ISSN 2070-0466, p-Adic Numbers, Ultrametric Analysis and Applications, 2009, Vol. 1, No. 4, pp. 338–351.

RESEARCH ARTICLES

Arithmetic Gravity: An Approach to Adelic Physics Via Algebraic Spaces

René Schmidt[**]

Mathematisches Institut der Westfälischen Wilhelms-Universität, D-48149 Münster, Germany

Received October 14, 2008

Abstract—In this paper we present an approach to adelic physics via algebraic spaces. Relative algebraic spaces $X \to S$ are considered as fundamental objects which describe space-time. This yields a number field invariant formulation of general relativity which, in the special case $S = \operatorname{Spec}\mathbb{C}$, may be translated back into the language of manifolds. With regard to adelic physics the case of an excellent Dedekind scheme S as base scheme is of interest (e.g. $S = \operatorname{Spec}\mathbb{Z}$). Some solutions of the arithmetic Einstein equations are studied.

DOI: 10.1134/S2070046609040062

Key words: *number field invariant physics, arithmetic gravity, Néron models.*

1. INTRODUCTION

Unless otherwise specified, let $K \subset \mathbb{R}$ be an algebraic number field (i.e. a finite algebraic extension of $\mathbb{Q}$), and let $\mathcal{O}_K$ be the ring of integral numbers of K (i.e. the integral closure of $\mathbb{Z}$ in K). For example, think of $\mathcal{O}_K = \mathbb{Z}$ and $K = \mathbb{Q}$.

Since 1987, there have been many interesting applications of p-adic numbers in physics. In his influential paper [11], I.V. Volovich draws the vision of number theory as the ultimate physical theory, where numbers are proposed as the fundamental entities of the universe. It is argued that the development of physics over arbitrary (number) fields might be necessary. In particular, this implies the incorporation of p-adic numbers in physical theories. Since then, many p-adic and even *adelic* models have been constructed. Adeles enable us to regard real and p-adic numbers simultaneously. More precisely, an adele is an infinite tuple

$$x = (x_2, \ldots, x_p, \ldots, x_\infty),$$

where $x_\infty \in \mathbb{R}$ and $x_p \in \mathbb{Q}_p$ with the restriction that one has $x_p \in \mathbb{Z}_p$ for all but a finite set of primes. In a certain way, these adelic models unify the ordinary (i.e. $\mathbb{R}$-valued) and p-adic models.

Adelic models of gravity are the starting point of this paper. But, instead of working directly with adeles and the respective adelic space-time models as it is usually done, we will study a new, purely geometric approach to adelic physics based on relative algebraic spaces $X \to S$, $S = \operatorname{Spec}\mathcal{O}_K$. However, there are close relations between these two approaches as it may be seen in the following example.

Example 7. Let us choose $K = \mathbb{Q}$. Consequently, $\mathcal{O}_K = \mathbb{Z}$ and $S = \operatorname{Spec}\mathbb{Z}$. Furthermore assume that the relative algebraic space X over S is representable by a smooth S-scheme, i.e. let us consider a smooth morphism $\pi : X \to \operatorname{Spec}\mathbb{Z}$ of schemes. Set-theoretically, $\operatorname{Spec}\mathbb{Z}$ consists of infinitely many closed points (one point for each prime number p) plus one generic point which we will denote by ∞, and which corresponds to the zero ideal of $\mathbb{Z}$. Furthermore, X may be viewed as union $\bigcup_p \pi^{-1}(p) \cup \pi^{-1}(\infty)$ of the fibres of π. In our arithmetic setting (and in analogy to complex algebraic geometry), a "physical point" x is given by an S-valued point of X, i.e. by a section $s : \operatorname{Spec}\mathbb{Z} \hookrightarrow X$ of the structure morphism π

[*] The text was submitted by the author in English.

[**] E-mail: rene.schmidt1@gmx.net

(i.e. $\pi \circ s = \mathrm{id}$). More precisely, x is given by the image of s. But, set-theoretically, s cuts out one closed point in each fibre. Thus, in analogy to the adelic situation, an S-valued point x may be viewed as a set of points:

$$x = \{x_2, \dots, x_p, \dots, x_\infty\}.$$

Furthermore, according to the point of view of adelic physics, each archimedean point (resp. each morphism over the archimedean prime spot at infinity) is only the archimedean component of an adelic point (resp. an adelic morphism). In short, everything in the archimedean world comes from the adelic level. If now $\varphi : Y \to \mathrm{Spec}\mathbb{Z}$ is an arbitrary smooth S-scheme and if we denote by Y_K the pre-image $\varphi^{-1}(\infty)$ of ∞ under φ, the above extension property (from the archimedean to the adelic level) reads as follows in algebraic geometry:

For every K-morphism $f_K : Y_K \to X_K$, there is an S-morphism $f : Y \to X$ which extends f_K. $(\star)$

All in all, instead of adeles, the set $X(S)$ of S-valued points of an algebraic space $X \to S$ is the set of interest in our approach. The objective of this paper is the investigation of a new approach to general relativity based on algebraic spaces. The condition $(\star)$ makes clear why Néron models will occur naturally. Let us finally remark that it is straightforward to include Yang-Mills fields in this framework using the notion of torsors (which are the algebraic geometric analogue of the differential geometric principal bundles).

2. THE ARITHMETIC SPACE-TIME

According to the theory of general relativity, space-time may be described by means of a differentiable manifold. Thereby, gravity is encoded in a metrical tensor g which satisfies the Einstein equations. More precisely, our starting point are the complex gravitational field equations. Then, any solution of the Einstein equations gives rise to a complex manifold. For technical reasons, we will once and for all assume that this classical space-time manifold may be realized as a compact complex manifold $\mathfrak{X}$ which is *Moishezon*. The latter condition means that

$$\mathrm{transdeg}_{\mathbb{C}}\Big(K(\mathfrak{X})\Big) = \dim_{\mathbb{C}} \mathfrak{X},$$

where $K(\mathfrak{X})$ denotes the field of meromorphic functions on $\mathfrak{X}$. For example, all algebraic manifolds fulfill this equation. Therefore, following the ideas of [1], where it is among other things argued that one should restrict to algebraic manifolds in quantum cosmology, our assumption seems not too restrictive. However, let us at least mention that there are Moishezon manifolds which are not algebraic. The technical reason why we restrict attention to Moishezon manifolds is the following beautiful theorem due to Artin.

Theorem 1. *There is an equivalence of categories*

$$\Big(\textit{Moishezon manifolds}\Big) \leftrightsquigarrow \Big(\textit{smooth, proper algebraic spaces over } \mathbb{C}\Big).$$

This theorem enables us to consider the ordinary complex space-time manifold $\mathfrak{X}$ as a complex algebraic space. Now the following observation is crucial. While, on the level of manifolds, the theory is essentially adapted to the complex numbers, the language of algebraic spaces offers to possibility to replace $\mathbb{C}$ by any commutative ring.

In 1987, I. V. Volovich suggested that a fundamental physical theory should be formulated in such a way that it is invariant under change of the underlying number field (see [11]). This motivates the following program which will be studied within the first part of this paper:

1. Replace the pair $(\mathfrak{X}, g)$ consisting of a (complex) manifold $\mathfrak{X}$ and a metric g by a pair

$$\Big(X \to S, g\Big),$$

where X is a smooth, separated algebraic space over a base S, and where g is a metric over X (see Definition 18).

2. Starting from exactly the same physical principles as in the realm of manifolds, deduce the equations of Einstein's theory of general relativity in the setting of algebraic spaces over an arbitrary base S (thus realizing a number field invariant theory). Determine the pair $(X \to S, g)$ in such a way that Einstein's equations are fulfilled.

3. Investigate properties of hypothetical space-time models $(X \to S, g)$ depending on the choice of the base S.

Remark 1. Principally, there are many interesting possible choices for S. For example, there is the case of positive characteristic, i.e. S might be chosen as the spectrum of a (finite) field or as function field of an algebraic curve over a finite field. However, in those models $(X \to S, g)$, which will be studied within the bounds of this paper, we will choose S to be representable by an excellent Dedekind-scheme with field of fractions K of characteristic zero. Then the following two cases are of interest:

1. S is Zariski zero-dimensional and given by the spectrum $\mathrm{Spec}K$ of a field K of characteristic zero. Especially in the case $K = \mathbb{C}$, everything may be translated back into the language of manifolds (by Theorem 1).

2. S is Zariski one-dimensional. In this case we are interested in the choice $S = \mathrm{Spec}\mathcal{O}_K$, where $\mathcal{O}_K \subset K$ is the ring of integral numbers of an algebraic number field K (e.g. $K = \mathbb{Q}$ and $\mathcal{O}_K = \mathbb{Z}$).

But what is the physics behind the choice $S = \mathrm{Spec}\mathcal{O}_K$? Why should we consider number fields instead of real or complex numbers? Following the ideas of B. Dragovich, V.S. Vladimirov, I.V. Volovich and many others (see, e.g. [1, 3–7, 10, 11]), let us state at least two arguments at this place. The first argument concerns the process of measurement. While it is not clear at all whether transcendental numbers can be the result of a measurement, integral (or rational) numbers can. Second, we know from Einstein that gravity is encoded in deformations of space-time scales (described by means of the metrical tensor g). Looking at the energy scale that we experience, it is an empiric fact that we may assume that gravity is completely encoded in the archimedean scale and that non-archimedean, $\mathfrak{p}$-adic scales may be neglected. Nevertheless, there is no reason why this should be true on all energy scales down to the Planck scale. It is an appealing project to study physical models where not only the ordinary, archimedean degrees of freedom are taken into consideration, but also the $\mathfrak{p}$-adic, non-archimedean degrees of freedom. Physically, the adelic approach means:

There is one degree of freedom per primespot and dimension. $(*)$

As already indicated in Example 7, the principle $(*)$ may as well be realized by considering algebraic spaces over $\mathcal{O}_K$. This motivates the following Definition 16 (whose physical motivation will be illustrated in Remark 2). Recall that, given two relative algebraic spaces $X \to S$ and $Y \to S$, we denote by $X(Y)$ the set of S-morphisms $Y \to X$. Furthermore recall that for an algebraic space $\pi : X \to S$ we denote the fibre of π over the generic point of S by X_K (physically this generic fibre represents the archimedean component of the algebraic space).

Definition 16. Let S be an excellent Dedekind scheme with field of fractions K of characteristic zero. Consider a pair $(X \to S, g)$ consisting of:

- a smooth, separated algebraic space $\pi : X \to S$ over S
- a metric g on X (see Definition 18)

such that the following conditions are fulfilled:

(i) g satisfies the Einstein equations, Definition 23.

(ii) For each smooth algebraic space $Y \to S$ and each K-morphism $u_K : Y_K \to X_K$ there is an S-morphism $u : Y \to X$ extending u_K.

Then the pair $(X \to S, g)$ is called a *model of type (GR)*.

Corollary 1. *In the setting of Definition 16, let us assume that the algebraic space $\pi : X \to S$ is representable by a smooth and separated S-scheme. Then, the morphism u in Definition 16 (ii) is uniquely determined. In other words, $X \to S$ is the* Néron model *of its generic fibre X_K (see [2], Def. 1.2/1). In particular, the following statements hold:*

1. *If u_K is an isomorphism so is u.*
2. *For each étale S-scheme S' with field of fractions K' the canonical map $X(S') \to X_K(K')$ is bijective.*

Proof. In order to prove the uniqueness assertion let us choose two morphisms u, v extending u_K. Using the separatedness of $X \to S$ we conclude from [9], Prop. 3.3.11, that u and v are equal if they coincide on a dense subset of Y. Therefore, it suffices to show that the generic fibre Y_K of Y is dense in Y. This may be done as follows: Due to smoothness, the structure morphism $f : Y \to S$ is an open map of topological spaces (use [2], Prop. 2.4/8). The openness of f implies that the pre-image $f^{-1}(D)$ of any dense subset D of S is dense in Y. As the generic point of S is dense in S we are done. Consequently, $X \to S$ is the Néron model of its generic fibre.

The statements 1. and 2. follow directly from the universal property of Néron models. For example, choose $Y = S'$ in order to see 2.. □

Remark 2. If $S = \mathrm{Spec}K$ is the spectrum of a field K, condition (ii) of Definition 16 is empty. If furthermore $K = \mathbb{R}$, any model of type (GR) induces a solution of Einstein's theory of general relativity (by evaluation at $\mathbb{R}$-valued points). This explains the label *model of type (GR)*, because (GR) shall remind of general relativity. However, in the case $S = \mathrm{Spec}\mathcal{O}_K$ we arrive at the following physical interpretation:

- Condition (ii) implements the "adelic" point of view.

  In order to see this, let us choose $S = \mathrm{Spec}\mathbb{Z}$ and therefore $K = \mathbb{Q}$. Recall that the generic fibre X_K of X represents the archimedean component. Then condition (ii) says that the archimedean world is only the projection from the "adelic" level to the archimedean component. In truth, everything is defined over all prime spots, and there is one degree of freedom per prime spot.

- We saw in Corollary 1 that condition (ii) implies a canonical bijection $X_K(K) = X(S)$. Recall that $X_K(K)$ is the set of archimedean points, and that $X(S)$ is the set of "adelic" points. In the special case $K = \mathbb{Q}$, the bijection $X_K(K) \cong X(S)$ means exactly that every archimedean point $x_\infty \in X_K(K)$ of X is in truth only the archimedean element x_∞ of an infinite set of points $x = \{x_2, \ldots, x_p, \ldots, x_\infty\} \in X(S)$. Finally, the first statement of Corollary 1 reflects the physically crucial statement that any "deformation" of the archimedean component by means of isomorphisms extends to the "adelic" level.

Furthermore, we immediately obtain the interesting result that the pair $(X \to S, g)$ cannot be the flat Minkowski space-time if we are in the "adelic" situation $S = \mathrm{Spec}\mathcal{O}_K$.

Proof. Let $S = \mathrm{Spec}\mathcal{O}_K$ and assume that $(X \to S, g)$ describes the flat, topologically trivial Minkowski space-time. Then

- $g = \mathrm{diag}(\pm 1, \pm 1, \pm 1, \pm 1)$ and
- $X = \mathbb{A}^n_S$ or $X = \mathbb{P}^n_S$ depending on whether we work projective or not. Recall that the affine space $\mathbb{A}^n_S$ may be regarded as the algebraic geometric analogue of flat space. In order to see this, let $S = \mathrm{Spec}R$ be the spectrum of a commutative ring R. Then $\mathbb{A}^n_S = \mathrm{Spec}R[T_1, \ldots, T_n]$ is the spectrum of a polynomial ring in n variables. Consequently, $\mathbb{A}^n_S(S) = \mathrm{Hom}_R(R[T_1, \ldots, T_n], R) = R^n$. In the special case $R = \mathbb{K}$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$, the space-time induced by $\mathbb{A}^n_S$ is the flat manifold $\mathbb{A}^n_S(S) = \mathbb{K}^n$.

But if $X = \mathbb{A}^n_S$, then $K^n \cong X_K(K) \neq X(S) \cong \mathcal{O}^n_K$, and if $X = \mathbb{P}^n_S$, not every morphism u_K extends to an S-morphism u (see [2], Example 3.5/5). Therefore, the flat, topologically trivial Minkowski space-time is impossible. □

Next, let us introduce a particularly simple class of models of type (GR).

Definition 17. Let $(X \to S, g)$ be a model of type (GR) in the sense of Definition 16, and let X_K be the generic fibre of X. Then the pair $(X \to S, g)$ is called a *model of type (SR)*, if in addition the following condition holds:

- (iii) X_K is a commutative K-group (see [2], Def. 4.1/2).

More precisely, the K-group X_K should be considered as a K-torsor under X_K (in the sense of [2], chapter 6.4). Physically, this means that the special choice of a zero element of the group is forgotten as it should be for physical reasons.

In order to generalize this notion slightly, one may also admit K-torsors X_K under K-groups $G_K \neq X_K$. However, we will restrict attention to the case $G_K = X_K$. Definition 17 is motivated by *special relativity* with electromagnetism: The Minkowski space-time of special relativity naturally carries an additive, commutative group structure, and the gauge group of electromagnetism is commutative, too. This explains the label *model of type (SR)*, because (SR) shall remind of special relativity. One can prove that the following statements are true for *all* models of type (SR).

Proposition 1. *Let $(X \to S, g)$ be a model of type (SR). Then the following statements are true:*

1. *$X \to S$ is étale-invariant. More precisely, this statement means the following: Let $\varphi : X \to X$ be an étale S-morhpism, and let $(X' \to S', g')$ be the pair obtained from $(X \to S, g)$ by base change with an étale morphism $S' \to S$. Then, $(X \to S, \varphi^* g)$ and $(X' \to S', g')$ are models of type (SR), too.*

   Proof. The crucial fact is that Néron models are compatible with étale base change ([2], Prop. 1.2/2 c)). □

2. *X cannot be the flat, topologically trivial Minkowski space (see above).*

3. *The archimedean component $X_K(K)$ is bounded with respect to all $\mathfrak{p}$-adic norms. In the special case $K = \mathbb{Q}$ and under the assumption that there is a closed immersion $X_K \hookrightarrow \mathbb{A}^n_K$, this is the following statement: For each prime number p, the p-adic manifold $X_K(\mathbb{Q}_p)$ is a bounded subset of some $\mathbb{Q}_p^n$ with respect to the canonical p-adic norm $|\cdot|_p$.*

   Proof. We know that X_K possesses a global Néron model. Consequently, the local Néron models exist ([2], Prop. 1.2/4). Therefore, due to [2], Thm. 10.2/1, it is necessary that $X_K(K)$ or even the continuum $X_K(\hat{K})$ is bounded. □

4. *The archimedean component $X_K(K)$ carries a discrete geometry, if X_K is quasi-compact, because in this case X_K is an Abelian variety (due to the following statement 5.)). Therefore, the Mordell-Weil theorem tells us that $X_K(K)$ is a finitely generated abelian group, i.e.*

   $$X_K(K) \cong \mathbb{Z}^d \oplus \mathbb{Z}/(p_1^{\nu_1}) \oplus \cdots \oplus \mathbb{Z}/(p_s^{\nu_s})$$

   for some prime numbers $p_i \in \mathbb{N}$ and integers $d, s, \nu_i \in \mathbb{N}$. In the special case $d = 0$, $X_K(K)$ consists of only finitely many points.

5. *If we do not demand the quasi-compactness of the archimedean component X_K of X, one can prove that X_K possesses a Néron model if and only if there is an exact sequence*

$$0 \longrightarrow T_K \longrightarrow X_K \longrightarrow A_K \longrightarrow 0$$

over some algebraic closure of K, where T_K is an algebraic torus and A_K is an abelian variety. While A_K is $\mathfrak{p}$-adically bounded, T_K is not. We interpret A_K as space part and the torus T_K as an internal, gauge group part which we therefore associate with electromagnetism. Thus, X_K should appear as A_K-torsor under T_K (which is the algebraic geometric analogue of the differential geometric principal bundle of gauge theory).

6. *On the "adelic" level there is some kind of entanglement of dimensions. For example, it is in general not possible to diagonalize the metric at the "adelic" points of $X(S)$.*

Let us remark that the statements 1. and 2. are also true for models of type (GR).

3. THE ARITHMETIC EINSTEIN EQUATIONS

Let $X \to S$ be a smooth, separated S-scheme of relative dimension n. The purpose of this section is the derivation of the fundamental equations of general relativity in our algebraic geometric setting. As the ordinary differential geometric Einstein equations are differential equation, we must expect that this holds in algebraic geometry, too. For the basic notions concerning smoothness and differential calculus in algebraic geometry we refer the reader to chapter 2 of [2]. Crucial are the following notions.

$\Omega^1_{X/S}$ sheaf of (relative) differential forms
$\mathcal{T}_{X/S} := \mathcal{H}om_{\mathcal{O}_X}\left(\Omega^1_{X/S}, \mathcal{O}_X\right)$ sheaf of (relative) vector fields
$T_{X/S} := \mathbb{V}\left(\Omega^1_{X/S}\right)$ (relative) tangent bundle .

One can prove that

$$\Gamma(T_{X/S}/U) := \mathrm{Hom}_X(U, T_{X/S}) \cong \mathcal{T}_{X/S}(U)$$

for every Zariski open subset $U \subset X$. Therefore vector fields correspond to sections of the tangent bundle (as one is used to from differential geometry).

3.1. The Metric Tensor

Due to smoothness, the sheaves $\Omega^1_{X/S}$ and $\mathcal{T}_{X/S}$ are locally free ([2], Prop. 2.2/5). Let us fix a local base $\{\omega^i\}$ of $\Omega^1_{X/S}$ which is dual to the local base $\{\partial_i\}$ of $\mathcal{T}_{X/S}$.

Definition 18. Let $g : T_{X/S} \times_X T_{X/S} \to \mathbb{A}^1_X$ be an X-morphism which is bilinear. Equivalently, g may be interpreted as a global section of $\Omega^{\otimes 2}_{X/S}$. Locally, we may write

$$g = \sum_{1 \le i,j \le n} g_{ij}\, \omega^i \otimes \omega^j \in \Omega^{\otimes 2}_{X/S}, \qquad g_{ij} \in \mathcal{O}_X.$$

Then g is called a *metric* if the following conditions hold for any sufficiently small open subset of X:

(i) The matrix (g_{ij}) is symmetric, i.e. $g_{ij} = g_{ji}$.

(ii) The matrix (g_{ij}) is invertible, i.e. $\det(g_{ij}) \in \mathcal{O}^*_X$.

3.2. Covariant Derivation

Definition 19. Let $\nabla : T_{X/S} \times_X T_{X/S} \to T_{X/S}$ be an X-morphism. Interpret ∇ as a map

$$\nabla : \mathcal{T}_{X/S}(X) \times \mathcal{T}_{X/S}(X) \to \mathcal{T}_{X/S}(X), \quad (\mathfrak{u}, \mathfrak{v}) \mapsto \nabla_{\mathfrak{u}}\mathfrak{v}.$$

Let us assume that ∇ is a $\mathcal{O}_S(S)$-bilinear map, where the $\mathcal{O}_X(X)$-module $\mathcal{T}_{X/S}(X)$ is viewed as $\mathcal{O}_S(S)$-module via the canonical morphism $\mathcal{O}_S(S) \to \mathcal{O}_X(X)$. Then ∇ is called a *covariant derivation* if the following conditions hold for all $f \in \mathcal{O}_X(X)$ and $\mathfrak{u}, \mathfrak{v} \in \mathcal{T}_{X/S}(X)$:

(i) $\nabla_{f\mathfrak{u}}\mathfrak{v} = f\nabla_{\mathfrak{u}}\mathfrak{v}$.

(ii) $\nabla_{\mathfrak{u}}(f\mathfrak{v}) = (\mathfrak{u}f)\mathfrak{v} + f\nabla_{\mathfrak{u}}\mathfrak{v}$.

Thereby, $\mathfrak{u}f := \mathfrak{u}(f) = (d_{X/S}f)(\mathfrak{u})$ is the canonical action of vector fields on functions (the differential $d_{X/S}$ is introduced and explained in [2], chap. 2.1).

Definition 20. Let ∇ be a covariant derivation, and let $\mathfrak{u}, \mathfrak{v}$ and $\mathfrak{w} \in \mathcal{T}_{X/S}(X)$.

1. $T(\mathfrak{u}, \mathfrak{v}) := \nabla_{\mathfrak{u}}\mathfrak{v} - \nabla_{\mathfrak{v}}\mathfrak{u} - [\mathfrak{u}, \mathfrak{v}]$ is called the *torsion* of ∇.
2. ∇ is called *torsion-free* if and only if $T(\mathfrak{u}, \mathfrak{v}) = 0$ for all $\mathfrak{u}, \mathfrak{v}$.
3. ∇ is called *metrical* if and only if $\mathfrak{u}g(\mathfrak{v}, \mathfrak{w}) = g(\nabla_{\mathfrak{u}}\mathfrak{v}, \mathfrak{w}) + g(\mathfrak{u}, \nabla_{\mathfrak{u}}\mathfrak{w})$ for all $\mathfrak{u}, \mathfrak{v}, \mathfrak{w}$.

In the same way as in differential geometry one proves that there exists a uniquely determined covariant derivation ∇ which is metrical and torsion-free, the *Levi-Civita connection*. The Levi-Civita connection is completely determined by the metrical tensor. More precisely, the Koszul formula holds.

$$2\,g(\nabla_{\mathfrak{u}}\mathfrak{v}, \mathfrak{w}) = \mathfrak{u}g(\mathfrak{v}, \mathfrak{w}) - \mathfrak{w}g(\mathfrak{u}, \mathfrak{v}) + \mathfrak{v}g(\mathfrak{w}, \mathfrak{u}) + g([\mathfrak{u}, \mathfrak{v}], \mathfrak{w}) + g([\mathfrak{w}, \mathfrak{u}], \mathfrak{v}) - g([\mathfrak{v}, \mathfrak{w}], \mathfrak{u})$$

3.3. Curvature

From now on let ∇ be the Levi-Civita connection. Then we may introduce the curvature tensor

$$R_{\mathfrak{u}\mathfrak{v}}(\mathfrak{w}) := \nabla_{\mathfrak{u}}\nabla_{\mathfrak{v}}\mathfrak{w} - \nabla_{\mathfrak{v}}\nabla_{\mathfrak{u}}\mathfrak{w} - \nabla_{[\mathfrak{u},\mathfrak{v}]}\mathfrak{w}.$$

Then the tensor $R_{\mathfrak{z}\mathfrak{w}\mathfrak{u}\mathfrak{v}} := g(R_{\mathfrak{u}\mathfrak{v}}(\mathfrak{w}), \mathfrak{z})$ is called the *Riemannian curvature tensor*. The Riemannian curvature tensor fulfills the following identities.

Proposition 2. *Let* $\mathfrak{u}, \mathfrak{v}, \mathfrak{w}, \mathfrak{z} \in \mathcal{T}_{X/S}(X)$. *Then:*

- $R_{\mathfrak{u}\mathfrak{v}\mathfrak{w}\mathfrak{z}} = -R_{\mathfrak{v}\mathfrak{u}\mathfrak{w}\mathfrak{z}}, \quad R_{\mathfrak{u}\mathfrak{v}\mathfrak{w}\mathfrak{z}} = -R_{\mathfrak{u}\mathfrak{v}\mathfrak{z}\mathfrak{w}}, \quad R_{\mathfrak{u}\mathfrak{v}\mathfrak{w}\mathfrak{z}} = R_{\mathfrak{w}\mathfrak{z}\mathfrak{u}\mathfrak{v}}$
- first Bianchi-identity: $R_{\mathfrak{z}\mathfrak{u}\mathfrak{v}\mathfrak{w}} + R_{\mathfrak{z}\mathfrak{v}\mathfrak{w}\mathfrak{u}} + R_{\mathfrak{z}\mathfrak{w}\mathfrak{u}\mathfrak{v}} = 0$
- second Bianchi-identity: $(\nabla_{\mathfrak{u}}R)_{\mathfrak{v}\mathfrak{w}} + (\nabla_{\mathfrak{v}}R)_{\mathfrak{w}\mathfrak{u}} + (\nabla_{\mathfrak{w}}R)_{\mathfrak{u}\mathfrak{v}} = 0$
  Thereby, $(\nabla_{\mathfrak{u}}R)_{\mathfrak{v}\mathfrak{w}}(\mathfrak{z}) := \nabla_{\mathfrak{u}}(R_{\mathfrak{v}\mathfrak{w}}(\mathfrak{z})) - R_{\nabla_{\mathfrak{u}}\mathfrak{v},\mathfrak{w}}(\mathfrak{z}) - R_{\mathfrak{v},\nabla_{\mathfrak{u}}\mathfrak{w}}(\mathfrak{z}) - R_{\mathfrak{v}\mathfrak{w}}(\nabla_{\mathfrak{u}}\mathfrak{z})$.

More generally, the covariant derivation of arbitrary tensor fields S and T with respect to a vector field $\mathfrak{v}$ may defined inductively as follows: $\nabla_{\mathfrak{v}}(S \otimes T) := \nabla_{\mathfrak{v}}S \otimes T + \nabla_{\mathfrak{v}}T \otimes S$.

Definition 21. The bi-quadratic form $k(\mathfrak{u}, \mathfrak{v}) := R_{\mathfrak{u}\mathfrak{v}\mathfrak{u}\mathfrak{v}}$ is called intersection curvature.

Proposition 3. *The Riemannian curvature tensor is completely determined by* k. *More precisely:*

1. $4 \cdot R_{\mathfrak{u}\mathfrak{v}\mathfrak{v}\mathfrak{w}} = k(\mathfrak{u} + \mathfrak{w}, \mathfrak{v}) - k(\mathfrak{u} - \mathfrak{w}, \mathfrak{v})$

2. $6 \cdot R_{\mathfrak{u}\mathfrak{v}\mathfrak{w}\mathfrak{z}} = R_{\mathfrak{u},\mathfrak{v}+\mathfrak{w},\mathfrak{v}+\mathfrak{w},\mathfrak{z}} - R_{\mathfrak{u},\mathfrak{v}-\mathfrak{w},\mathfrak{v}-\mathfrak{w},\mathfrak{z}} - R_{\mathfrak{v},\mathfrak{u}+\mathfrak{w},\mathfrak{u}+\mathfrak{w},\mathfrak{z}} + R_{\mathfrak{v},\mathfrak{u}-\mathfrak{w},\mathfrak{u}-\mathfrak{w},\mathfrak{z}}$

Corollary 2. *Let $X \to S$ be a smooth S-scheme with metric g. Then X is* flat*, i.e. $k(\mathfrak{u}, \mathfrak{v}) = 0$ for all $\mathfrak{u}, \mathfrak{v} \in \mathcal{T}_{X/S}(X)$, if and only if the Riemannian curvature tensor vanishes, i.e. $R = 0$.*

Definition 22. Let $X \to S$ be a smooth S-scheme with metric g and consider a local base $\{\partial_i\}$ of $\mathcal{T}_{X/S}$ and a local base $\{\omega^i\}$ of $\Omega^1_{X/S}$. The curvature tensor $R_{\mathfrak{u}\mathfrak{v}}(\mathfrak{w})$ is trilinear in $\mathfrak{u}, \mathfrak{v}, \mathfrak{w}$ and therefore induces linear maps $R_{\bullet\mathfrak{v}}(\mathfrak{w})$. Taking the trace finally yields the symmetric bilinear form ric

$$\mathrm{ric}(\mathfrak{v}, \mathfrak{w}) := \mathrm{Tr}(R_{\bullet\mathfrak{v}}(\mathfrak{w}))$$

which is called the *Ricci-form* of $(X \to S, g)$. Now consider the uniquely determined tensor Ric which is given by $g(\mathrm{Ric}(\mathfrak{u}), \mathfrak{v}) = \mathrm{ric}(\mathfrak{u}, \mathfrak{v})$ for all vector-fields $\mathfrak{u}, \mathfrak{v}$. The *scalar curvature* sc is by definition the trace

$$\mathrm{sc} := \mathrm{Tr}(\mathrm{Ric}(\bullet))$$

of the linear map $\mathrm{Ric}(\bullet)$. Furthermore, the *divergence* $\mathrm{div}(T)$ of any symmetric $(0, q)$-tensor $T := \sum T_{i_1 \ldots i_q} \omega^{i_1} \otimes \ldots \otimes \omega^{i_q}$ is defined as follows: The covariant derivation ∇T of T is a $(0, q+1)$-tensor $\nabla T := \sum T_{i_1 \ldots i_q ; j} \omega^{i_1} \otimes \ldots \otimes \omega^{i_q} \otimes \omega^j$. Then $\mathrm{div}(T)$ is the $(0, q-1)$-tensor which is obtained by lifting the new variable and contracting it:

$$(\mathrm{div}(T))_{i_1 \ldots i_{q-1}} = g^{i_q j} T_{i_1 \ldots i_q ; j}.$$

3.4. Einstein's Equation

Let $X \to S$ be a smooth S-scheme with metric g, and let ∇ be the Levi-Civita connection on X. Furthermore, let T denote the energy-stress tensor. This is a symmetric $(0, 2)$-tensor on X with $\mathrm{div}(T) = 0$. Then the equations of general relativity in our arithmetic setting are given by the following system of equations:

Definition 23.

$$\mathrm{ric} - \frac{1}{2} \mathrm{sc} \cdot g = \kappa T \,,$$

where $\kappa \in \mathcal{O}_S(S)$ is a constant. Now, having written down the equations of general relativity in the setting of arithmetic algebraic geometry, one can ask for solutions. Choosing $S = \mathrm{Spec}\mathbb{R}$ and assuming that there exists a solution of the corresponding algebraic geometric Einstein equations, it follows that this solution gives rise to a differential geometric solution of the ordinary, differential geometric Einstein equations. This follows from the purely algebraic nature of the notions metric, covariant derivation and curvature.

In order to solve the Einstein equations, it is most convenient to perform all computations locally and to glue the local solutions in a second step. We will see that these local computations may be performed in essentially the same way as in differential geometry. On the one hand, this is due to the fact that the local ring $\mathcal{O}_{X,x}$ at a rational point x may be embedded into a ring of formal power series.

Proposition 4. *Let $X \to S$ be a smooth morphism of locally Noetherian schemes. Let $s \in S$ and $x \in X_s$ be a $k(s)$-rational point. Then there exists an isomorphism of $\widehat{\mathcal{O}}_{S,s}$-algebras*

$$\widehat{\mathcal{O}}_{X,x} = \widehat{\mathcal{O}}_{S,s}[[x_1, \ldots, x_n]]$$

where $(x_1, \ldots, x_n)$ is a set of variables and $n = \dim \mathcal{O}_{X_s,x}$.

Proof. [9], Ex. 6.3.1 □

Furthermore, we know from [2], Prop. 2.2/11, that each point $x \in X$ possesses an open environment U which is étale over some affine space $\mathbb{A}^n_S = \mathrm{Spec}\mathcal{O}_S[x_1, \ldots, x_n]$. Therefore, the module $\Omega^1_{U/S}$ of differential forms over U is the free $\mathcal{O}_U$-module generated by the differentials $dx_1, \ldots, dx_n$ ([2], Cor. 2.2/10), and we may choose the base $\{\omega^i := dx_i\}$ of $\Omega^1_{U/S}$ together with the corresponding dual base $\{\partial_i\}$ of $\mathcal{T}_{U/S}$.

The functions $f \in \mathcal{O}_U$ on U are algebraic over the polynomial ring $\mathcal{O}_S[x_1, \ldots, x_n]$, because we may assume that U is standard étale over $\mathbb{A}^n_S$ (see [2], Prop. 2.3/3). Consequently, there is a canonical differential calculus on U with respect to the coordinates $x_1, \ldots, x_n$. More precisely, the vector field ∂_i acts on f by means of ordinary partial derivation with respect to the i-th coordinate x_i. This may be seen as follows: On polynomials we have clearly $\partial_i x_j^n = \delta_{ij} \cdot n x_j^{n-1}$ due to the Leibniz rule (vector-fields may be identified with derivations; see [2], chap. 2.1). If $f \in \mathcal{O}_U$ is arbitrary, there is an algebraic equation $\sum_{j=0}^m c_j f^j = 0$ with polynomials $c_j \in \mathcal{O}_S[x_1, \ldots, x_n]$, $c_m \neq 0$. It follows that $0 = \partial_i(\sum_{j=0}^m c_j f^j) = \sum_{j=0}^m (f^j \partial_i c_j + c_j j f^{j-1} \partial_i f)$ which is a linear equation in $\partial_i f$ and thus may be solved uniquely for $\partial_i f$ on the locus where $\sum_{j=0}^m c_j j f^{j-1} \neq 0$. However, by what we already know, $\partial_i c_j$ is the ordinary partial derivation of c_j with respect to the i-th coordinate x_i, and so we are done.

Therefore, we obtain the following local formulas on U (where we make use of Einstein's summation convention):

- $g = g_{ij} \omega^i \otimes \omega^j$ with $g_{ij} = g_{ji} \in \mathcal{O}_U$, $\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$;
- $\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k$, where the functions $\Gamma^k_{ij} \in \mathcal{O}_U$ are called the *Christoffel-symbols*;
- $R_{\partial_i \partial_j}(\partial_k) = R^l_{kij} \partial_l$, $R^l_{ijk} = \partial_j \Gamma^l_{ki} - \partial_k \Gamma^l_{ij} + \Gamma^r_{ki} \Gamma^l_{jr} - \Gamma^l_{kr} \Gamma^r_{ij}$;
- $R_{ik} := \mathrm{ric}_{ik} = R^l_{ilk}$, $R := \mathrm{sc} = g^{ik} R_{ik}$.

Now, the Einstein equations take their well known form

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} \quad \text{or equivalently} \quad R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right), \; T := g^{\mu\nu} T_{\mu\nu},$$

and Theorem 2 follows from the corresponding differential geometric result.

Theorem 2. *Let g be a metric on a smooth S-scheme $X \to S$. The Einstein equations on X are universal in the following sense: For $x \in X$, let $\{\partial_i\}$ be a base of $\Omega^1_{X/S,x}$, and let $g_{ij} \in \mathcal{O}_{X,x}$ be the components of the metric tensor at x. Assume that there exists a tensor G of rank two such that for all $x \in X$ the following statements hold at x:*

1. *G is a polynomial over K in the variables g_{ij}, $\partial_k g_{ij}$ and $\partial_k(\partial_l g_{ij})$ which is linear in $\partial_k(\partial_l g_{ij})$.*
2. *G is a symmetrical tensor.*
3. *$div(G) = 0$.*

Then, G coincides with the Einstein tensor $ric - \frac{1}{2} sc \cdot g$.

4. SOLUTIONS OF THE ARITHMETIC EINSTEIN EQUATIONS

In section 3, we deduced the algebraic geometric analogue of the differential geometric Einstein equations. Let us point to the crucial fact that the class of functions, which is available in order to solve the equations of arithmetic general relativity, is much smaller than in the differential geometric setting, because the local functions $f \in \mathcal{O}_U$ are *algebraic* functions. Nevertheless we will see that there actually exist models of type (GR) $(X \to S, g)$.

4.1. The Case of Zariski Zero-Dimensional Base

Within this section, let $S = \mathrm{Spec}K$ be the spectrum of a field K. We are looking for models of type (GR) $(X \to S, g)$ (see Definition 16). However, the condition (ii) of Definition 16 is empty in this case. Therefore, from the adelic point of view, $S = \mathrm{Spec}K$ is not the physically interesting case, but rather a toy model. The interesting "adelic" models, where all conditions of Definition 16 are non-trivial will be considered in section 4.2 . At least there are many models of type (GR) for the choice $S = \mathrm{Spec}\, K$, because things are particularly easy in this case. The most easiest example is the Minkowski solution

$$(\mathbb{A}^n_K, g_0), \qquad g_0 := \text{ diag}\,(\pm 1, \pm 1, \pm 1, \pm 1).$$

However, there are further, less trivial examples which correspond to certain solutions of the classical differential geometric Einstein equations.

4.1.1. Kasner solution. Again we choose $X_K := \mathbb{A}^n_K$ with coordinates $(t = x^0, x^1, \ldots, x^{n-1})$, but this time we choose the following non-trivial metric g_K

$$g_{00} = 1, \qquad g_{0i} = 0, \quad i \neq 0, \qquad g_{ij} := c\delta_{ij} \cdot t^{2k_i}, \quad i \neq 0 \neq j,$$

where δ_{ij} denotes the Kronecker delta, and where $c \in K$ and $k_i \in \mathbb{Q}$ are constants. Then g_K is well defined in the category of algebraic spaces, and it remains to show that we can choose the constants in such a way that the Einstein equations are fulfilled. We may do this on stalks. Recalling the remarks below Proposition 4 and the formulas stated there, we may compute the Christoffel symbols corresponding to the given metric. Exactly the same computation as in differential geometry shows:

Lemma 1. *The Ricci tensor is diagonal in the given coordinates, i.e.* $R_{\mu\nu} = \delta_{\mu\nu} R_{\mu\nu}$*. For the diagonal elements one obtains:*

$$R_{00} = \frac{1}{t^2} \sum_{j \neq 0} \left(k_j - k_j^2\right), \quad R_{ii} = -ck_i t^{2(k_i-1)} \left(\sum_{j\neq 0} k_j - 1\right), \; i \neq 0.$$

Corollary 3. *Let* $X_K := \mathbb{A}^n_K$ *and* g_K *be as stated above. Furthermore, choose the constants* $k_i \in \mathbb{Q}$ *such that*

$$\sum_{j\neq 0} k_j = 1 = \sum_{j \neq 0} k_j^2.$$

Then (X_K, g_K) *is a model of type (GR).*

4.1.2. Schwarzschild solution. The example of the Schwarzschild metric will show very clearly the general phenomenon that the Zariski topology is too coarse for physical applications and that it is necessary to work within the context of the étale topology. However, let us again start from the affine space $\mathbb{A}^n_K$ with coordinates $(t = x^0, x^1, \ldots, x^{n-1})$, but this time we consider the K-scheme $X_K := \mathrm{Spec}K[t, x^1, \ldots, x^{n-1}, r, r^{-1}]/(r^2 - \sum_{i\neq 0}(x^i)^2)$, whereby $r := \sqrt{\sum_{i \neq 0}(x^i)^2}$ should be interpreted as a spacial radius. By construction, X_K is étale over $\mathbb{A}^n_K$. In particular, the respective differential calculi "coincide". We choose the following metric g_K on X_K:

$$g_{00} = \frac{1}{1 + \frac{2m}{r}}, \qquad g_{0i} = 0, \qquad g_{ij} := -\left(1 + \frac{2m}{r}\right)^2 \delta_{ij} + \frac{x^i x^j}{r^2}\left(1 + \frac{2m}{r}\right)\frac{2m}{r}, \quad i \neq, 0$$

where δ_{ij} denotes the Kronecker delta, and $m \in K$ is a constant. By means of a longer but standard calculation, one can prove that the metric above solves the vacuum Einstein equations. Thus (X_K, g_K) is indeed a model of type (GR). The metric g_K corresponds to the Schwarzschild metric: (X_K, g_K) describes the exterior of a black hole. More precisely, r scales the distance from the event horizon of the black hole, and the constant m turns out to be the Schwarzschild diameter of the black hole.

4.1.3. Robertson-Walker models. Last, but not least, let us briefly mention the Robertson-Walker models. In differential geometry, these are defined in spherical coordinates by

$$g_{\mu\nu} = \mathrm{diag}\left(1, \frac{-S(t)^2}{1-kr^2}, -S(t)^2r^2, -S(t)^2r^2sin^2(\theta)\right)$$
$$T_{\mu\nu} = \mathrm{diag}\left(\rho(t), \frac{-p(t)S(t)^2}{1-kr^2}, -p(t)S(t)^2r^2, -p(t)S(t)^2r^2sin^2(\theta)\right)$$

where $S(t)$ is a so called scale factor, and where ρ resp. p denote the density resp. pressure of energy. In the special case $p = 0$, the divergence div$T$ of the energy-stress tensor vanishes if and only if the product $C := \rho(t) \cdot S(t)^3$ is constant with respect to t. Then the Einstein equations $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}$ yield the differential equation

$$\left(\frac{\partial S(t)}{\partial t}\right)^2 + k = \frac{\kappa C}{3S(t)}.$$

Whenever the solution $S(t)$ of this differential equation is an algebraic function, the corresponding Robertson-Walker metric would make sense in algebraic geometry. However, in general, $S(t)$ will not be algebraic. So, the classical Robertson-Walker models have no algebraic geometric analogues. But at least in the case $k = 0$, we find the algebraic solution

$$S(t) = t^{\frac{2}{3}} \sqrt[3]{4/3 \cdot \kappa C}.$$

Therefore, we obtain a model of type (GR) if we choose $X_K := \mathbb{A}^n_K$ with coordinates $(t = x^0, x^1, \ldots, x^{n-1})$ as well as the following metric and energy-stress tensor:

$$g_{\mu\nu} = \mathrm{diag}\left(1, -ct^{\frac{4}{3}}, -ct^{\frac{4}{3}}, -ct^{\frac{4}{3}}\right), \qquad c := (4/3 \cdot \kappa C)^{\frac{2}{3}} \in K, \qquad T_{\mu\nu} = \left(\frac{4}{3t^2}, 0, 0, 0\right).$$

4.2. The Case of Zariski One-Dimensional Base

Within this section, let $S = \mathrm{Spec}\mathcal{O}$ be the spectrum of a Dedekind ring which is not a field. From the "adelic" point of view, this is the physically interesting case, because the condition (ii) of Definition 16 is no longer empty. Consequently, it is much harder to construct models of type (GR).

In section 4.2.1, we will first consider the low dimensional case, because then the Einstein equations are trivial. But as soon as the tangent spaces exceed three dimensions, this is no longer true. Then, the conditions (i) and (ii) of Definition 16 are both non-trivial. This the physically interesting situation where we are looking for models of type (GR) $(X \to S, g)$. An example, which may be interpreted as the "adelic" Minkowski space, is presented in section 4.2.2.

4.2.1. The low dimensional case. Let $X \to S$ be a smooth S-scheme of relative dimension one or two with metric g. We will show that the Einstein equations are trivial in this case. As it suffices to show this locally, we may choose an open sub-scheme U of X such that $\Omega^1_{U/S}$ is free with base $\{\omega^i\}$. In the one dimensional case, our assertion is clear, because the curvature tensor has got only a single component, and this component vanishes due to the symmetries of the curvature tensor (see Proposition 2). Thus, also the Einstein tensor ric $- \frac{1}{2}\, g \cdot$ sc vanishes . In the two dimensional case, a small computation is necessary. Again making use of the identities of the curvature tensor several times, we obtain with respect to the given base:

$$R := \mathrm{sc} = g^{\mu\nu}\mathrm{ric}_{\mu\nu} = g^{\mu\nu}R^1_{\mu 1\nu} + g^{\mu\nu}R^2_{\mu 2\nu} = g^{\mu 2}R^1_{\mu 12} + g^{\mu 1}R^2_{\mu 21} = g^{\mu 2}g^{\nu 1}R_{\nu\mu 12} + g^{\mu 1}g^{\nu 2}R_{\nu\mu 21}$$
$$= g^{11}g^{22}R_{1212} + g^{21}g^{12}R_{2112} + g^{11}g^{22}R_{2121} + g^{21}g^{12}R_{1221} = 2\mathrm{det}g \cdot R_{2121}.$$

Therefore, one derives that

$$R^\mu_\nu := \mathrm{Ric}^\mu_\nu = g^{\mu\lambda}\mathrm{ric}_{\lambda\nu} = g^{\mu\lambda}R^1_{\lambda 1\nu} + g^{\mu\lambda}R^2_{\lambda 2\nu} = g^{\mu\lambda}g^{1\iota}R_{\iota\lambda 1\nu} + g^{\mu\lambda}g^{2\iota}R_{\iota\lambda 2\nu}$$
$$= g^{\mu 1}g^{12}R_{211\nu} + g^{\mu 2}g^{11}R_{121\nu} + g^{\mu 1}g^{22}R_{212\nu} + g^{\mu 2}g^{21}R_{122\nu}$$
$$= R_{211\nu}(g^{\mu 1}g^{12} - g^{\mu 2}g^{11}) + R_{212\nu}(g^{\mu 1}g^{22} - g^{\mu 2}g^{21})$$

$$= \delta_{2\mu}\delta_{\nu 2}(g^{21}g^{12} - g^{22}g^{11})R_{2112} + \delta_{\mu 1}\delta_{\nu 1}(-g^{21}g^{12} + g^{22}g^{11})R_{2121}$$
$$= R_{2121}\det g \cdot (\delta_{2\mu}\delta_{\nu 2} + \delta_{\mu 1}\delta_{\nu 1}) = \frac{1}{2}\delta_{\mu\nu}R.$$

Consequently, $R^{\mu}_{\nu} - \frac{1}{2}\delta_{\mu\nu}R = 0$ or equivalently $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$. In particular, there are many models of type (GR) in the low dimensional case.

Corollary 4. *Let $X \to S$ be a smooth S-scheme of relative dimension one or two such that X is the Néron model of its generic fibre. Then X gives rise to a model of type (GR).*

4.2.2. A higher dimensional solution. Again, let $X \to S$ be smooth over the Zariski one-dimensional base scheme S. If the relative dimension exceeds two, the Einstein equations are no longer trivial. Therefore, it is not easy to find models of type (GR) in the "adelic" situation. However, in this section we will state at least one example, namely the fibred product $X := E_1 \times_S \ldots \times_S E_n$ of smooth elliptic curves E_i over S. In a certain way, this is the "adelic" analogue of Minkowski space-time. However, the idea of the proof that X is indeed a model of type (GR) (Theorem 3) is as follows: Locally, every smooth curve C over S may be embedded in some affine space $\mathbb{A}^n_S$ (see [2], Def. 2.2/3). Pulling back the flat metric $\mathrm{diag}(\pm 1, \ldots, \pm 1)$ on $\mathbb{A}^n_S$ to C and pushing forward this metric on C to $\mathbb{A}^1_S$ by means of an étale morphism $C \to \mathbb{A}^1_S$ (which is possible by [2], Prop. 2.2/11 b) and Cor. 2.2/10), we obtain the first fundamental form on C in local coordinates. Analogously, we obtain the first fundamental form on a product of curves in local coordinates. (Recall that the first fundamental form is in general defined as follows: Due to smoothness, each point $x \in X$ possesses an open environment $U \subset X$ such that U may be embedded into some affine space $\mathbb{A}^m_S$ for some m. Pulling back the flat metric $\mathrm{diag}(\pm 1, \ldots, \pm 1)$ on $\mathbb{A}^m_S$ to U, we obtain the first fundamental form on U. The first fundamental form on X is obtained by gluing.) This metric is diagonal, because $\Omega^1_{(X_1 \times_S X_2)/S} \cong \bigoplus_i p_i^* \Omega^1_{X_i/S}$, where p_i denotes the projection onto the i-th factor ([2], Prop. 2.1/4). It follows that the corresponding curvature tensor vanishes (see Corollary 5). Consequently, a product of elliptic curves is a vacuum solution of the Einstein equations and it is even a model of type (GR), because it is the Néron model of its generic fibre. However, let us now make the indicated steps of the proof more explicit.

Lemma 2. *Let $C_1, \ldots, C_n$ be n smooth curves over S. Provide $X := C_1 \times_S \ldots \times_S C_n$ with the first fundamental form g as metric. Then g is diagonal.*

Proof. Instead of proving this lemma in full generality, let us restrict attention to the special case of a product of elliptic curves, because we will only make use of Lemma 2 in the case that X is a product of elliptic curves. The proof of the general case may be performed in a similar (but more abstract) way. First, we have to compute the first fundamental form. In order to do this, let us consider a smooth elliptic curve E over a field K which is not of characteristic two. Let us now restrict E to the affine open subset $\mathrm{Spec}\, K[X, Y, Y^{-1}] \subset \mathbb{P}^2_K$, and let us therefore assume that E is described by an equation $P(Y, X) := Y^2 - X^3 - g_2 X - g_3 = 0$. Then we have canonical morphisms of K-algebras

$$\begin{array}{ccccc} K[X,Y,Y^{-1}] & \underset{j^*}{\longrightarrow} & \mathcal{O}_E := K[X,Y,Y^{-1}]/(P) & \underset{g^*}{\longleftarrow} & K[X] \\ X & \mapsto & \overline{X} & \hookleftarrow & X \\ Y & \mapsto & \overline{Y} & & \end{array}$$

where g^* is étale. Therefore, we obtain on the level of differential forms

$$\begin{array}{ccccc} \Omega^1_{K[X,Y,Y^{-1}]/K} \otimes_K \mathcal{O}_E & \underset{j^*}{\longrightarrow} & \Omega^1_{\mathcal{O}_E/K} & \underset{(g^*)^{-1}}{\overset{\sim}{\longrightarrow}} & \Omega^1_{K[X]/K} \otimes_K \mathcal{O}_E \\ dX & \mapsto & d\overline{X} & \mapsto & dX \\ dY & \mapsto & d\overline{Y} & \mapsto & \frac{3X^2+g_2}{2Y} dX \end{array}$$

where we made use of the identity $(3\overline{X}^2 + g_2)d\overline{X} = d\overline{Y^2} = 2\overline{Y}d\overline{Y}$ in $\Omega^1_{\mathcal{O}_E/K}$. The dual of this map is the $\mathcal{O}_E$-linear map $\partial_X \mapsto \partial_X + \frac{3X^2+g_2}{2Y}\partial_Y$, where ∂_X (resp. ∂_Y) denotes the dual of dX (resp. dY). In particular, the tangent vectors of E in the affine open subset $\mathrm{Spec}K[X,Y,Y^{-1}] \subset \mathbb{P}^2_K$ may be written as vectors

$$\begin{pmatrix} 1 \\ \frac{3X^2+g_2}{2Y} \end{pmatrix}.$$

Providing $\mathrm{Spec}K[X,Y,Y^{-1}]$ with the trivial metric $g_0 = \mathrm{diag}(1,1)$ and interpreting g_0 as bilinear form, we derive the first fundamental form g on E:

$$g = g_0\left(\begin{pmatrix} 1 \\ \frac{3X^2+g_2}{2Y} \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{3X^2+g_2}{2Y} \end{pmatrix}\right) = 1 + \frac{(3X^2+g_2)^2}{4Y^2}.$$

This is manifestly the same result as in differential geometry. The procedure in the case of a product of n elliptic curves is straight forward. For example, if $n = 2$, we have to compose the above homomorphism of K-algebras with the projection map $K[X,Y,Y^{-1},Z,W] \to K[X,Y,Y^{-1}]$, $Z, W \mapsto 0$, where Z and W are the variables of the second elliptic curve. In this case we obtain the two tangent vectors

$$\begin{pmatrix} 1 \\ \frac{3X^2+g_2}{2Y} \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ \frac{3Z^2+g_2'}{2W} \end{pmatrix}.$$

In particular, the metric is diagonal. □

Corollary 5. *Let X be as in Lemma 2. Then X is flat, i.e. the curvature tensor vanishes.*

Proof. As the curvature tensor is a global section of a sheaf, it suffices to prove the statement locally. Therefore, we may assume that the components g_{ij} of the metric tensor g are algebraic over some polynomial ring, say with variables $x_1, \ldots, x_n$. Due to Lemma 2, the metric is diagonal and the statement may be proven in exactly the same way as in differential geometry. □

Theorem 3. *Let $X := E_0 \times_S \ldots \times_S E_n$ be the fibred product of smooth elliptic curves E_i over S. Let g be the first fundamental form on X. Then $(X \to S, g)$ is a model of type (GR).*

Proof. By Corollary 5, X is flat, i.e. the curvature tensor vanishes. In particular, (X, g) is a solution of the vacuum Einstein equations. Due to the fact that Néron models fulfill the property (ii) of models of type (GR) (see Definition 16), it suffices to show that X is the Néron model of its generic fibre. Now notice that the fibred product of Néron models over S is again the Néron model of its generic fibre, because the universal property of fibred products implies the universal property of Néron models. Consequently, we are reduced to the proof that an elliptic curve over S is the Néron model of its generic fibre. Thus we are done by [2], Theorem 1.4/3. □

Remark 3. Let $X := E_0 \times_S \ldots \times_S E_n$ be the fibred product of n smooth elliptic curves E_i over S provided with first fundamental form g. In truth, $(X \to S, g)$ is even a model of type (SR) (see Definition 17). In particular, all results, which are stated in 1, are true for X.

Let us now choose $S = \mathrm{Spec}\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integral numbers of an algebraic number field $K \subset \mathbb{R}$. Let us compare X with the Minkowski space-time $\mathbb{A}^n_S$. Both describe a space-time without gravity, because the curvature tensor vanishes identically. Furthermore, both carry a canonical, commutative group structure. The inverse of the respective group laws may be interpreted as a simultaneous space and time reflection. The difference between X and Minkowski space-time is of topological nature.

While $\mathbb{A}^n_S$ is a product of curves of genus zero, X is a product of curves of genus one. Therefore, X carries a non-trivial vacuum structure.

This difference in the global topology has some interesting consequences: In the Minkowski case, the set of archimedean points $\mathbb{A}^n_K(K) = K^n$ is not a finitely generated abelian group. There is even no finitely generated abelian subgroup of $\mathbb{A}^n_K(K)$ which is invariant under all K-isomorphisms of $\mathbb{A}^n_K$. But, if we consider instead the model X something interesting happens: Due to a theorem of Mordell, the set $X_K(K)$ of archimedean points of X is a finitely generated abelian group. Furthermore, $X_K(K)$ is invariant under arbitrary K-isomorphisms of X_K. Therefore, its makes sense to consider $X_K(K)$ as a vacuum.

ACKNOWLEDGEMENT

This work was supported by a fellowship of the Graduiertenkolleg "Analytische Topologie und Metageometrie" of the Westfälische Wilhelms-Universität Münster.

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