \int_{x+B_\gamma} \omega(y)dy \le C \text{ ess.inf } \omega(y),$$

where the essential infimum is taken over all y belonging to the ball $x + B_\gamma$. Over local fields, a theory of A_ℓ weights ($1 \le \ell < \infty$) and weighted weak and strong type norm inequalities for the Hardy-Littlewood maximal operator, was systematically introduced in [12]. In particular, there is a local field version of Muckenhoupt's theorem, which states that $\omega \in \mathcal{A}_\ell$ class is necessary and sufficient in order that the following weighted norm inequalities hold

$$\int_{\mathbb{K}^d} |Mf(x)|^\ell \, \omega(x)dx \le C_\ell \int_{\mathbb{K}^d} |f(x)|^\ell \, \omega(x)dx \; (1 < \ell < \infty), \tag{1.7}$$

$$\omega\left(\left\{x \in \mathbb{K}^d : \; Mf(x) > \alpha\right\}\right) \le \frac{C}{\alpha^\ell} \int_{\mathbb{K}^d} |f(x)|^\ell \omega(x)dx \; (1 \le \ell < \infty). \tag{1.8}$$

Another useful result is the following

Lemma 1. *If* $\omega \in \mathcal{A}_\ell$ *for* $1 < \ell < \infty$, *then* $\omega \in \mathcal{A}_r$ *for any* $\ell \le r < \infty$ *and there is* $1 < \ell' < \ell$ *so that* $\omega \in \mathcal{A}_{\ell'}$.

For a proof of Lemma 1, see [12].

2. SOME BASIC COVERING LEMMAS AND FEFFERMAN-STEIN'S DUALITY INEQUALITY

$\mathbb{K}^d$ has many interesting properties, which differ from those of the Euclidean case.

Lemma 2. *(i)* $\mathbb{K}^d = \bigcup_{\gamma \in \mathbb{Z}} x + B_\gamma = \bigcup_{x \in \mathbb{K}^d, \, \gamma \in \mathbb{Z}} x + B_\gamma$,

(ii) $y \in x + B_\gamma$ *for* $x, y \in \mathbb{K}^d$ *if and only if* $x + B_\gamma = y + B_\gamma$,

(iii) *If* $x + B_\gamma \cap x' + B_{\gamma'} \neq \emptyset$ *then* $x + B_\gamma \subset x' + B_{\gamma'}$ *or* $x' + B'_\gamma \subset x + B_\gamma$,

(iv) $x + B_\gamma, x + S_\gamma$ *are closed and open subsets in* $\mathbb{K}^d$.

The following Lemma, which we shall use in the sequel, is the Wiener covering Lemma over local fields.

Lemma 3. *Let $E \subset \mathbb{K}^d$ be measurable and suppose that E is covered by balls $\{x + B_k : (x, k) \in P_E\}$, where P_E is a nonempty subset of $\mathbb{K}^d \times \mathbb{Z}$ satisfying $\sup_{(x,k)\in P_E} k \le k_0 < +\infty$. Then, there is a disjoint and countable subcover $\{x_j + B_{k_j} : j = 1, 2, \ldots\}$ of E such that*

$$|E| \le q^d \cdot \sum_{j=1}^{\infty} |x_j + B_{k_j}|. \tag{2.1}$$

In case $d = 1$, a proof of Lemma 3 was given in [30], but for the general case the proof can be done by a similar way. Some techniques for proving (2.1) are as follows: any two balls in $\mathbb{K}^d$ are either distinct or contained in each other, so we could define an equivalence relation on the elements of balls $\{x + B_k : (x, k) \in P_E\}$. We choose the maximal ball for each equivalence class. Then inequality (2.1) follows from the subadditivity of the measure.

Next we shall obtain the local field versions of the Calderón–Zygmund decomposition of any function $f \in L^1(\mathbb{K}^d)$, the idea of which is to split the function f into its small and large parts. In [28] a local compact group version of the decomposition was introduced, and [30] contains another version of the decomposition but for spheres. The proofs of the two following lemmas employ the characterizing properties ((ii)–(iv) Lemma 2) of local fields.

Lemma 4. *Let $f \in L^1(\mathbb{K}^d)$ and $\alpha > 0$. Then, there exist functions g, $b_j \in L^1(\mathbb{K}^d)$, and a finite or countable collection of pairwise disjoint balls $\{B^j\}_{j\ge1}$, such that $f = g + \sum_{j=1}^{\infty} b_j$ with supp $b_j \subset B^j_\star$. These functions and balls satisfy additionally the following conditions:*

(a) $|g(x)| \le \alpha$,

(b) $||b_j||_{L^1(\mathbb{K}^d)} \le 2q^d \alpha |B^j|$,

(c) $\int_{B^j} b_j dx = 0$,

(d) $\bigcup_{j=1}^{\infty} B^j \subset E_\alpha = \{Mf > \alpha\} \subset \bigcup_{j=1}^{\infty} B^j_\star$,

(e) $\sum_{j=1}^{\infty} |B^j_\star| \le \frac{q^{2d}}{\alpha} \cdot ||f||_{L^1(\mathbb{K}^d)}$,

where $B^j_\star$ is the ball with the same center with B^j but with the radius being q times of B^j's radius.

Proof. Let $E_\alpha = \{x \in \mathbb{K}^d : Mf > \alpha\}$, then E_α is an open subset in $\mathbb{K}^d$. For each $x \in E_\alpha$, inequality (1.5) implies that E_α has finite measure. Thus, there exists an integer γ such that $x + B_\gamma \cap E^c_\alpha \ne \emptyset$ (where E^c is the complement of the set E_α and it is trivial that $E^c_\alpha \ne \emptyset$). Because of $\bigcap_{\gamma\in\mathbb{Z}} x + B_\gamma = \{x\}$ and since E_α is open, we could choose the smallest $\gamma = \gamma(x)$ so that $x + B_\gamma \cap E^c_\alpha \ne \emptyset$. We now have $x + B_{\gamma(x)-1} \subset E_\alpha$ for any $x \in E_\alpha$. Hence, the balls $\{x + B_{\gamma(x)} : x \in E_\alpha\}$ are of uniformly bounded measure. Thus by applying Lemma 3 to $\{x + B_{\gamma(x)} :\ x \in E_\alpha\}$, we obtain a countable and disjoint subcover $\{B^j_\star :\ j = 1, 2, \ldots\}$ of E_α such that

$$\bigcup_{j=1}^{\infty} B^j \subset E_\alpha = \{x \in \mathbb{K}^d :\ Mf > \alpha\} \subset \bigcup_{j=1}^{\infty} B^j_\star.$$

This shows the part (d).

The function $g(x)$ is determined as follows

$$g(x) = \begin{cases} f(x) & \text{if } x \notin E_\alpha \\ \frac{1}{|B^j_\star|} \int\limits_{B^j_\star} f(y)dy & \text{if } x \in B^j_\star, \end{cases} \tag{2.2}$$

and let $b_j(x) = \chi_{B^j_\star}\left(f(x) - \frac{1}{|B^j_\star|} f(y)dy\right)$, where $\chi_{B^j_\star}$ is the characteristic function of $B^j_\star$. The balls $\{B^j_\star\}$ are pairwise disjoint, so

$$f(x) = g(x) + \sum_{j=1}^{\infty} b_j(x) = g(x) + b(x).$$

If $x \notin E_\alpha$, then $\sup\limits_{\gamma\in\mathbb{Z}} \frac{1}{q^{d\gamma}} \int\limits_{x+B_\gamma} f(y)dy \le \alpha$. Applying the Lebesgue differential theorem (see [30]), we have

$$\frac{1}{q^{d\gamma}} \int\limits_{x+B_\gamma} f(y)dy \to f(x) \quad \text{almost everywhere } x \in \mathbb{K}^d \text{ when } \gamma \to -\infty.$$

Thus $f(x) \le \alpha$ almost everywhere $x \notin E_\alpha$. On the other hand, if $x \in E_\alpha$, then there exists j for which $x \in B^j_\star$, and so we have $g(x) = \frac{1}{|B^j_\star|} \int\limits_{B^j_\star} f(y)dy$. Because of $B^j_\star \cap E^c_\alpha \ne \emptyset$, there exists $x' \in B^j_\star \cap E^c_\alpha$. By Lemma 2-ii, we could consider $B^j_\star$ as a ball with its center at x'. Since $x' \in E^c_\alpha$, it follows that $Mf(x') \le \alpha$. Thus, $\frac{1}{|B_\gamma|} \int\limits_{x'+B_\gamma} f(y)dy \le \alpha$ for any $\gamma \in \mathbb{Z}$. Let γ be an integer such that B_γ has the same radius as $B^j_\star$, then $x' + B_\gamma = B^j_\star$. This implies

$$g(x) = \frac{1}{|B^j_\star|} \int\limits_{B^j_\star} f(y)dy = \frac{1}{|B_\gamma|} \int\limits_{x'+B_\gamma} f(y)dy \le \alpha.$$

Thus, the part (a) is proved.

The part (b) is proved as follows

$$||b_j||_{L^1(\mathbb{K}^d)} = \int\limits_{B^j_\star} \left| f(x) - \frac{1}{|B^j_\star|} \int_{B^j_\star} f(y)dy \right| dx \le 2 \int\limits_{B^j_\star} |f(y)|dy \le 2q^d\alpha|B^j|.$$

The part (c) is trivial. For (e), from the definition of balls $B^j_\star$ and the fact that M is of weak type $(1,1)$, it is obvious that

$$\sum_{j=1}^{\infty} |B^j_\star| = q^d \sum_{j=1}^{\infty} |B^j| \le q^d \cdot |E_\alpha| \le \frac{q^{2d}}{\alpha} ||f||_{L^1(\mathbb{K}^d)}.$$

□

Lemma 5. *Let $f \in L^1(\mathbb{K}^d)$ and $\alpha > 0$. There exists a finite or countable collection of pairwise disjoint balls $\{B^j\}_{j\ge1}$ such that* *(a)* $E_\alpha = \{x \in \mathbb{K}^d : Mf(x) > \alpha\} = \bigcup_{j=1}^{\infty} B^j$,

(b) $\alpha < \frac{1}{|B^j|} \int_{B^j} |f(y)|dy \leq q^d \alpha$ *for any* j.

Proof. Let $x \in E_\alpha$. Since $Mf(x) > \alpha$ and $f \in L^1(\mathbb{K}^d)$, there exists the largest integer $\gamma = \gamma(x)$ such that $\frac{1}{q^{d\gamma(x)}} \int_{x+B_{\gamma(x)}} |f(y)|dy > \alpha$. This means $\frac{1}{q^{d(\gamma(x)+1)}} \int_{x+B_{\gamma(x)+1}} |f(y)|dy \leq \alpha$ and thus $q^{d\gamma(x)} < \frac{1}{\alpha}||f||_{L^1(\mathbb{K}^d)} < \infty$. The collection of balls $\{x + B_{\gamma(x)} : x \in E_\alpha\}$ are of uniformly bounded measure. Thus, by Lemma 3, we could extract a countable collection of pairwise disjoint balls $\{B^j\}$ so that $E_\alpha \subset \bigcup_{j=1}^{\infty} B^j$. We observe that, for any $x \in E_\alpha$, then $x + B_{\gamma(x)} \subset E_\alpha$. Indeed, since $y \in x + B_{\gamma(x)}$ and Lemma 2-(ii) imply $y + B_{\gamma(x)} = x + B_{\gamma(x)}$. Obviously,

$$Mf(y) \geq \frac{1}{q^{d\gamma(x)}} \int_{y+B_{\gamma(x)}} |f(z)|dz = \frac{1}{q^{d\gamma}} \int_{x+B_{\gamma(x)}} |f(z)|dz > \alpha.$$

Thus $y \in E_\alpha$. So we have proved that $E_\alpha \supset \bigcup_{j=1}^{\infty} B^j$. Thus, claim (a) is proved.

Note that $\frac{1}{q^{d(\gamma(x)+1)}} \int_{x+B_{\gamma(x)+1}} |f(y)|dy \leq \alpha$, implying

$$\alpha < \frac{1}{|B^j|} \int_{B^j} |f(y)|dy = \frac{q^d}{|B^j_\star|} \int_{B^j} |f(y)|dy \leq \frac{q^d}{|B^j_\star|} \int_{B^j_\star} |f(y)|dy \leq q^d \alpha,$$

where $B^j_\star$ is the ball which has the same center as B^j but with the radius being q times the one of B^j. □

Corollary 3. *If* $f \in L^1(\mathbb{K}^d)$ *and* $\alpha > 0$*, then there exists a decomposition of* $\mathbb{K}^d$ *such that*

(a) $\mathbb{K}^d = \Omega \bigcup F$ *and* $\Omega \cap F = \emptyset$,

(b) $|\{x \in F : |f(x)| > \alpha\}| = 0$,

(c) $\Omega = \bigcup_{j=1}^{\infty} B^j$ *is the countable union of pairwise disjoint balls* $\{B^j\}$ *satisfying*

$$\alpha \leq \frac{1}{|B^j|} \int_{B^j} |f(x)|dx \leq q^d \alpha.$$

The following theorem gives one of the keys to our main results in the next sections. It contains the local field version of the Fefferman-Stein's duality inequality. From its proof, it is a bit surprise to know that, over local fields, the norm of the Hardy-Littlewood maximal operator from $L^1(\mathbb{K}^d)$ to $L^{1,\infty}(\mathbb{K}^d)$ is not greater than 1, which differs from the Euclidean case.

Theorem 3. (Fefferman-Stein's duality inequality) *For each* $1 < \ell < \infty$ *there exists a constant* $c_\ell > 0$ *such that, for arbitrary measurable functions* $\phi \geq 0$ *and* f *on* $\mathbb{K}^d$ *the following estimate holds true :*

$$\int_{\mathbb{K}^d} |Mf(x)|^\ell \phi(x)dx \leq c_\ell \cdot \int_{\mathbb{K}^d} |f(x)|^\ell M\phi(x)dx. \tag{2.3}$$

Here we assume that $0 \cdot (+\infty) = +\infty$.

Proof. If $M\phi(x) = \infty$ a.e. in a positive measure subset of $\mathbb{K}^d$ then (2.3) is trivial. If not, $M\phi(x)$ is the density of a positive measure μ, $d\mu(x) = M\phi(x)dx$ and ϕ is the density of a positive measure ν so that $d\nu(x) = \phi(x)dx$. So (2.3) means that M is bounded from $L^\ell(\nu)$ to $L^\ell(\mu)$. Now by theorem 1, (2.3) follows if we are able to show the (∞, ∞) result and M is of weak type $(1, 1)$.

Proof for the case (∞, ∞): If there exists $x \in \mathbb{K}^d$ so that $M\phi(x) = 0$, then $\phi(y) = 0$ a.e. $y \in \mathbb{K}^d$. Then $L^\infty(\nu) = \{0\}$, so there is nothing to prove. If $M\phi(x) > 0$ for some x and let $\alpha > ||f||_{L^\infty(\mu)}$, then $\int\limits_{\{|f|>\alpha\}} M\phi(y)dy = 0$, hence $|\{|f| > \alpha\}| = 0$. Thus $|f| \leq \alpha$ a.e. in $\mathbb{K}^d$, this implies $Mf(x) \leq \alpha$ and then $||Mf||_{L^\infty(\nu)} \leq \alpha$. Thus $||Mf||_{L^\infty(\nu)} \leq ||f||_{L^\infty(\mu)}$.

Proof for the case M is of weak type $(1, 1)$: We shall show that

$$\int\limits_{\{Mf(x)>\alpha\}} \phi(x)dx \leq \frac{1}{\alpha} \int\limits_{\mathbb{K}^d} |f(x)|(M\phi)(x)dx. \tag{2.4}$$

There exists a sequence of integrable functions f_γ such that $f_\gamma \to f$ a.e. and

$$\{x \in \mathbb{K}^d : \ Mf(x) > \alpha\} = \bigcup_{\gamma=1}^{\infty} \{x \in \mathbb{K}^d : \ Mf_\gamma(x) > \alpha\}.$$

So we can assume that f is integrable, has compact support, and $f \geq 0$. Applying Lemma 5 to f, there exists a finite or countable collection of pairwise disjoint balls $\{B^j\}$ satisfying

$$E_\alpha = \Big\{x \in \mathbb{K}^d : \ Mf(x) > \alpha\Big\} = \bigcup_{j=1}^{\infty} B^j$$

and $\alpha < \frac{1}{|B^j|} \int\limits_{B^j} |f(y)|dy \leq q^d\alpha$ for any j. Hence

$$\int_{\{Mf(x)>\alpha\}} \phi(x)dx \leq \sum_{j=1}^{\infty} \int\limits_{B^j} \phi(x)dx \leq \sum_{j=1}^{\infty} \frac{1}{\alpha} \cdot \frac{1}{|B^j|} \int\limits_{B^j} |f(y)|dy \int\limits_{B^j} \phi(x)dx$$

$$\leq \frac{1}{\alpha} \sum_{j=1}^{\infty} \int\limits_{B^j} |f(y)| \left(\frac{1}{|B^j|} \cdot \int\limits_{B^j} \phi(x)dx \right) dy \leq \frac{1}{\alpha} \sum_{j=1}^{\infty} \int\limits_{B^j} |f(y)|M\phi(y)dy \leq \frac{1}{\alpha} \int\limits_{E_\alpha} |f(y)|M\phi(y)dy.$$

So M is of weak type $(1, 1)$ from $L^\ell(\nu)$ to $L^\ell(\mu)$. □

We observe that if we take $\phi = 1$ on $\mathbb{K}^d$, then inequality (2.4) implies that the norm of the Hardy-Littlewood maximal operator from $L^1\left(\mathbb{K}^d\right)$ to $L^{1,\infty}\left(\mathbb{K}^d\right)$ is not greater than 1.

3. A WEIGHT PROBLEM OF MUCKENHOUPT

A natural question posed by B. Muckenhoupt (see [27]), concerning the Hardy-Littlewood maximal operator M, is the following: what is the characterization of the weight v for which M is bounded from $L^\ell(u)$ to $L^\ell(v)$ for some non-trivial u? Another problem is characterizing the weight u for which there is non-trivial weight v. In the Euclidean case, the complete answer to the first problem dues to Wo-Sang Young [33] and independently by A.E. Gatto, C.E. Gutiérrez [18]. The second problem was solved independently by J. L. Rubio de Francia [14] and L. Carleson, P. W. Jones [4]. For local field settings, recently in [12], we have given a necessary condition in a form of series of weight function v such that the Hardy-Littlewood maximal operator is bounded from $L^\ell(u)$ to $L^\ell(v)$ for some u. In what follows, we study the first problem over a local field and obtain characterizations of the weight $v \geq 0$ such that M is bounded from $L^\ell(u)$ to $L^\ell(v)$ for some non-trivial u.

Lemma 6. *Let v be a non-negative measurable function on $\mathbb{K}^d$ and $1 < \ell < \infty$. The following conditions are equivalent:*

(a) $\int\limits_{\mathbb{K}^d} \frac{v(x)}{(1+|x|^d)^\ell} < \infty$,

(b) $\sum\limits_{\gamma \in \mathbb{Z}} \frac{q^{d\gamma(\ell-1)}}{(1+q^{d\gamma})^\ell} \int\limits_{S_0} v(\beta^{-\gamma}x)dx < \infty$, *where β is an element of $\mathbb{K}^d$ such that $B_{-1} = \beta B_0$*

(c) $\begin{cases} \int\limits_{B_0} v(x)dx < \infty \\ \int\limits_{(B_0)^c} \frac{v(x)}{|x|^{d\ell}} dx < \infty, \end{cases}$ *where $(B_0)^c$ is the complement of the unit ball B_0.*

If v satisfies one of the above conditions, then we shall say that v is in $\mathcal{W}_\ell$ class.

Proof. If (a) or (b) holds, from (2.1), it follows that

$$\int\limits_{\mathbb{K}^d} \frac{v(x)}{(1+|x|^d)^\ell}\, dx = \sum_{\gamma\in\mathbb{Z}} \int_{S_\gamma} \frac{v(x)}{(1+q^{d\gamma})^\ell} = \sum_{\gamma\in\mathbb{Z}} \frac{q^{\gamma d(\ell-1)}}{(1+q^{d\gamma})^\ell} \int_{S_0} v(\beta^{-\gamma}x)dx < \infty,$$

so (a) and (b) are equivalent.

Next, it is clear that

$$\int\limits_{\mathbb{K}^d} \frac{v(x)}{(1+|x|^d)^\ell}\, dx = \sum_{\gamma\in\mathbb{Z}} \int_{S_\gamma} \frac{v(x)}{(1+q^{d\gamma})^\ell}$$

$$= \sum_{\gamma=1}^{+\infty} \frac{1}{(1+q^{d\gamma})^\ell} \int_{S_\gamma} v(x)dx + \sum_{\gamma=0}^{+\infty} \frac{q^{d\gamma}}{(1+q^{d\gamma})^\ell} \int\limits_{S_{-\gamma}} v(x)dx < \infty.$$

There exists a constant $c > 0$ so that $(1+q^{d\gamma})^\ell \geq q^{d\gamma\ell} \geq c(1+q^{d\gamma})^\ell$ for any $\gamma \geq 0$. This means

$$\int\limits_{\mathbb{K}^d} \frac{v(x)}{(1+|x|^d)^\ell}\, dx \approx \left(\int\limits_{(B_0)^c} \frac{v(x)}{|x|^{d\ell}} dx + \int\limits_{\mathcal{B}_0} v(x)dx \right).$$

Thus (a) and (c) are equivalent. □

It is clear that the sum of two functions in $\mathcal{W}_\ell$ class also belongs to the $\mathcal{W}_\ell$ class. Furthermore, from Lemma 6, we shall observe that $L^\infty(\mathbb{K}^d)$ is contained in $\mathcal{W}_\ell$ class. It is a direct consequence of the following Lemma.

Lemma 7. *If $1 < \ell < \infty$, then $\int\limits_{\mathbb{K}^d} \frac{dx}{(1+|x|^d)^\ell} < \infty$.*

Indeed, from (2.1) we obtain

$$\int\limits_{\mathbb{K}^d} \frac{dx}{(1+|x|^d)^\ell} = \sum_{\gamma\in\mathbb{Z}} \int\limits_{S_\gamma} \frac{dx}{(1+|x|^d)^\ell}.$$

Splitting the last sum into two summands for $\gamma > 0$ and $\gamma \leq 0$, and replacing $-\gamma$ by γ we get

$$\int\limits_{\mathbb{K}^d} \frac{dx}{(1+|x|^d)^\ell} = \sum_{0<\gamma\in\mathbb{Z}} \frac{q^{d\gamma}}{(1+q^{d\gamma})^\ell} \Big(1 - \frac{1}{q^d}\Big) + \sum_{0\leq\gamma\in\mathbb{Z}} \frac{q^{d\gamma(\ell-1)}}{(1+q^{d\gamma})^\ell} \Big(1 - \frac{1}{q^d}\Big) < \infty.$$

Theorem 4. *Let v be a non-negative measurable function on $\mathbb{K}^d$ with values in $[0;+\infty]$ and $1<\ell<\infty$. Then, the necessary and sufficient condition for v belonging to the $\mathcal{W}_\ell$ class is that there exists a finite almost everywhere, non-negative, measurable function u on $\mathbb{K}^d$, with values in $[0;+\infty]$, such that*

$$\int\limits_{\mathbb{K}^d} |Mf|^\ell v dx \leq C \int\limits_{\mathbb{K}^d} |f|^\ell u dx \tag{3.1}$$

for all $f \in L^\ell(u)$, where C is a constant depending only on ℓ, q, and d.

Proof. Suppose that there exits a finite almost everywhere, non-negative, and measurable function u such that (3.1) is valid. Let us consider the level sets of u, $E_\alpha = \{x \in \mathbb{K}^d : \ |u(x)| \leq \alpha\}$, where $\alpha > 0$. There is at least one $\alpha > 0$ so that E_α has positive measure, since if not, then $u = \infty$ a.e. in $\mathbb{K}^d$. So we can find a subset E of $\mathbb{K}^d$ with positive measure on which u is bounded. Furthermore, we can assume that $E \subset B_\gamma$ for some non-negative integer γ, because we could replace E by $E \cap B_\gamma$. Let $f = \chi_E$, then

$$\int\limits_{\mathbb{K}^d} |f(x)|^\ell u dx = \int\limits_{E} u dx \leq |E| \cdot \text{ess.sup}|u| < +\infty.$$

So $f \in L^\ell(u)$. On the other hand, let us take any $x \in \mathbb{K}^d$. If $x \in B_\gamma$ then $x + B_\gamma = B_\gamma$ (see Lemma 2-(ii)), so

$$Mf(x) \geq \frac{1}{q^{d\gamma}} \int\limits_{B_\gamma} |f(y)| dy \geq \frac{1}{q^{d\gamma}} \cdot \int\limits_{E} dx = \frac{|E|}{q^{d\gamma}}.$$

If $x \notin B_\gamma$ then $|x| = q^{\gamma'} > q^\gamma$, consequently

$$Mf(x) \geq \frac{1}{q^{d\gamma'}} \int\limits_{x+B'_\gamma} |f(y)| dy \geq \frac{|E|}{|x|^d}.$$

Thus

$$Mf(x) \geq \frac{|E|}{\max\{q^{\gamma d}, |x|^d\}} \geq \frac{1}{1+|x|^d} \cdot \frac{|E|}{q^{\gamma d}} \qquad (\forall x \in \mathbb{K}^d). \tag{3.2}$$

So

$$\int\limits_{\mathbb{K}^d} |Mf|^\ell v dx \geq \Big(\frac{|E|}{q^{d\gamma}}\Big)^\ell \int\limits_{\mathbb{K}^d} \frac{v(x)}{(1+|x|^d)^\ell} dx.$$

Since $f \in L^\ell(u)$, then (3.1) and (3.2) give v belonging to $\mathcal{W}_\ell$ class.

Conversely, let v be in $\mathcal{W}_\ell$ class. Let $v_1(x) = \max\{v(x), 1\}$. Then by Lemma 7, v_1 also belongs to $\mathcal{W}_\ell$. Since $v \leq v_1$, it is enough to prove that there exist $u < \infty$ a.e. and a constant $C = C(\ell, q, d) > 0$ such that

$$\int\limits_{\mathbb{K}^d} |Mf|^\ell v_1 dx \leq C \int\limits_{\mathbb{K}^d} |f|^\ell u dx \tag{3.3}$$

for any $f \in L^\ell(u)$.

Put $w(x) = (1+|x|^d)^{1-\ell}$. We shall prove that $M(wv_1) < \infty$ almost everywhere. Taking any $x \in \mathbb{K}^d$, then we can choose $\gamma_0 \geq 0$ so that $x \in B_{\gamma_0}$. For any $\gamma \in \mathbb{Z}$, let us consider two cases:

If $\gamma \geq \gamma_0$, then

$$\frac{1}{q^{d\gamma}} \int\limits_{x+B_\gamma} wv_1 dy \leq \frac{1}{q^{d\gamma}} \int\limits_{x+B_\gamma} \frac{v_1(y)}{(1+|y|^d)^{\ell-1}} dy \leq \frac{1+q^{d\gamma}}{q^{d\gamma}} \int\limits_{x+B_\gamma} \frac{v_1(y)}{(1+|y|^d)^\ell} dy \leq 2 \int\limits_{\mathbb{K}^d} \frac{v_1(y)}{(1+|y|^d)^\ell} dy < +\infty.$$

If $\gamma < \gamma_0$, then

$$\frac{1}{q^{d\gamma}} \int\limits_{x+B_\gamma} wv_1 dy \leq \frac{1}{q^{d\gamma}} \int\limits_{x+B_\gamma} \frac{v_1(y)}{(1+|y|^d)^{\ell-1}} dy \leq \frac{1}{q^{d\gamma}} \int\limits_{x+B_\gamma} v_1(y) \cdot \chi_{B_{\gamma_0}} dy \leq M(v_1 \chi_{B_{\gamma_0}})(x) < \infty.$$

The last inequality holds for almost everywhere $x \in \mathbb{K}^d$, because M is of weak type $(1,1)$. Thus we have showed that, for a.e. $x \in \mathbb{K}^d$, then

$$M(wv_1)(x) \leq \max \left\{ 2 \int\limits_{\mathbb{K}^d} \frac{v_1(y)}{(1+|y|^d)^\ell} dy,\ M(v_1 \chi_{B_{\gamma_0}})(x) \right\} < +\infty.$$

Now put $u = w^{-3} \cdot M(wv_1) \cdot \chi_{(B_{-1})^c} + |x|^{2d(1-\ell)} \cdot M(wv_1) \cdot \chi_{B_{-1}}$, where $B_{-1} = \{x \in \mathbb{K}^d :\ |x| \leq q^{-1}\}$, and $(B_{-1})^c$ is the complement of the ball B_{-1}. The function u is clearly measurable and finite a.e. in $\mathbb{K}^d$. Let $\gamma \in \mathbb{Z}$. We put $f_\gamma = f \cdot \chi_{S_\gamma}$ for $\gamma \geq 0$ and $f_{-1} = f \cdot \chi_{B_{-1}}$. Fefferman-Stein's duality inequality (2.3) implies

$$\int\limits_{|x| \leq q^\gamma} |Mf_\gamma|^\ell v_1 dx = \int\limits_{|x| \leq q^\gamma} |Mf_\gamma|^\ell \cdot (wv_1) \cdot (1+|x|^d)^{\ell-1} \leq C \cdot \left(1+q^{d\gamma}\right)^{\ell-1} \cdot \int\limits_{\mathbb{K}^d} |f_\gamma|^\ell M(wv_1).$$

Case 1. If $\gamma \geq 0$, then

$$\int\limits_{|x| \leq q^\gamma} |Mf_\gamma|^\ell v_1 dx \leq C \cdot \left(1+q^{d\gamma}\right)^{\ell-1} \cdot \int\limits_{\mathbb{K}^d} |f_\gamma|^\ell u(1+|x|^d)^{3(1-\ell)} dx$$

$$\leq C \cdot \left(1+q^{d\gamma}\right)^{2(1-\ell)} \cdot \int\limits_{S_\gamma} |f|^\ell u dx \leq C \cdot q^{2d\gamma(1-\ell)} \cdot \int\limits_{S_\gamma} |f|^\ell u dx = C q^{-2d|\gamma|(\ell-1)} \cdot \int\limits_{S_\gamma} |f|^\ell u dx.$$

Case 2. If $\gamma = -1$, then

$$\int\limits_{|x| \leq q^\gamma} |Mf_\gamma|^\ell v_1 dx \leq C \cdot \left(1+q^{d\gamma}\right)^{\ell-1} \cdot \int\limits_{\mathbb{B}_{-1}} |f|^\ell u \cdot |x|^{2d(\ell-1)} dx$$

$$\leq C q^{-2d(\ell-1)} \int_{\mathbb{K}^d} |f|^\ell u dx = C q^{-2d|\gamma|(\ell-1)} \int_{\mathbb{K}^d} |f|^\ell u dx.$$

Thus we have showed that for any integer $\gamma \geq -1$, then

$$\int\limits_{|x| \leq q^\gamma} |Mf_\gamma|^\ell v_1 dx \leq C q^{-2d|\gamma|(\ell-1)} \int_{\mathbb{K}^d} |f|^\ell u dx. \tag{3.4}$$

Let γ, γ' be integers with $\gamma \geq -1$, and let $x \in \mathbb{K}^d$ such that $|x| \geq q^{\gamma+1}$. We put $S' = S_\gamma$ if $\gamma \geq 0$, and $S' = B_{-1}$ if $\gamma = -1$. Then, for each $y \in x + B_{\gamma'} \cap S'$, we have

$$q^{\gamma'} \geq |x-y| \geq |x| - q^\gamma \geq \left(1 - \frac{1}{q}\right)|x|.$$

By Hölder's inequality

$$Mf_\gamma(x) = \sup_{\gamma' \in \mathbb{Z}} \frac{1}{q^{d\gamma'}} \int\limits_{x+B_{\gamma'}} |f_\gamma(y)| dy \leq \left(\frac{q}{q-1}\right)^d \cdot \frac{1}{|x|^d} \cdot \sup_{\gamma' \in \mathbb{Z}} \int\limits_{x+B_{\gamma'} \cap S_\gamma} |f(y)| dy$$

$$\leq \left(\frac{q}{q-1}\right)^d \cdot \frac{1}{|x|^d} \cdot \int_{S_\gamma} |f(y)|dy \leq \frac{C}{|x|^d} \cdot \left(\int_{S_\gamma} |f|^\ell u dx\right)^{1/\ell} \cdot \left(\int_{S_\gamma} u^{-\frac{1}{\ell-1}}\right)^{1-\frac{1}{\ell}}$$

for any $\gamma \geq 0$. In case $\gamma = -1$, the above inequality S_γ is replaced by B_{-1}.

Therefore

$$\int_{|x|\geq q^{\gamma+1}} |Mf_\gamma|^\ell v_1 dx \leq C \int_{S_\gamma} |f|^\ell u dx \cdot \left(\int_{S_\gamma} u^{-\frac{1}{\ell-1}}\right)^{\ell-1} \cdot \int_{|x|\geq q^{\gamma+1}} \frac{v_1(x)}{|x|^{d\ell}} dx.$$

for $\gamma \geq 0$ and S_γ is replaced by B_{-1} in case $\gamma = -1$. Since $\gamma \geq -1$ so $\{|x| \geq q^{\gamma+1}\}$ is contained in $(B_{-1})^c = S_0 \bigcup (B_0)^c$. From this, by v_1 belonging to $\mathcal{W}_\ell$ class, and by Lemma 6, the term $\int\limits_{|x|\geq q^{\gamma+1}} \frac{v_1(x)}{|x|^{d\ell}} dx$ is bounded by some constant C.

We now want to estimate the terms

$$\left(\int_{S_\gamma} u^{-\frac{1}{\ell-1}}\right)^{\ell-1} \quad \text{for} \quad \gamma \geq 0 \quad \text{and} \left(\int_{B_{-1}} u^{-\frac{1}{\ell-1}}\right)^{\ell-1} \quad \text{in case } \gamma = -1.$$

To do this, note that from $y \in S_\gamma \bigcup B_{-1}$ and $v_1(z) \geq 1$, we obtain

$$M(wv_1)(y) = \sup_{\gamma' \in \mathbb{Z}} \frac{1}{q^{d\gamma'}} \int_{y+B_{\gamma'}} \frac{v_1(z)}{(1+|z|^d)^{\ell-1}} \geq \frac{1}{q^{d\gamma}} \int_{B_\gamma} \frac{v_1(z)}{(1+|z|^d)^{\ell-1}} \geq \frac{1}{(1+q^{d\gamma})^{\ell-1}}.$$

Hence $M(wv_1)(y)^{-\frac{1}{\ell-1}} \leq 1 + q^{d\gamma}$, for all $y \in S_\gamma$ $(\gamma \geq 0)$, and for all $y \in B_{-1}$ when $\gamma = -1$.

From the definition of u, we now consider two cases:

Case 1. If $\gamma \geq 0$, then

$$\left(\int_{S_\gamma} u^{-\frac{1}{\ell-1}}\right)^{\ell-1} \leq \left(1+q^{d\gamma}\right)^{\ell-1} \left(\int_{S_\gamma} \left(1+q^{d\gamma}\right)^{-3} dx\right)^{\ell-1}$$

$$= \left(1+q^{d\gamma}\right)^{2(1-\ell)} q^{d\gamma(\ell-1)} \left(1-\frac{1}{q^d}\right)^{\ell-1} \leq Cq^{d\gamma(1-\ell)}.$$

Case 2. If $\gamma = -1$, then

$$\left(\int_{B_{-1}} u^{-\frac{1}{\ell-1}}\right)^{\ell-1} \leq \left(1+q^{-d}\right)^{\ell-1} \left(\int_{B_{-1}} |y|^{2d} dy\right)^{\ell-1} \leq Cq^{-d(\ell-1)},$$

where note that $\int\limits_{B_{-1}} |y|^{2d} dy = \sum\limits_{\gamma\leq -1} \int\limits_{S_\gamma} |y|^{2d} dy = \left(1-\frac{1}{q^d}\right) \sum\limits_{\gamma \geq 1} \frac{1}{q^{3d\gamma}} < \infty$.

Therefore

$$\int_{|x|\geq q^{\gamma+1}} |Mf_\gamma|^\ell v_1 dx \leq Cq^{-d|\gamma|(\ell-1)} \int_{\mathbb{K}^d} |f|^\ell u dx \tag{3.5}$$

for any $\gamma \geq -1$. From (3.4) and (3.5) we obtain

$$\int_{\mathbb{K}^d} |Mf_\gamma|^\ell v_1 dx \leq Cq^{-d|\gamma|(\ell-1)} \int_{\mathbb{K}^d} |f|^\ell u dx. \tag{3.6}$$

Applying Minkowski's inequality, we have

$$\left(\int_{\mathbb{K}^d} |Mf|^\ell v_1 dx\right)^{1/\ell} \le \sum_{\gamma=-1}^{\infty} \left(\int_{\mathbb{K}^d} |Mf_\gamma|^\ell v_1 dx\right)^{1/\ell}$$

$$\le C\left(\sum_{\gamma=-1}^{\infty} q^{-d|\gamma|(\ell-1)/\ell}\right) \cdot \int_{\mathbb{K}^d} |f|^\ell u dx \le C \int_{\mathbb{K}^d} |f|^\ell u dx.$$

This proves (3.3), and so the proof of the theorem is complete. □

By means of Theorem 4 and Theorem 3.5 in [12] stating that the Hardy-Littlewood maximal operator M is bounded from $L^\ell(\omega)$ to $L^\ell(\omega)$ if and only if ω is a $\mathcal{A}_\ell$ weight, we can obtain such an interesting result as $\mathcal{A}_\ell \subset \mathcal{W}_\ell$. In the Euclidean case, this is known as a result of Hunt, Muckenhoupt and Wheenden (see Lemma 1 in [19]).

Corollary 4. *Let $1 < \ell < \infty$ and ω be a $\mathcal{A}_\ell$ weight, then ω is in the $\mathcal{W}_\ell$ class.*

4. WEIGHTED INEQUALITIES FOR THE FEFFERMAN-STEIN VECTOR-VALUED MAXIMAL OPERATOR

Let $f = \{f_k\}_{k=1}^{\infty}$ (or $\{f_k\}$ for short) be a sequence of locally integrable functions on $\mathbb{K}^d$, $\overrightarrow{M}f = \{Mf_k\}$ and $|f(x)|_r = \left(\sum_{k=1}^{\infty} |f_k(x)|^r\right)^{1/r}$. Let $1 \le t, r < \infty$ and ω be a weight function on $\mathbb{K}^d$. We denote by $L^t_\omega(\ell^r)$ the space of all sequences $f = \{f_k\}$ of measurable functions on $\mathbb{K}^d$ with *norm*

$$||f||_{L^t_\omega(\ell^r)} := \left(\int_{\mathbb{K}^d} |f(x)|^t_r \omega(x) dx\right)^{1/t} < \infty.$$

The purpose of this section is to obtain the weighted weak and strong type norm inequalities for the Fefferman-Stein vector-valued maximal operator over local fields. These were first proved by C. Fefferman and E. Stein in [15], by Kenneth F. Andersen and Russel T. John in [1] (with weighted forms) for Euclidean spaces and by Loukas Grafakos, Liguang Liu, Dachun Yang in [17] for spaces of homogeneous type. The characterizing properties of local fields are significant in the proof of the following results.

Theorem 5. *Let M be the Hardy-Littlewood maximal operator.*

(a) Let $1 \le \ell \le r < \infty$. Then $\omega \in \mathcal{A}_\ell$, if and only if, there exists a positive constant $C = C(r, \ell, q, d)$ such that

$$\omega\left(\left\{x \in \mathbb{K}^d : |\overrightarrow{M}f|_r > \alpha\right\}\right) \le \frac{C}{\alpha^\ell} \int_{\mathbb{K}^d} |f(x)|^\ell_r \omega(x) dx, \tag{4.1}$$

for any $f = \{f_j\} \in L^\ell_\omega(\ell^r)$.

(b) If $1 < \ell \le r < \infty$, then $\omega \in \mathcal{A}_\ell$, if and only if, there is a positive constant $C = C(r, \ell, q, d)$ such that

$$\int_{\mathbb{K}^d} |\overrightarrow{M}f(x)|^\ell_r \omega(x) dx \le C \int_{\mathbb{K}^d} |f(x)|^\ell_r \omega(x) dx, \tag{4.2}$$

for any $f = \{f_j\} \in L^\ell_\omega(\ell^r)$.

Proof. Concerning the necessity of $\omega \in \mathcal{A}_\ell$, if $f = \{f_k\}$, where $f_k(x) = 0$ for $k = 2, 3, \ldots$, then according to (1.7) and (1.8), there is nothing to show. If not, then inequalities (4.1) and (4.2) will be shown as follows.

If $\ell = r$: then (4.2) is an easy consequence of (1.7) because

$$\int_{\mathbb{K}^d} |\overrightarrow{M} f(x)|_r^r \omega(x)dx = \int_{\mathbb{K}^d} \sum_{k=1}^{\infty} |Mf_k(x)|^r \omega(x)dx \tag{4.3}$$

$$= \sum_{k=1}^{\infty} \int_{\mathbb{K}^d} |Mf_k(x)|^r \omega(x)dx \le C_{r,q,d} \sum_{k=1}^{\infty} \int_{\mathbb{K}^d} |f_k(x)|^r \omega dx = C_{r,q,d} \int_{\mathbb{K}^d} |f(x)|_r^r \omega(x)dx.$$

If $\ell < r$ and $\alpha > 0$, then we can assume without loss of generality that $f \in \mathcal{S}(\ell^r)$, and the general case follows by a standard limiting argument. Since $|f(x)|_r$ is integrable on $\mathbb{K}^d$, Lemma 5 yields a finite or countable collection of pairwise disjoint balls $\{B_\star^j\}$ such that

$$|f(x)|_r \le \alpha \qquad \text{almost everywhere } x \notin B = \bigcup_{j=1}^{\infty} B_\star^j, \tag{4.4}$$

$$\alpha < \frac{1}{|B_\star^j|} \int_{B_\star^j} |f(x)|_r dx \le q^d \alpha \qquad \text{for } j = 1, 2, 3, \ldots. \tag{4.5}$$

Let $f = f' + f''$ where $f' = \{f'_k\}$, $f'_k(x) = f_k(x)\chi_{\mathbb{K}^d - B}(x)$. Applying Minkowski's inequality, we have $|\overrightarrow{M} f|_r \le |\overrightarrow{M} f'|_r + |\overrightarrow{M} f''|_r$ Then, (4.1) will be obtained if we can show that

$$\omega\left(\left\{x \in \mathbb{K}^d : \ |\overrightarrow{M} f'|_r > \alpha\right\}\right) \le \frac{C_{r,\ell,q,d}}{\alpha^\ell} \int_{\mathbb{K}^d} |f(x)|_r^\ell \omega(x)dx, \tag{4.6}$$

and

$$\omega\left(\left\{x \in \mathbb{K}^d : \ |\overrightarrow{M} f''|_r > \alpha\right\}\right) \le \frac{C_{r,\ell,q,d}}{\alpha^\ell} \int_{\mathbb{K}^d} |f(x)|_r^\ell \omega(x)dx. \tag{4.7}$$

Since $\omega \in \mathcal{A}_\ell$ and $\ell < r$, $\omega \in \mathcal{A}_r$ (Lemma 1). From (4.4), it follows that $|f'(x)|_r^r \le \alpha^{r-\ell} \cdot |f'(x)|_r^\ell$. By Chebyshev's inequality (see for example [16], [31] for these type of inequalities and for Riemann-Stieltjes integrals and distribution functions) and by (4.3) it is not difficult to see

$$\omega\left(\left\{x \in \mathbb{K}^d : \ |\overrightarrow{M} f'|_r > \alpha\right\}\right) \le \frac{1}{\alpha^r} \int_{\mathbb{K}^d} |\overrightarrow{M} f'(x)|_r^r \omega(x)dx \le \frac{C_{r,\ell,q,d}}{\alpha^r} \int_{\mathbb{K}^d} |f'(x)|_r^r \omega(x)dx$$

$$\le \frac{C_{r,\ell,q,d}}{\alpha^\ell} \int_{\mathbb{K}^d} |f'(x)|_r^\ell \omega(x)dx \le \frac{C_{r,\ell,q,d}}{\alpha^\ell} \int_{\mathbb{K}^d} |f(x)|_r^\ell \omega(x)dx.$$

In order to prove (4.7), we define $\overline{f} = \{\overline{f}_k\}$ as follows :

$$\overline{f}_k(x) = \begin{cases} \frac{1}{|B_\star^j|} \int_{B_\star^j} |f_k(y)|dy & \text{if } x \in B_\star^j, \ j = 1, 2, \ldots \\ 0 & \text{if } x \notin B. \end{cases}$$

For $x \in B_\star^j$

$$|\overline{f}(x)|_r = \left(\sum_{k=1}^{\infty} \left(\frac{1}{|B_\star^j|} \int\limits_{B_\star^j} |f_k(y)| dy \right)^r \right)^{1/r}$$

$$\leq \frac{1}{|B_\star^j|} \int\limits_{B_\star^j} \left(\sum_{k=1}^{\infty} |f_k(y)|^r \right)^{1/r} dy \leq \frac{1}{|B_\star^j|} \int\limits_{B_\star^j} |f(y)|_r dy \leq q^d \alpha,$$

(here we use (4.5) and Hölder's inequality).

For $x \notin B$, all $\overline{f}_k(x)$ are zero, so $|\overline{f}(x)|_r = 0$. Hence, the function $|\overline{f}|_r$ has support in B and is bounded by $q^d\alpha$. So if we use a similar argument to the proof of (4.6), and we replace the function f' by $\overline{f}$, then we shall obtain

$$\omega \left(\left\{ x \in \mathbb{K}^d : \ |\overrightarrow{M}\overline{f}|_r > \alpha \right\} \right) \leq \frac{C_{r,\ell,q,d}}{\alpha^\ell} \int\limits_{\mathbb{K}^d} |\overline{f}(x)|_r^\ell \omega(x) dx \leq C_{r,\ell,q,d}\, \omega(B). \tag{4.8}$$

Now we estimate $\omega(B)$. If $\ell = 1$, then the $\mathcal{A}_1$ condition and (4.5) yield

$$\omega(B_\star^j) \leq \frac{\omega(B_\star^j)}{|B_\star^j|} \cdot \frac{1}{\alpha} \int\limits_{B_\star^j} |f(x)|_r dx \leq \frac{C_{r,\ell,q,d}}{\alpha} \int\limits_{B_\star^j} |f(x)|_r \omega(x) dx,$$

thus $\omega(B) \leq \frac{C_{r,\ell,q,d}}{\alpha} \int\limits_B |f(x)|_r \omega(x) dx$.

If $\ell > 1$, then Hölder's inequality and (4.5) give

$$\omega(B_\star^j) \leq \frac{1}{\alpha^\ell} \cdot \frac{1}{|B_\star^j|^\ell} \left(\int_{B_\star^j} |f(x)|_r dx \right)^\ell \cdot \int_{B_\star^j} \omega dx$$

$$\leq \frac{1}{\alpha^\ell} \left(\int_{B_\star^j} |f(x)|_r^\ell \omega dx \right) \cdot \left(\frac{1}{|B_\star^j|} \int_{B_\star^j} \omega(x)^{-\frac{1}{\ell-1}} dx \right)^{\ell-1} \cdot \int_{B_\star^j} \omega dx \leq \frac{C_{r,\ell,q,d}}{\alpha^\ell} \int_{B_\star^j} |f(x)|_r^\ell \omega dx,$$

because ω is in $\mathcal{A}_\ell$ class.

Thus, we have proved that

$$\omega(B) \leq \frac{C_{r,\ell,q,d}}{\alpha^\ell} \int\limits_{\mathbb{K}^d} |f(x)|_r^\ell \omega dx. \tag{4.9}$$

Hence, (4.8) and (4.9) give

$$\omega \left(\left\{ x \in \mathbb{K}^d : \ |\overrightarrow{M}\overline{f}|_r > \alpha \right\} \right) \leq \frac{C_{r,\ell,q,d}}{\alpha^\ell} \int\limits_{\mathbb{K}^d} |f(x)|_r^\ell \omega dx. \tag{4.10}$$

We shall now prove that $\left|\overrightarrow{M}f''(x)\right|_r \leq \left|\overrightarrow{M}\overline{f}(x)\right|_r$ for $x \notin B = \bigcup\limits_{j=1}^{\infty} B_\star^j$. In fact, it is enough to prove the inequalities $Mf_k''(x) \leq M\overline{f_k}(x)$ for any positive integer k and $x \notin B$. An interesting property on local fields, which differs from the Euclidean case, will be used here to verify this. Indeed,

$$Mf_k''(x) = \sup_{\gamma \in \mathbb{Z}} \frac{1}{|B_\gamma|} \int\limits_{x+B_\gamma} |f_k''(y)| dy,$$

and for any fixed $\gamma \in \mathbb{Z}$

$$\frac{1}{|B_\gamma|}\int\limits_{x+B_\gamma} |f_k''(y)|dy = \frac{1}{|B_\gamma|}\sum_{j\in J}\int\limits_{B_\star^j\cap x+B_\gamma} |f_k''(y)|dy,$$

where $J = \{j = 1, 2, \dots :\ B_\star^j \cap x + B_\gamma \neq \emptyset\}$. For each $y \in B_\star^j \cap x + B_\gamma$,

$$\overline{f}_k(y) = \frac{1}{|B_\star^j|}\int\limits_{B_\star^j} |f_k(z)|dz \geq \frac{1}{|B_\star^j|}\int\limits_{B_\star^j\cap x+B_\gamma} |f_k''(z)|dz$$

So

$$\frac{1}{|B_\gamma|}\int\limits_{x+B_\gamma} |f_k''(y)|dy \leq \frac{1}{|B_\gamma|}\sum_{j\in J}\int\limits_{B_\star^j} |\overline{f}_k(y)|dy. \tag{4.11}$$

Note that for $j \in J$, Lemma 2-iii implies that $B_\star^j \subset x + B_\gamma$ or $B_\star^j \supset x + B_\gamma$. Since $x \notin B$, then $B_\star^j \subset x + B_\gamma$. Thus for any $x \notin B$, we get $\bigcup\limits_{j\in J} B_\star^j \subset x + B_\gamma$. From this and (4.11) it follows that

$$\frac{1}{|B_\gamma|}\int\limits_{x+B_\gamma} |f_k''(y)|dy \leq \frac{1}{|B_\gamma|}\int\limits_{x+B_\gamma} |\overline{f}_k(y)|dy \qquad \text{for } x \notin B.$$

This means that $Mf_k''(x) \leq M\overline{f_k}(x)$ for any $x \notin B$ and positive integer k.

Now from (4.8) and (4.9) we obtain

$$\omega\left(\left\{x \in \mathbb{K}^d :\ |\overrightarrow{M}f''|_r > \alpha\right\}\right) \leq \omega(B) + \omega\left(\left\{x \in \mathbb{K}^d :\ |\overrightarrow{M}\overline{f}(x)|_r > \alpha\right\}\right) \leq \frac{C_{r,\ell,q,d}}{\alpha^\ell}\int\limits_{\mathbb{K}^d} |f(x)|_r^\ell \omega(x)dx.$$

Thus, (4.7) holds and so does (5). So we have proved that, if $\omega \in \mathcal{A}_\ell$ then (4.1) holds for any $\ell \leq r < \infty$.

If $r > \ell > 1$ and $\omega \in \mathcal{A}_\ell$, then $\omega \in \mathcal{A}_{\ell'}$ for any $\ell < \ell' \leq r$ and so (4.1) holds for any $\ell < \ell' \leq r$. Again, by Lemma 1, $\omega \in \mathcal{A}_{\ell''}$ for some $1 < \ell'' < \ell$. So (4.1) holds for $\ell = \ell''$. Hence, theorem 2 yields (4.2) for ℓ. □

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