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RESEARCH ARTICLES

p-Adic Physics, Non-Well-Founded Reality and Unconventional Computing

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Abstract—We consider perspectives of application of coinductive and corecursive methods of non-well-founded mathematics to modern physics, especially to adelic and p-adic quantum mechanics. We also survey perspectives of relationship between modern physics and unconventional computing.

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1. INTRODUCTION

In the paper we consider an opportunity of physical measurement based on the non-Kolmogorovian probabilistic model with non-well-founded probabilities, see [23–29] for p-adic and adelic theoretical physics. Surprisingly such a departure from the conventional probability model generates departure from conventional set theory, namely, toward non-well-founded set theory and additionally departure from the conventional computing based on the Church-Turing thesis toward unconventional computing based on coinduction.

A non-well-founded (non-WF) set theory belongs to axiomatic set theories that violate the rule of WF-ness and, as an example, allow sets to contain themselves: $X \in X$. In non-WF set theories, the foundation axiom of Zermelo-Fraenkel set theory is replaced by axioms implying its negation. The theory of non-WF sets has been explicitly applied in diverse fields such as logical modelling non-terminating computational processes and behavior of interactive systems in computer science (process algebra, coalgebra, logical programming based on coinduction and corecursion), linguistics and natural language semantics (situation theory), logic (analysis of semantic paradoxes). The set theory with anti-foundation axiom (which denies the axiom of foundation) is considered in [2, 7–9].

Non-WF sets have been also implicitly used in non-standard (more precisely, *non-Archimedean*) analysis like infinitesimal and p-adic analysis [17, 30, 37]. The point is that denying the foundation axiom in number systems implies setting the *non-Archimedean ordering structure*. Recall that Archimedes' axiom affirms the existence of an integer multiple of the smaller of two numbers which exceeds the greater: for any positive real or rational number y, there exists a positive integer n such that $y > 1/n$ or $ny > 1$. The informal sense of Archimedes' axiom is that anything can be measured by a rigid scale. Refusing the Archimedean axiom entails the existence of infinitely large numbers (in the case of field this means additionally the existence of infinitely small numbers). So, there is the following true implication: if the field has a Dedekind completeness, which affirms the existence of a supremum for every bounded set, then this field has the Archimedean property. It is known that the field $\mathbf{R}$ of real numbers satisfies the Dedekind completeness (i.e. any nonempty set of real numbers which has an upper bound has a least upper bound), then it satisfies Archimedes' axiom, too. Refusing the latter implies refusing the standard

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completeness. Meanwhile, there is no upper bound for the set $\mathbf{Z} \subset \mathbf{R}$ of all integers; no matter how large a number we choose for the upper bound, there will always be some integer bigger than it. Without the standard completeness we obtain infinite real numbers (infinitely large real numbers) and infinitesimals (infinitely small real numbers). For them there are no least upper bounds in the general case. The field with infinite numbers and infinitesimals is called the field of hyperreal numbers and denoted by $^*\mathbf{R}$. This field is not complete in the standard sense, because, for example, the set of infinitesimals does not have a least upper bound. On the other hand, the set $\mathbf{N}$ of positive integers is bounded above by the member $t \in {}^*\mathbf{N}\backslash\mathbf{N}$, where $^*\mathbf{N}$ is the set of hypernatural numbers and t is the sequence given by $t_n = n$ for all $n \in \mathbf{N}$. But $\mathbf{N}$ can have no least upper bound: if $n \leq c$ for all $n \in \mathbf{N}$ then $n \leq c - 1$ for all $n \in \mathbf{N}$.

There exists also a different version of mathematical analysis in that Archimedes' axiom is rejected, namely, p-adic analysis (e.g., see [30]). In this analysis, one investigates the properties of the completion of the field $\mathbf{Q}$ of rational numbers (respectively, the properties of the completion of the ring $\mathbf{Z}$ of integers) with respect to the metric $\rho_p(x, y) = |x - y|_p$, where the norm $|\cdot|_p$ called p-adic is defined as follows: (1) $|y|_p = 0 \leftrightarrow y = 0$, (2) $|x \cdot y|_p = |x|_p \cdot |y|_p$, (3) $|x + y|_p \leqslant \max(|x|_p, |y|_p)$ (non-Archimedean triangular inequality). That metric over the field $\mathbf{Q}$ (respectively, over the ring $\mathbf{Z}$) is non-Archimedean, because $|n \cdot 1|_p \leqslant 1$ for all $n \in \mathbf{Z}$. This completion of the field $\mathbf{Q}$ (respectively, of the ring $\mathbf{Z}$) is called the field $\mathbf{Q}_p$ of p-adic numbers (respectively, the ring $\mathbf{Z}_p$ of p-adic integers). $\mathbf{Q}_p$ is not an ordered field. This means that there is no total ordering of its elements that agrees with the field operations. On the other hand, the ring $\mathbf{Z}_p$ includes $\mathbf{Z}$ and satisfies the equality $\mathbf{Z}_p = \{x \in \mathbf{Q}_p \colon |x \cdot 1|_p \leqslant 1\}$. As a result, in $\mathbf{Z}_p$ there are integers that are naturally to be interpreted as infinitely large numbers.

A connection between denying the axiom of foundation and denying the Archimedean axiom may be shown as follows. Recall that a binary relation, R, is WF on X if and only if every non-empty subset of X has a minimal element with respect to R. In other words, if R is WF, then the *induction principle* is valid: if R is a WF relation on X and $P(x)$ is some property of elements of X, then to show $P(x)$ holds for all elements of X, it suffices to show that if x is a member of X and $P(y)$ is true for all y such that yRx, then $P(x)$ must also be true: $\forall x \in X((\forall y \in X(yRx \to P(y))) \to P(x)) \to \forall x \in XP(x)$. For instance, if a membership relation, $\in$, is WF, then the epsilon-induction ($\in$-induction) holds, i.e. for any formula φ:

$$\forall x((\forall y(y \in x \to \varphi(y))) \to \varphi(x)) \to \forall x \varphi(x).$$

More generally, one can define objects by induction on any WF relation R. This generalized kind of induction is called sometimes Noetherian induction.

The negation of the axiom of foundation causes that there are objects that cannot be defined by Noetherian induction. This means that there exists a non-empty subset of X that has no minimal element with respect to a relation R. The latter is called a non-WF relation. We remember that there are no least upper bounds for non-Archimedean numbers in the general case[1]. This means that those sets of upper bounds have no least (minimal) element. Therefore we cannot use Noetherian induction for the set of infinitesimals or for the set of infinitely large integers (notice that sets $^*\mathbf{Z}\backslash\mathbf{Z}$ and $\mathbf{Z}_p\backslash\mathbf{Z}$ of infinitely large integers have a different meaning). For example, there are no $\in$-induction for these sets. Taking into account this circumstance, we can state that some initial objects of non-Archimedean mathematics are objects obtained implicitly by denying the axiom of foundation. Non-Archimedean numbers are non-WF.

Instead of Noetherian induction, we can use coinduction as the dual notion applied to non-WF objects. In this case, a binary relation, R, is non-WF on X if and only if every non-empty subset of X has a maximal element with respect to R (this definition is one of the possible formulation of anti-foundation axiom). Transfinite coinduction is implicitly used in the definition of spherically complete ultrametric spaces: the ultrametric field K is spherically complete if and only if it is maximal among ultrametric fields with the same value group and residue field.

The conventional approach to the physical measurement for statistical phenomena uses classical (Kolmogorov's) probability theory built in the language of WF mathematics of real numbers. It sets a framework of modern physics, taking into account that physical reality is regarded in modern science as reality of stable repetitive phenomena (phenomena that have probabilities, i.e. do not fluctuate in the standard real metric). However, *we can assume another approach to the physical measurement for*

[1] Notice that the completeness axiom is equivalent to the principle of continuous induction [44].

statistical phenomena that uses tools of non-WF mathematics, in particular uses the coinduction principle and non-WF probability theory on non-Archimedean structures.

2. ALGORITHMS VS. COALGORITHMS, INDUCTION VS. COINDUCTION

Beyond all doubt, the most basic notion of mathematics and physics is an algorithm. It plays a significant role providing, e.g., a correct (from the standpoint of logic) reasoning in mathematics and a well-defined measurement by rigid scales in physics. Its simplest definition is as follows: the algorithm is a set of instructions for solving a problem. In computer sciences, the algorithm is regarded either as the one implemented by a computer program or simulated by a computer program. In other words, the algorithm is reduced to the computer's process instructions, telling the computer what specific steps and in what specific order to perform in order to carry out a specified task.

There are at least two general approaches to explicate mathematically the notion of algorithm: conventional and unconventional. The *conventional* approach is presented by the following well-known statement: every algorithm can be simulated by a Turing machine and formally expressed by the lambda calculus. This statement is known as the *Church-Turing thesis*. Recall that it was initially formulated as follows: the intuitive notion of effective computability for functions and algorithms is well explicated by Turing machines (Turing) or the lambda calculus (Church). This thesis equated lambda calculus, Turing machines and algorithmic computing as equivalent mechanisms of problem solving. Later, in the conventional approach to algorithms, it was reinterpreted as a uniform complete mechanism for solving all computational problems.

Notice that the standard name of Turing machines actually refers, in Turing's words, to automatic machines, or a-machines. He also proposed other models of computation: c-machines (choice machines) and u-machines (unorganized machines). Turing argued for the claim (Turing's thesis) that whenever there is an effective method for obtaining the values of a mathematical function, the function can be computed by a Turing a-machine. At the same time, Church formulated the following thesis: a function of positive integers is effectively calculable only if it is recursive. If attention is restricted to functions of positive integers then Church's thesis and Turing's thesis are equivalent. It is important to distinguish between the Turing-Church thesis and the different proposition that whatever can be calculated by a machine can be calculated by a Turing machine [13]. The two propositions are sometimes confused. Gandy termed the second proposition '*Thesis M*': whatever can be calculated by a machine is Turing-machine-computable [19]. The latter thesis is a fundamental of conventional approach for explicating the notion of algorithm.

So, conventionally, *the algorithm is associated with step-by-step processing information and with mapping the initial input to the final output, ignoring the external world while it executes*. As we see, we accept here the five of requirements for an algorithm:

1. the finiteness, an algorithm must always terminate after a finite number of steps;
2. the definiteness, each step of an algorithm must be precisely defined;
3. the determinateness by input (initial data), i.e. by quantities which are given to it initially before the algorithm begins;
4. the finiteness of output (desired result), i.e. of quantities which have a specified relation to the inputs, they have no history dependence for multiple computations;
5. the effectiveness, all of the operations to be performed in the algorithm can in principle be done exactly and in a finite length of time.

Recall that Markov formulated the following three features characteristic of algorithms [32]:

1. the definiteness of the algorithm, consisting in the universal comprehensibility and precision of prescription, leaving no place to arbitrariness;
2. the generality of the algorithm, the possibility of starting out with initial data, which may vary within given limits;

3. the conclusiveness of the algorithm, the orientation of the algorithm toward obtaining some desired result, which is indeed obtained in the end with proper initial data.

The most basic notion in the conventional approach to algorithms implicitly used here is the least fixed point presupposing the *induction principle* (Noetherian induction), e.g. this notion is fundamental for *recursion*.

However, this notion is violated fully in the *unconventional approach* to explicate the notion of algorithm. The conventional treatment of algorithm is unapplied, broken in unconventional computing models [16]. The main originality of unconventional computing is that just as conventional models of computation make a distinction between the structural part of a computer, which is fixed, and the data on which the computer operates, which are variable, so unconventional models assume that both structural parts and computing data are variable ones [3]. Therefore the essential point of conventional computation is that the physics is segregated once and for all within the logic primitives. For instance, reaction-diffusion computing [4] is one of the unconventional models built on spatially extended chemical systems, which process information using interacting growing patterns, of excitable and diffusive waves. In reaction-diffusion processors, both the data and the results of the computation are encoded as concentration profiles of the reagents. The computation is performed via the spreading and interaction of wave fronts.

Unconventional computing may be presented as massive-parallel locally-connected mathematical machines. They cannot have a single centralized source exercising precise control over vast numbers of heterogeneous devices. Therefore the conventional definition of algorithm is unapplied for unconventional computing models [20]. Instead of recursion, one applies there corecursion as a type of operation that is dual to recursion [51]. Corecursion is typically used to generate infinite data structures. The rule for primitive corecursion on codata is the dual to that for primitive recursion on data. Instead of descending on the argument, we ascend on the result. Notice that corecursion creates potentially infinite codata, whereas ordinary recursion analyzes necessarily finite data. By induction and recursion we use the notion of least fixed points, whereas by coinduction and corecursion we use the notion of greatest fixed points (for more details concerning coinduction and corecursion see [6, 22, 34]).

Induction and recursion are firmly entrenched as fundamentals for proving properties of inductively defined objects, e.g. of finite or enumerable objects. Discrete mathematics and computer science abound with such objects, and mathematical induction is certainly one of the most important tools. However, we cannot use the principle of induction for non-WF objects. Instead of this principle, the notion of *coinduction* appears as the dual to induction.

The difference between induction and coinduction may be well defined as follows. Firstly, let an operation $\Phi : \mathcal{P}(A) \to \mathcal{P}(A)$, where $\mathcal{P}(A)$ is the powerset of A, be defined as monotone iff $X \subseteq Y$ implies $\Phi(X) \subseteq \Phi(Y)$ for $X, Y \subseteq A$. Any monotone operation Φ has the least and the greatest fixed point, X_Φ and X^Φ respectively, that is, $\Phi(X_\Phi) = X_\Phi$, $\Phi(X^\Phi) = X^\Phi$, and for any other fixed point $Y \subseteq A$ of Φ (i.e. $\Phi(Y) = Y$) we have $X_\Phi \subseteq Y \subseteq X^\Phi$. The sets X_Φ and X^Φ can be defined by $X_\Phi := \bigcap\{Y : Y \subseteq A, \Phi(Y) \subseteq Y\}$, $X^\Phi := \bigcup\{Y : Y \subseteq A, Y \subseteq \Phi(Y)\}$. It is easy to see that the monotonicity of Φ implies the required properties of X_Φ and X^Φ.

On the one hand, by definition of X_Φ, we have for any set $Y \subseteq A$ that $\Phi(Y) \subseteq Y$ implies $X_\Phi \subseteq Y$. This principle is called *induction*. On the other hand, by definition of X^Φ, we have for any set $Y \subseteq A$ that $Y \subseteq \Phi(Y)$ implies $Y \subseteq X^\Phi$. This principle is called *coinduction*.

Thus, the difference of applying two kinds of fixed points may by illustrated as follows. The least fixed point of the equation $S = A \times S$ is the empty set (i.e. it is so from the standpoint of conventional approach), while the greatest fixed point is the set of all streams over S (i.e. it is so from the standpoint of unconventional approach).

The greatest fixed point (respectively, coinduction and corecursion) allows us to describe the behavior of computing without a priori presuppositions. As a result, unconventionally, *the algorithm can be viewed as processing information that may be massive-parallel and as mapping the initial input to the final output, whose values depend on interaction with the open unpredictable environment*, i.e. identical inputs may provide different outputs, as the system learns and adapts to its history of interactions. The algorithm with such interpretation is said to be *coalgorithm* (it is an algorithm based on coinduction and corecursion). Its basic properties are as follows:

1. the infiniteness, an algorithm is simulated behaviorally by differential equations;
2. the indefiniteness, there can be a massive-parallel change of system;
3. the determinateness not only by input (initial data), but also by interaction with the open unpredictable environment;
4. the infiniteness of output;
5. the bisimulation[2] effectiveness, when there is no prespecified endpoint.

The notion of coalgorithm appeared due to applications of non-WF mathematics in computer sciences proposed within the framework of *interactive-computing/concurrency paradigm*, e.g. the latter assumes coinductive and corecursive methods to be used in computations [20]. Although in such paradigm, computation models may be abstract in the same measure as Turing machines, they assume a combination of physics and logic in the computation, as well as that in cellular automata or in other unconventional computing models. In conventional computing, the computation is performed in a closed-box fashion, transforming a finite input, determined by the start of the computation, to a finite output, available at the end of the computation, in a finite amount of time. Therefore physical implementation does not play role in such computation that may be considered as process in a black box. As opposed to the conventional approach, in interactive models inputs and outputs may be infinite and computation in each point of lattice may proceed simultaneously and independently.

Massive-parallel locally-connected mathematical machines cannot have a single centralized source. Interactive-computing paradigm is able to describe concurrent/parallel computations whose configuration may change during the computation and is decentralized as well. Within the framework of this paradigm, so-called concurrency calculi also called process algebras are built. They are typically presented using systems of equations. These formalisms for concurrent systems are formal in the sense that they represent systems by expressions and then reason about systems by manipulating the corresponding expressions.

One of the most useful non-WF mathematical object defined by coinduction is a stream — a corecursive data-type of the form $s = \langle a, s' \rangle$, where s' is another stream. A stream can be exemplified as a succession of days, unfolding in a cyclic pattern:

$$days = \langle Monday, \langle Tuesday, \langle Wednesday, \ldots \langle Sunday, days \rangle \ldots \rangle \rangle \rangle.$$

It is an example of the explicit non-WF object that has a self-reference structure. 'Days' is not a finite set {Monday, ..., Sunday}, because it has an infinite number of members of the kind 'Monday', of the kind 'Sunday', etc. Therefore it is natural to represent 'days' as an infinite tuple ⟨Monday, ⟨ ..., ⟨Sunday, ⟨Monday, ⟨...⟩⟩⟩⟩⟩. The stream of such a kind is circular: after 'Sunday' we repeat again and again: 'Monday', ..., 'Sunday'. A circular stream we cannot define by means of conventional mathematics. For example, in conventional mathematics there is the Russellian paradox that appears if we have $a \in a$, here we obtain a circular definition too: $a = \{a\}$, then $a = \{a\} = \{\{a\}\} = \{\{\{a\}\}\} = \ldots$

Another example of stream is an adele. Adeles $a \in A$ are constructed by real $a_1 \in \mathbf{Q}_\infty$ and p-adic $a_p \in \mathbf{Q}_p$ numbers

$$a = \langle a_1, a_2, a_3, a_5, \ldots, a_p, \ldots \rangle,$$

with restriction that $a_p \in \mathbf{Z}_p = \{x \in \mathbf{Q}_p \colon |x|_p \leq 1\}$ for all but a finite set F of primes p. The notion of adeles is used for explicating structure of space-time at the Planck scale. This approach treats simultaneously real (Archimedean) and p-adic (non-Archimedean) aspects. As a result, adelic quantum mechanics can be viewed as quantum mechanics on an adelic space and contained standard as well as all p-adic quantum mechanics.

The notion of adele is very interesting, because it assumes using the coalgorithmic approach (i.e. the unconventional approach to algorithms) in physics.

[2]Bisimulation is a binary relation between labelled transition systems with the same set of actions, associating those systems if one of them simulates the other and vice-versa. According to bisimulation, two systems are equivalent if both have the same behavior [21].

Typically, the measurement in physics is understood within the framework of conventional approach to algorithms. This means that the measurement is based on least fixed points, therefore the following classical entities: mass, size, distance, weight, force, etc. are regarded inductively. However, we can assume that *the measurement in physics might be presented in the coinductive form too, supposing greatest fixed points*. As an example, we can consider adeles as results of physical measurements.

3. PHYSICAL MEASUREMENT AND NON-WELL-FOUNDED PROBABILITIES

The measurement provides theoretical statements with empirical data. These data are collected due to *experiments*. One of the important roles that experiments play is to test theories and to provide them with findings concerning the existence of the entities involved in theories.

However, experiments may not always give clear-cut results, so they may disagree for a time. Galison exemplified this by the histories of measurements of the gyromagnetic ratio of electrons, the discovery of the muon and the discovery of weak neutral currents [18]. It is well known that the theoretical presuppositions of the experimenters may enter into the decision to end an experiment and report the result. Another probable imperfection of experiments lays in that experimental interpretations seem to be more traditionary than theories that change very often. Experimental apparatus, instruments seem to create an invariant relationship between results of measurement and the world. But when our theories change, we may expect that experimental interpretations may change too. However, usually, we observe the same reading of data and measurements after a change in theory, even though we may take the reading to be no longer important [1].

Using the greatest-fixed point measurement in modern physics, especially in adelic or p-adic quantum mechanics, we can expect the change of paradigm in the physical measurement.

From the philosophical point of view, there are at least two distinct approaches to measurement: the classical visual tradition (observational study) and the statistical (probability) one. The *visual tradition* uses detectors, which provide detailed information about an individual event (e.g. the mass, force, distance, weight, etc. of a thing). At the same time, the *statistical tradition* using, for instance, the electronic detectors, such as geiger counters and spark chambers, detects more events and, therefore, provides detailed information about a collective of individual events of the same kind.

Let us consider some opportunities of using the greatest-fixed point notion in the statistical approach to measurement ([23–29, 43]). To do it, let us make some basic definitions of probability applications of the greatest-fixed point measurement. In the sequel we will use coalgorithms, more precisely the coinductive principle.

Let A be any set. We define the set A^ω of all streams over A as $A^\omega = \{\sigma\colon \{0,1,2,\dots\} \to A\}$. For a stream σ, we call $\sigma(0)$ the initial value of σ. We define the *derivative* of a stream σ, for all $n \geq 0$, by $\sigma'(n) = \sigma(n+1)$. For any $n \geq 0$, $\sigma(n)$ is called the n-th element of σ. It can also be expressed in terms of higher-order stream derivatives, defined, for all $k \geq 0$, by $\sigma^{(0)} = \sigma$; $\sigma^{(k+1)} = (\sigma^{(k)})'$. In this case the n-th element of a stream σ is given by $\sigma(n) = \sigma^{(n)}(0)$. Also, the stream is understood as an infinite sequence of derivatives. It will be denoted by an infinite sequence of values or by an infinite tuple: $\sigma = \sigma(0) :: \sigma(1) :: \sigma(2) :: \cdots :: \sigma(n-1) :: \sigma^{(n)}$, $\sigma = \langle \sigma(0), \sigma(1), \sigma(2), \dots \rangle$. The state stream $a :: a :: a :: \dots$ is denoted by $[a]$.

It can be easily shown that p-adic numbers may be represented as potentially infinite data structures such as streams. Each stream of the form $\sigma = \sigma(0) :: \sigma(1) :: \sigma(2) :: \cdots :: \sigma(n-1) :: \sigma^{(n)}$, where $\sigma(n) \in \{0, 1, \dots, p-1\}$ for every $n \in \mathbf{N}$, may be converted into a p-adic integer by the following rule:

$$\forall n \in \mathbf{N}, \sigma(n) = \sum_{k=0}^{n} \sigma(k) \cdot p^k \wedge \sigma(n) = \sigma(0) :: \sigma(1) :: \cdots :: \sigma(n). \tag{3.1}$$

And vice versa, each p-dic integer may be converted into a stream taking rule (3.1). Such a stream is called *p-adic*.

Streams are defined by coinduction: two streams σ and τ in A^ω are equal if they are *bisimilar*: (i) $\sigma(0) = \tau(0)$ (they have the same *initial value*) and (ii) $\sigma' = \tau'$ (they have the same *differential equation*). To set addition and multiplication by coinduction, we should use the following facts about differentiation of sums and products by applying the basic operations: $(\sigma + \tau)' = \sigma' + \tau'$, $(\sigma \times \tau)' =$

Table 1. Coinductive definitions of sum, product and inverse.

Differential equation	Initial value	Name
$(\sigma + \tau)' = \sigma' + \tau'$	$(\sigma + \tau)(0) = \sigma(0) + \tau(0)$	Sum
$(\sigma \times \tau)' = (\lvert\sigma(0)\rvert \times \tau') + (\sigma' \times \tau)$	$(\sigma \times \tau)(0) = \sigma(0) \times \tau(0)$	Product
$(\sigma^{-1})' = \lvert -1\rvert \times \lvert\sigma(0)^{-1}\rvert \times \sigma' \times \sigma^{-1}$	$(\sigma^{-1})(0) = \sigma(0)^{-1}$	Inverse

$(|\sigma(0)| \times \tau') + (\sigma' \times \tau)$, where $|\sigma(0)| = \langle \sigma(0), 0, 0, 0, \dots \rangle$. Now we can define them and as well as one another stream operation, see Table 1.

So, we examine streams as dynamical entities, whose behavior consists of the repeatedly offering of the next element of the stream. Using coinduction streams are defined by specifying their behavior, and the equality of two streams can be established by proving that they have the same behavior (in other words, that they are 'behaviorally' equivalent).

The main reason for choosing to use streams as the coinductive datatype consists in a possibility to give definitions very close in style to those with self-reference corresponding to $\sigma = a :: \sigma$.

We can try to get non-WF probabilities on a non-WF algebra $F^V(A^\omega)$ of fuzzy subsets $Y \subset A^\omega$ that consists of the following: (1) union, intersection, and difference of two *non-WF fuzzy* subsets of A^ω; (2) $\emptyset$ and A^ω. In this case a *finitely additive non-WF probability measure* is a nonnegative set function $P(\cdot)$ defined for sets $Y \in F^V(A^\omega)$ that runs the set V (for example, $V = \mathbf{Z}_p$) and satisfies the following properties: (1) $P(A) \geq [0]$ for all $A \in F^V(A^\omega)$, (2) $P(A^\omega) = |1|$ and $P(\emptyset) = [0]$, (3) if $A \in F^V(A^\omega)$ and $B \in F^V(A^\omega)$ are disjoint, then $P(A \cup B) = P(A) + P(B)$, (4) $P(\neg A) = |1| + |-1| \times P(A)$ for all $A \in F^V(A^\omega)$.

This probability measure is called *non-WF probability*. Their main originality is that conditions 3, 4 are independent. As a result, in a probability space $\langle X, F^V(X), P\rangle$ some Bayes' formulas do not hold in the general case.

Suppose that the ordering relation on $\mathbf{Z}_p$ is defined digit by digit. Then the number -1 is the greatest in $\mathbf{Z}_p$. As an example of *trivial non-WF probability* we can introduce the following function defined on p-adic streams by coinduction: $P(\sigma) = \inf(\sigma, |-1|) \times |-1|^{-1}$ for every $\sigma \in \mathbf{Z}_p$. Notice that p-adic probabilities used in adelic or p-adic quantum mechanics are particular cases of non-WF probabilities.

If we take the p-adic case of non-WF probability theory, then we observe essentially new properties of relative frequencies that do not appear on real numbers. For example, consider two attributes α_1 and α_2. Suppose that in the first $N := N_k = (\sum_{j=0}^{k} 2^j)^2$ tests the label α_1 has $n_N(\alpha_1; x) = 2^k$ realizations, α_2 has $n_N(\alpha_2; x) = \sum_{j=0}^{k} 2^j$ realizations. According to our intuition, their probabilities should be different, but in real probability theory we obtain: $P_x(\alpha_1) = \lim n_N(\alpha_1; x)/N = P_x(\alpha_2) = \lim n_N(\alpha_2; x)/N = 0$. In 2-adic probability theory we have $P_x(\alpha_1) = 0 \neq P_x(\alpha_2) = -1$, because in $\mathbf{Q}_2$, $2^k \to 0$, $k \to 0$, and $-1 = 1 + 2 + 2^2 + \dots + 2^n + \dots$

This example shows that in p-adic probability theory there are statistical phenomena for that relative sequences of observed events have non-zero probabilities in the p-adic metric, but do not have positive probabilities in the standard real metric.

Now p-adic numbers are applied intensively in different domains of physics — quantum logic, string theory, cosmology, quantum mechanics, quantum foundations, see, e.g., [11, 49, 50], dynamical systems [26, 47, 48], biological and cognitive models [26, 27]. Recall that p-adic valued probabilities were introduced in [25] to serve p-adic theoretical physics. In some quantum physical models [15] a wave function (which is a complex probability amplitude in ordinary QM) takes vales in $\mathbf{Q}_p$ (for some prime number p) or its quadratic extensions. Such a wave function can be interpreted probabilistically in the framework of p-adic probability theory. So, p-adic models of classical statistical mechanics were considered recently and some preliminary results about invariant p-adic valued measures for dynamical systems were obtained.

4. p-ADIC PROBABILITIES AND NON-WELL-FOUNDED REALITY

Note that p-adic analysis as a kind of non-WF mathematics is a mathematical approach independent of real analysis and it has now a lot of practical applications. Traditionally, since Newton proposed his mechanics, real analysis was used in physical investigations as the only mathematical approach. Let us remember that the third Newtonian rule of reasoning in natural philosophy says: "The qualities of bodies, which admit neither intension nor remission of degrees, and which are found to belong to all bodies within reach of our experiments, are to be esteemed the universal qualities of all bodies whatsoever" [35]. This means that the whole physical reality is to be described by mathematics. But real analysis was the only mathematics which was known in classical physics.

According to the forth Newtonian rule, "In experimental philosophy we are to look upon propositions collected by general induction from phenomena as accurately or very nearly true, notwithstanding any contrary hypotheses that may be imagined, till such time as other phenomena occur, by which they may either be made more accurate, or liable to exceptions" [35]. This means that *physical reality is to be regarded as reality of stable repetitive phenomena*. Therefore physical measurements are to be based on the principle of the statistical stabilization of relative frequencies in the long run of trials. In the mathematical model this principle is represented by the law of large numbers.

Let us assume that any observational measurement is based on rational numbers [48]. Then experiments considering a stabilization of empirical phenomena provide us with a topology as completion of the field of rational numbers. The conventional meaning of stabilization is based on real numbers with respect to the standard real metric $\rho_{\mathbf{R}}(x,y) = |x-y|$ (the distance between points x and y on the real line $\mathbf{R}$). In modern physics the real physical reality (i.e., reality based on the $\rho_{\mathbf{R}}$-stability) is, in fact, identified with the whole physical reality. However, we need not more identify any stabilization just with the $\rho_{\mathbf{R}}$-stabilization. We can identify it also with new forms of stabilization in physical experiments, namely with p-adic forms [24, 25], because besides the usual real topology as a completion of the field $\mathbf{Q}$ of rational numbers, for which $\rho_{\mathbf{R}}(x,y) = |x-y|$, there exist only the p-adic topologies $p = 2, 3, 5, \ldots$ as completions of $\mathbf{Q}$, where $\rho_p(x,y) = |x-y|_p$ (for more details see [30]). Therefore we can introduce a special physical theory, p-adic physics [23, 48, 49]. The main reason is that p-adic numbers are, in fact, a unique alternative to real numbers: there is no other possibility to complete the field of rational numbers and obtain a new number field (Ostrovski's theorem, see, for example, [30]).

The probability of an event may be defined as the limit of the relative frequencies of the occurrence of the event when the volume of the statistical sample tends to infinity. For example, J. Bernoulli affirmed that the probability of a visitation of a plague in a given year was equal to the ratio of the number of these visitations during a long period of time to the number of years in that period. Also, the probability of an event E is defined as the limit of the sequence of frequencies $\nu^{(E)} = n/N$, where n is the number of cases in which the event E is detected in the first N tests. In the meantime, statistical stabilization of relative frequencies $\{\nu = n/N\}$ can be considered not only in the real topology on $\mathbf{Q}$, but also in any other topology on $\mathbf{Q}$, namely in p-adic topologies, $p = 2, 3, 5, \ldots$, see [24, 25, 29].

Real probabilities are obtained as a result of a limiting process for rational frequencies in real topology by means of the law of large numbers. Using these probabilities we accept only WF phenomena. In p-adic physics and in p-adic probability theory we assume that reality is non-WF. Since statistical stabilization (the limiting process) can be considered not only in the real topology on the field of rational numbers $\mathbf{Q}$ but also in p-adic topologies on $\mathbf{Q}$, we see that reality can be considered as non-WF too.

It is known (see the previous section, for more details [25]) that *in p-adic probability theory there are statistical phenomena for that relative sequences of observed events stabilize in the p-adic metric, but fluctuate in the standard real metric*. This means that some physical phenomena are probable in p-adic metric, but they are improbable in real metric. More formally, *the element of reality that it would be impossible to observe by using real measurement procedure might be observed by using p-adic measurement procedure*. Therefore we can suppose that p-adic physics is to be regarded as alternative approach to physical reality (to treatment of experimental data).

If in the real topology statistical stabilization is absent, then it is not possible to obtain any physical constants in the language of ordinary probability theory. But these constants could exist and be, for example, p-adic numbers. If a collective has not only a real probability distribution but an entire spectrum of other distributions, then, besides real constants corresponding to physical properties of the investigated objects, we obtain an entire spectrum of new constants corresponding to physical properties that were hidden from the real statistics. *In accordance with assuming that reality is non-WF, experimental results may be analyzed not only in the field of real numbers but also in p-adic fields (or more generally, on streams)*. Some recent applications of unconventional (non-WF) approach to experimental data using distribution on p-adic fields are as follows: a) in cognitive science and neurophysiology [5, 28]; b) logical foundation of p-adic probability [29]; c) p-adic cosmology and quantum physics [48–50].

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