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On Existence and Numerical Solution of a New Class of Nonlinear Second Degree Integro-Differential Volterra Equation with Convolution Kernel

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Abstract—This paper considers a new class of nonlinear second degree integro-differential Volterra equation with a convolution kernel. We derive some sufficient conditions to establish the existence and uniqueness of solutions by using Schauder fixed point theorem. Moreover, the Nyström method is applied to obtain the approximate solution of the proposed Volterra equation. A numerical examples are given to validate the adduced results.

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1. INTRODUCTION

In recent years, the theory of Volterra (Fredholm) integral and integro-differential equations have become as an interesting and popular field of research, because of their applications in many engineering and scientific disciplines, such as: mechanical phenomena, control technology, electrical engineering, models of population growth and fluid dynamics. See for details [1–7].

For this reason, a lot of works have been developed for studding a different kinds of integral equations. For example, we list some types of these equations with only two references to each one: Linear and nonlinear Volterra and Fredholm equations [8, 9]; integro-differential equations [10, 11]; integral equations in the complex plane [12, 13]; equations with weakly singular kernels [14, 15]; equations with Toeplitz plus Hankel Kernels [16, 17]; integral equations involving constant delay [18, 19]; equations in two-dimensional space [20, 21]; Chandrasekhar integral equation [22, 23]; Abel's integral equation [24, 25]; fuzzy integral equations [26, 27]; fractional integral equations [28, 29]; etc.

In this study, we are interested in a new kind of Volterra equation, which have a nonlinear convolution kernel that involves the first and second derivatives of solution. This equation is presented in the following form:

$$u(t) = \int_a^t g(t-s)\varphi(t,s,u(s),u'(s),u''(s))ds + f(t), \quad \forall t \in \mathcal{I} = [a,b],$$

where $f \in C^2(\mathcal{I})$, $g \in C^2(\mathcal{I}^2)$, $g(0) = 0$, $\partial_t g(0) = \lambda \in \mathbb{R}$, and $\varphi \in C^2(\mathcal{I}^2 \times \mathbb{R}^3)$ are given functions and u is the unknown to be found in the space $C^2(\mathcal{I})$.

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On the other hand, we mention that the unknown function u and its derivatives appear nonlinearly under the integral operator. Therefore, in order to control the solution of the proposed equation and its derivatives, we need to derive both sides of equation twice. Which allows us to convert our equation after doing some simple calculus to the following system:

$$u(t) = \int_a^t g(t-s)\varphi(t,s,u(s),u'(s),u''(s))ds + f(t), \quad \forall t \in \mathcal{I}, \tag{1}$$

$$u'(t) = \int_a^t \left(\partial_t g(t-s)\varphi(t,s,u(s),u'(s),u''(s)) + g(t-s)\partial_t\varphi(t,s,u(s),u'(s),u''(s))\right) ds$$
$$+f'(t), \quad \forall t \in \mathcal{I}, \tag{2}$$

$$u''(t) = \lambda\varphi(t,t,u(t),u'(t),u''(t)) + \int_a^t \left(2\partial_t g(t-s)\partial_t\varphi(t,s,u(s),u'(s),u''(s))\right.$$
$$\left. + \partial_t^2 g(t-s)\varphi(t,s,u(s),u'(s),u''(s)) + g(t-s)\partial_t^2\varphi(t,s,u(s),u'(s),u''(s))\right) ds$$
$$+f''(t), \quad \forall t \in \mathcal{I}. \tag{3}$$

Furthermore, Eqs. (1)–(3) of this system will serve an important role throughout the study.

The paper is structured as follows: In Section 2, we prove the existence of solution to the proposed problem by means of fixed point theorem of Schauder. Section 3, contains the uniqueness results of problem's solution. In Section 4, we discuss the Nyström method to give an approximate solution of our equation. In the last section, we present some illustrative examples.

2. EXISTENCE RESULTS VIA SCHAUDER'S FIXED POINT THEOREM

In this section, we present the existence results of solution of the proposed Eq. (1) by using Schauder fixed point theorem. Before proving the main result, we need to make the following assumptions:

($\mathcal{A}_1$): let $\varphi(t,s,x,y,z)$ be a function belongs to $C^2(\mathcal{I}^2 \times \mathbb{R}^3)$ and there exists a constant $M_1 > 0$ such that $\forall t,s \in \mathcal{I}$, $\forall x,y,z \in \mathbb{R}$

$$\max\left(|\varphi(t,s,x,y,z)|, |\partial_t\varphi(t,s,x,y,z)|, |\partial_t^2\varphi(t,s,x,y,z)|\right) \le M_1;$$

($\mathcal{A}_2$): let $g(t,s)$ be a function belongs to $C^2(\mathcal{I}^2)$ that satisfies $g(0)=0, \partial_t g(0) = \lambda$, and there exists a constant $M_2 > 0$ such that $\forall t,s \in \mathcal{I}$

$$\max\left(|g(t-s)|, |\partial_t g(t-s)|, |\partial_t^2 g(t-s)|\right) \le M_2.$$

Theorem 1. *Let* ($\mathcal{A}_1$) *and* ($\mathcal{A}_2$) *be verified. Then the Volterra equation* (1) *has at least one solution in the space* $C^2(\mathcal{I})$.

Proof. Let $\Phi : C^2(\mathcal{I}) \to C^2(\mathcal{I})$ be an integral operator defined by the following form: $\forall \xi \in C^2(\mathcal{I})$, $\forall t \in \mathcal{I}$

$$\Phi(\xi)(t) = \int_a^t g(t-s)\varphi\left(t,s,\xi(s),\xi'(s),\xi''(s)\right) ds + f(t).$$

It's clear that Eq. (1) has at least one solution in the space $C^2(\mathcal{I})$, if and only if the operator Φ has a fixed point. Which we will prove by using the Schauder fixed point theorem.

First, we can see easily that Φ is continuous from $C^2(\mathcal{I})$ to it self. Consider the subset $F \subset C^2(\mathcal{I})$ defined by the following way:

$$F := \left\{ \forall \xi \in C^2(\mathcal{I}) \text{ such that } \begin{array}{l} \xi\,(a) = f(a),\ \xi'(a) = f'(a), \\ |\xi(t) - f(t)| \le M_1 M_2\,(b-a)\,, \\ |\xi'(t) - f'(t)| \le 2M_1 M_2\,(b-a)\,, \\ |\xi''(t) - f''(t)| \le M_1 M_2 \left(4(b-a) + \dfrac{|\lambda|}{M_2}\right), \\ \forall \epsilon > 0,\ \exists\, \delta_\epsilon > 0,\ \forall t_1, t_2 \in \mathcal{I},\ |t_1 - t_2| < \delta_\epsilon, \\ \text{then } |\xi''(t_1) - \xi''(t_2)| < \epsilon \end{array} \right\}.$$

Before applying the Schauder fixed point theorem, the subset F must be nonempty, convex and closed. Obviously, F is nonempty and convex, we just prove that is closed. Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence in F, assume that it converges to some $\widetilde{\xi} \in C^2(\mathcal{I})$ in the norm of the space $C^2(\mathcal{I})$ as follows:

$$\|\xi_n - \widetilde{\xi}\| = \left(\|\xi_n - \widetilde{\xi}\|_\infty + \|\xi'_n - \widetilde{\xi}'\|_\infty + \|\xi''_n - \widetilde{\xi}''\|_\infty\right) \to 0, \text{ where } \|\xi\|_\infty = \sup_{t\in\mathcal{I}} |\xi(t)|.$$

Then, we need to verify that $\widetilde{\xi} \in F$, in order to confirm the closedness of F.

It is clear that the convergence in the space $C^2(\mathcal{I})$ means simultaneously uniform convergence of functions, of their derivatives and of their second derivatives, which permits us to write

$$\forall n \in \mathbb{N} \quad \xi_n(a) = f(a) \Rightarrow \lim_{n\to\infty} \xi_n(a) = f(a) \Rightarrow \widetilde{\xi}(a) = f(a),$$

$$\forall n \in \mathbb{N} \quad \xi'_n(a) = f'(a) \Rightarrow \lim_{n\to\infty} \xi'_n(a) = f'(a) \Rightarrow \widetilde{\xi}'(a) = f'(a).$$

Also,

$$\begin{aligned} \forall n \in \mathbb{N} \quad |\xi_n(t) - f(t)| \le M_1 M_2\,(b-a) \ &\Rightarrow\ \lim_{n\to\infty} |\xi_n(t) - f(t)| \le M_1 M_2\,(b-a) \\ &\Rightarrow\ \left|\lim_{n\to\infty} \xi_n(t) - f(t)\right| \le M_1 M_2\,(b-a) \\ &\Rightarrow\ \left|\widetilde{\xi}(t) - f(t)\right| \le M_1 M_2\,(b-a)\,. \end{aligned}$$

Similarly, we obtain:

$$|\,\widetilde{\xi}'(t) - f'(t)\,| \le 2M_1 M_2\,(b-a) \quad \text{and} \quad |\,\widetilde{\xi}''(t) - f''(t)\,| \le M_1 M_2 \left(4(b-a) + \frac{|\,\lambda\,|}{M_2}\right).$$

Now, from the last condition of F, it is clear that $\forall \epsilon > 0$, $\exists\,\delta_\epsilon > 0$, $\forall t_1, t_2 \in \mathcal{I}$, $|t_1 - t_2| < \delta_\epsilon$,

$$\left|\xi''_n(t_1) - \xi''_n(t_2)\right| < \frac{\epsilon}{3} \quad \forall n \in \mathbb{N}.$$

On the other hand, we have

$$\begin{aligned}\left|\widetilde{\xi}''(t_1)-\widetilde{\xi}''(t_2)\right| &= \left|\widetilde{\xi}''(t_1)-\xi_n''(t_1)+\xi_n''(t_1)-\xi_n''(t_2)+\xi_n''(t_2)-\widetilde{\xi}''(t_2)\right|\\ &\le \left|\widetilde{\xi}''(t_1)-\xi_n''(t_1)\right|+\left|\xi_n''(t_1)-\xi_n''(t_2)\right|+\left|\xi_n''(t_2)-\widetilde{\xi}''(t_2)\right|.\end{aligned}$$

Also, as ξ_n'' converges uniformly to $\widetilde{\xi}''$, we write:

$$\text{if } \forall\epsilon>0,\ \exists N_\epsilon\in\mathbb{N},\ \forall t\in\mathcal{I},\ \forall n\ge N_\epsilon,\quad \text{then} \quad \mid \xi_n''(t)-\widetilde{\xi}''(t)\mid<\frac{\epsilon}{3}.$$

By passing to infinity limit (i.e., $n\ge N_\epsilon$), the previous inequality gives us:

$$\left|\widetilde{\xi}''(t_1)-\widetilde{\xi}''(t_2)\right|<\frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon.$$

Thus $\widetilde{\xi}$ satisfies all conditions of subset F. Which means that F is closed.

We pass now to proving that Φ is completely continuous on the subset F.

First, from (1) and (2) we get directly $\Phi(\xi)(a)=f(a)$ and $\Phi(\xi)'(a)=f'(a)$. Now for all $\xi\in F$ and all $t\in\mathcal{I}$, we have

$$|\Phi(\xi)(t)-f(t)|=\left|\int_a^t g(t-s)\varphi\left(t,s,\xi\left(s\right),\xi'\left(s\right),\xi''\left(s\right)\right)ds\right|\le M_1M_2\left(b-a\right).$$

Also,

$$\begin{aligned}|\Phi(\xi)'(t)-f'(t)| &\le \left|\int_a^t \partial_t g(t-s)\varphi\big(t,s,\xi(s),\xi'(s),\xi''(s)\big)ds\right|\\ &\quad+\left|\int_a^t g(t-s)\partial_t\varphi\big(t,s,\xi(s),\xi'(s),\xi''(s)\big)ds\right|\le 2M_1M_2\left(b-a\right).\end{aligned}$$

In the same way:

$$\begin{aligned}\left|\Phi(\xi)''(t)-f''(t)\right| &\le \left|\lambda\varphi\big(t,t,\xi(t),\xi'(t),\xi''(t)\big)+\int_a^t 2\partial_t g(t-s)\partial_t\varphi\big(t,s,\xi(s),\xi'(s),\xi''(s)\big)ds\right|\\ &\quad+\left|\int_a^t \partial_t^2 g(t-s)\varphi\big(t,s,\xi(s),\xi'(s),\xi''(s)\big)ds\right|\\ &\quad+\left|\int_a^t g(t-s)\partial_t^2\varphi\big(t,s,\xi(s),\xi'(s),\xi''(s)\big)ds\right|\\ &\le M_1M_2 4(b-a)+|\lambda|M_1\le M_1M_2\left(4(b-a)+\frac{|\lambda|}{M_2}\right).\end{aligned}$$

Now we want to verify that if $\forall\epsilon>0,\ \exists\delta_\epsilon>0,\ \forall t_1,t_2\in\mathcal{I}$ with $|t_1-t_2|<\delta_\epsilon$ then $|\Phi(\xi)''(t_1)-\Phi(\xi)''(t_2)|<\epsilon$. For $t_1,t_2\in\mathcal{I}$, $t_1\le t_2$, we have

$$\begin{aligned}&\left|\Phi(\xi)''(t_1)-\Phi(\xi)''(t_2)\right|\\ &\le \left|\lambda\varphi\big(t_1,t_1,\xi(t_1),\xi'(t_1),\xi''(t_1)\big)-\lambda\varphi\big(t_2,t_2,\xi(t_2),\xi'(t_2),\xi''(t_2)\big)\right|+|f''(t_1)-f''(t_2)|\\ &\quad+2\left|\int_a^{t_1}\partial_t g(t_1-s)\partial_t\varphi\big(t_1,s,\xi(s),\xi'(s),\xi''(s)\big)ds-\int_a^{t_2}\partial_t g(t_2-s)\partial_t\varphi\big(t_2,s,\xi(s),\xi'(s),\xi''(s)\big)ds\right|\end{aligned}$$

$$+\left|\int_a^{t_1}\partial_t^2 g(t_1-s)\varphi\big(t_1,s,\xi(s),\xi'(s),\xi''(s)\big)ds-\int_a^{t_2}\partial_t^2 g(t_2-s)\varphi\big(t_2,s,\xi(s),\xi'(s),\xi''(s)\big)ds\right|$$
$$+\left|\int_a^{t_1} g(t_1-s)\partial_t^2\varphi\big(t_1,s,\xi(s),\xi'(s),\xi''(s)\big)ds-\int_a^{t_2} g(t_2-s)\partial_t^2\varphi\big(t_2,s,\xi(s),\xi'(s),\xi''(s)\big)ds\right|.$$

By subdividing the integration interval we obtain:

$$\begin{aligned}
&\big|\Phi(\xi)''(t_1)-\Phi(\xi)''(t_2)\big| \\
&\le \big|\lambda\varphi\big(t_1,t_1,\xi(t_1),\xi'(t_1),\xi''(t_1)\big)-\lambda\varphi\big(t_2,t_2,\xi(t_2),\xi'(t_2),\xi''(t_2)\big)\big|+\big|f''(t_1)-f''(t_2)\big| \\
&\quad+2\left|\int_a^{t_1}\partial_t g(t_1-s)\big(\partial_t\varphi\big(t_1,s,\xi(s),\xi'(s),\xi''(s)\big)-\partial_t\varphi\big(t_2,s,\xi(s),\xi'(s),\xi''(s)\big)\big)ds\right| \\
&\quad+2\left|\int_a^{t_1}\partial_t\varphi\big(t_2,s,\xi(s),\xi'(s),\xi''(s)\big)\big(\partial_t g(t_1-s)-\partial_t g(t_2-s)\big)ds\right| \\
&\quad+2\left|\int_{t_1}^{t_2}\partial_t g(t_2-s)\partial_t\varphi\big(t_2,s,\xi(s),\xi'(s),\xi''(s)\big)ds\right| \\
&\quad+\left|\int_a^{t_1}\partial_t^2 g(t_1-s)\big(\varphi\big(t_1,s,\xi(s),\xi'(s),\xi''(s)\big)-\varphi\big(t_2,s,\xi(s),\xi'(s),\xi''(s)\big)\big)ds\right| \\
&\quad+\left|\int_a^{t_1}\varphi\big(t_2,s,\xi(s),\xi'(s),\xi''(s)\big)\big(\partial_t^2 g(t_1-s)-\partial_t^2 g(t_2-s)\big)ds\right| \\
&\quad+\left|\int_{t_1}^{t_2}\partial_t^2 g(t_2-s)\varphi\big(t_2,s,\xi(s),\xi'(s),\xi''(s)\big)ds\right| \\
&\quad+\left|\int_a^{t_1} g(t_1-s)\big(\partial_t^2\varphi\big(t_1,s,\xi(s),\xi'(s),\xi''(s)\big)-\partial_t^2\varphi\big(t_2,s,\xi(s),\xi'(s),\xi''(s)\big)\big)ds\right| \\
&\quad+\left|\int_a^{t_1}\partial_t^2\varphi\big(t_2,s,\xi(s),\xi'(s),\xi''(s)\big)\big(g(t_1-s)-g(t_2-s)\big)ds\right| \\
&\quad+\left|\int_{t_1}^{t_2} g(t_2-s)\partial_t^2\varphi\big(t_2,s,\xi(s),\xi'(s),\xi''(s)\big)ds\right|.
\end{aligned}$$

The application of the mean value theorem on functions φ, $\partial_t\varphi$, g, and $\partial_t g$, respectively gives us:

$$\begin{aligned}
\big|\Phi(\xi)''(t_1)-\Phi(\xi)''(t_2)\big| \le\ &(|\lambda|M_1+4M_1M_2+6M_1M_2(b-a))|t_1-t_2| \\
&+\big|f''(t_1)-f''(t_2)\big|+M_1\int_a^{t_1}\big|\partial_t^2 g(t_1-s)-\partial_t^2 g(t_2-s)\big|ds \\
&+M_2\int_a^{t_1}\big|\partial_t^2\varphi\big(t_1,s,\xi(s),\xi'(s),\xi''(s)\big)-\partial_t^2\varphi\big(t_2,s,\xi(s),\xi'(s),\xi''(s)\big)\big|ds.
\end{aligned}$$

Let $\epsilon>0$. If we took $\mid t_1-t_2\mid<\delta_\epsilon^1$, where $\delta_\epsilon^1=\frac{\epsilon}{4(|\lambda|M_1+4M_1M_2+6M_1M_2(b-a))}$, clearly we get

$$\begin{aligned}
\big|\Phi(\xi)''(t_1)-\Phi(\xi)''(t_2)\big| <\ &\frac{\epsilon}{4}+\big|f''(t_1)-f''(t_2)\big|+M_1\int_a^{t_1}\big|\partial_t^2 g(t_1-s)-\partial_t^2 g(t_2-s)\big|ds \\
&+M_2\int_a^{t_1}\big|\partial_t^2\varphi\big(t_1,s,\xi(s),\xi'(s),\xi''(s)\big)-\partial_t^2\varphi\big(t_2,s,\xi(s),\xi'(s),\xi''(s)\big)\big|ds.
\end{aligned}$$

Moreover, since f'', $\partial_t^2 g$ and $\partial_t^2\varphi$ are uniformly continuous as functions of t over the interval $\mathcal{I}$, then there exist $\delta_\epsilon^2>0$, $\delta_\epsilon^3>0$, and $\delta_\epsilon^4>0$, respectively, where $\forall t_1,t_2\in\mathcal{I}$, with $\mid t_1-t_2\mid<\delta_\epsilon^2$, $\mid t_1-t_2\mid<\delta_\epsilon^3$, and $\mid t_1-t_2\mid<\delta_\epsilon^4$. We have

$$\begin{aligned}
&\big|f''(t_1) - f''(t_2)\big| < \frac{\epsilon}{4},\\
&\big|\partial_t^2 g(t_1 - s) - \partial_t^2 g(t_2 - s)\big| < \frac{\epsilon}{4M_1(b-a)},\\
&\big|\partial_t^2 \varphi\big(t_1, s, \xi(s), \xi'(s), \xi''(s)\big) - \partial_t^2 \varphi\big(t_2, s, \xi(s), \xi'(s), \xi''(s)\big)\big| < \frac{\epsilon}{4M_2(b-a)}.
\end{aligned}$$

By choosing $\delta_\epsilon = \min\{\delta_\epsilon^1,\, \delta_\epsilon^2,\, \delta_\epsilon^3,\, \delta_\epsilon^4\}$, we get $\forall t_1, t_2 \in \mathcal{I}$, with $\mid t_1 - t_2 \mid < \delta_\epsilon$,

$$\begin{aligned}
\big|\Phi(\xi)''(t_1) - \Phi(\xi)''(t_2)\big| \;&<\; \frac{\epsilon}{4} + \frac{\epsilon}{4} + M_1 \int_a^{t_1} \frac{\epsilon}{4M_1(b-a)} ds + M_2 \int_a^{t_1} \frac{\epsilon}{4M_2(b-a)} ds\\
&=\; \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon.
\end{aligned}$$

So, we conclude that $\Phi(F) \subset F$. Now to prove that operator Φ is compact, it is enough to prove that F is a compact subset. In order to show that F is compact, it is necessary to prove that F is uniformly bounded and equicontinuous. The uniform boundedness is evident according to the form of subset F which gives us:

$$\begin{aligned}
|\xi(t)| \;&\leq\; M_1 M_2\,(b-a) + \max_{s\in\mathcal{I}} |f(s)|,\\
|\xi'(t)| \;&\leq\; 2M_1 M_2\,(b-a) + \max_{s\in\mathcal{I}} |f'(s)| = \varrho_1,\\
|\xi''(t)| \;&\leq\; M_1 M_2 \left(4(b-a) + \frac{|\lambda|}{M_2}\right) + \max_{s\in\mathcal{I}} |f''(s)| = \varrho_2.
\end{aligned}$$

We verify now the uniform equicontinuity. From the last property of subset F, from the boundedness of ξ' and ξ'' described above and by applying the mean value theorem, directly we get: $\forall \xi \in F$, $\forall \epsilon > 0$, $\exists\, \widetilde{\delta}_\epsilon = \min\left\{\frac{\epsilon}{\varrho_1},\, \frac{\epsilon}{\varrho_2},\, \delta_\epsilon\right\} > 0$, $\forall t_1, t_2 \in \mathcal{I}$ with $|t_1 - t_2| < \widetilde{\delta}_\epsilon$,

$$|\xi(t_1) - \xi(t_2)| < \epsilon, \quad |\xi'(t_1) - \xi'(t_2)| < \epsilon, \quad |\xi''(t_1) - \xi''(t_2)| < \epsilon.$$

Which means that F is uniformly equicontinuous. Hence along with the Arzela–Ascoli theorem [9] we confirm the compactness of subset F. So, we conclude that Φ is completely continuous. Finally, the application of Schauder's theorem shows that Φ has a fixed point $\xi = \Phi(\xi)$ in F, which represents a solution of the Volterra equation (1), as well as, its derivatives verify the Eqs. (2) and (3), respectively. □

3. UNIQUENESS RESULTS

Clearly, using Schauder fixed point theorem, only the existence of solution of the previous equation (1) have been guaranteed. So, to prove the uniqueness of this solution, we need the following auxiliary lemma.

Lemma. *Let $\gamma(t)$ be a continuous and positive function on $[a, b]$, which satisfies*:

$$\exists L > 0, \;\; \gamma(t) \leq L \int_a^t \gamma(s) ds,$$

then $\gamma(t) = 0$, $\forall t \in [a, b]$.

Proof. See [30]. □

On the other hand, we need also to introduce the following assumption:

($\mathcal{A}_3$): There exist constants $A, B, C, \overline{A}, \overline{B}, \overline{C}, \tilde{A}, \tilde{B}, \tilde{C} > 0$ such that $\forall t, s \in \mathcal{I}$, $\forall x, y, z, \overline{x}, \overline{y}, \overline{z} \in \mathbb{R}$,

$$\begin{aligned}
|\varphi(t,s,x,y,z) - \varphi(t,s,\overline{x},\overline{y},\overline{z})| &\leq A|x-\overline{x}| + B|y-\overline{y}| + C|z-\overline{z}|,\\
|\partial_t\varphi(t,s,x,y,z) - \partial_t\varphi(t,s,\overline{x},\overline{y},\overline{z})| &\leq \overline{A}|x-\overline{x}| + \overline{B}|y-\overline{y}| + \overline{C}|z-\overline{z}|,\\
|\partial_t^2\varphi(t,s,x,y,z) - \partial_t^2\varphi(t,s,\overline{x},\overline{y},\overline{z})| &\leq \widetilde{A}|x-\overline{x}| + \widetilde{B}|y-\overline{y}| + \widetilde{C}|z-\overline{z}|.
\end{aligned}$$

Theorem 2. *Let* ($\mathcal{A}_1$)–($\mathcal{A}_3$) *be verified. In addition, we assume that*:

$$|\,\lambda\,|\,C < 1,$$

then the Volterra equation (1) *has a unique solution in the space* $C^2(\mathcal{I})$.

Proof. Suppose that $u(t), v(t) \in C^2(\mathcal{I})$ are two solutions of Eq. (1). Let $\gamma(t)$ be a positive function defined by

$$\gamma\,(t) = |u(t) - v(t)| + |u'(t) - v'(t)| + |u''(t) - v''(t)|.$$

Going now to prove that $\gamma\,(t) = 0$ based on the previous lemma. Which means that $u(t) = v(t)$, $u'(t) = v'(t)$, and $u''(t) = v''(t)$.

First, we put

$$\theta = M_2 \max(A,B,C), \quad \overline{\theta} = M_2 \max(\overline{A},\overline{B},\overline{C}), \quad \widetilde{\theta} = M_2 \max(\widetilde{A},\widetilde{B},\widetilde{C}).$$

For all $t \in \mathcal{I}$, we have

$$\begin{aligned}
|u(t)-v(t)| &= \left|\int_a^t g(t-s)\left(\varphi(t,s,u(s),u'(s),u''(s)) - \varphi(t,s,v(s),v'(s),v''(s))\right)ds\right|\\
&\leq M_2\int_a^t \left(A|u(s)-v(s)| + B|u'(s)-v'(s)| + C|u''(s)-v''(s)|\right)ds\\
&\leq M_2\max(A,B,C)\int_a^t \left(|u(s)-v(s)| + |u'(s)-v'(s)| + |u''(s)-v''(s)|\right)ds\\
&= \theta\int_a^t \gamma(s)ds. \qquad (4)
\end{aligned}$$

In the same way, we obtain:

$$\begin{aligned}
|u'(t)-v'(t)| &\leq \left|\int_a^t \partial_t g(t-s)\big(\varphi\big(t,s,u(s),u'(s),u''(s)\big) - \varphi\big(t,s,v(s),v'(s),v''(s)\big)\big)ds\right|\\
&+\left|\int_a^t g(t-s)\left(\partial_t\varphi\big(t,s,u(s),u'(s),u''(s)\big) - \partial_t\varphi\big(t,s,v(s),v'(s),v''(s)\big)\right)ds\right|\\
&\leq M_2\int_a^t \left(A|u(s)-v(s)| + B|u'(s)-v'(s)| + C|u''(s)-v''(s)|\right)ds
\end{aligned}$$

$$+M_2\int_a^t \left(\overline{A}|u(s)-v(s)|+\overline{B}|u'(s)-v'(s)|+\overline{C}|u''(s)-v''(s)|\right)ds$$

$$\leq M_2\max(A,B,C)\int_a^t (|u(s)-v(s)|+|u'(s)-v'(s)|+|u''(s)-v''(s)|)ds$$

$$+M_2\max(\overline{A},\overline{B},\overline{C})\int_a^t \left(|u(s)-v(s)|+|u'(s)-v'(s)|+|u''(s)-v''(s)|\right)ds$$

$$=\left(\theta+\overline{\theta}\right)\int_a^t \gamma(s)ds. \tag{5}$$

Then similar as before, we get:

$$\begin{aligned}
\left|u''(t)-v''(t)\right| \leq\ & |\lambda|\left|\left(\varphi\big(t,t,u(t),u'(t),u''(t)\big)-\varphi\big(t,t,v(t),v'(t),v''(t)\big)\right)\right| \\
&+\left|\int_a^t 2\partial_t g(t-s)\left(\partial_t\varphi\big(t,s,u(s),u'(s),u''(s)\big)-\partial_t\varphi\big(t,s,v(s),v'(s),v''(s)\big)\right)ds\right| \\
&+\left|\int_a^t \partial_t^2 g(t-s)\left(\varphi\big(t,s,u(s),u'(s),u''(s)\big)-\varphi\big(t,s,v(s),v'(s),v''(s)\big)\right)ds\right| \\
&+\left|\int_a^t g(t-s)\left(\partial_t^2\varphi\big(t,s,u(s),u'(s),u''(s)\big)-\partial_t^2\varphi\big(t,s,v(s),v'(s),v''(s)\big)\right)ds\right| \\
\leq\ & |\lambda|\left(A|u(t)-v(t)|+B|u'(t)-v'(t)|+C|u''(t)-v''(t)|\right) \\
&+2M_2\int_a^t \left(\overline{A}|u(s)-v(s)|+\overline{B}|u'(s)-v'(s)|+\overline{C}|u''(s)-v''(s)|\right)ds \\
&+M_2\int_a^t \left(A|u(s)-v(s)|+B|u'(s)-v'(s)|+C|u''(s)-v''(s)|\right)ds \\
&+M_2\int_a^t \left(\widetilde{A}|u(s)-v(s)|+\widetilde{B}|u'(s)-v'(s)|+\widetilde{C}|u''(s)-v''(s)|\right)ds.
\end{aligned}$$

Thus

$$\begin{aligned}
|u''(t)-v''(t)| \leq\ & |\lambda|A|u(t)-v(t)|+|\lambda|B|u'(t)-v'(t)|+|\lambda|C|u''(t)-v''(t)| \\
&+2M_2\max(\overline{A},\overline{B},\overline{C})\int_a^t \left(|u(s)-v(s)|+|u'(s)-v'(s)|+|u''(s)-v''(s)|\right)ds \\
&+M_2\max(A,B,C)\int_a^t \left(|u(s)-v(s)|+|u'(s)-v'(s)|+|u''(s)-v''(s)|\right)ds \\
&+M_2\max(\widetilde{A},\widetilde{B},\widetilde{C})\int_a^t \left(|u(s)-v(s)|+|u'(s)-v'(s)|+|u''(s)-v''(s)|\right)ds \\
=\ & |\lambda|A|u(t)-v(t)|+|\lambda|B|u'(t)-v'(t)|+|\lambda|C|u''(t)-v''(t)|+(2\overline{\theta}+\theta+\widetilde{\theta})\int_a^t \gamma(s)ds.
\end{aligned}$$

We obtain from inequalities (4) and (5) the fact that:

$$\left|u''(t)-v''(t)\right|\leq|\lambda|C\left|u''(t)-v''(t)\right|+\left(|\lambda|A\theta+|\lambda|B(\theta+\overline{\theta})+2\overline{\theta}+\theta+\widetilde{\theta}\right)\int_a^t \gamma\left(s\right)ds.$$

By the property $|\lambda|C<1$ we find:

$$\left|u''(t)-v''(t)\right| \leq \left(\frac{\theta(|\lambda|A+|\lambda|B+1)+\overline{\theta}(|\lambda|B+2)+\widetilde{\theta}}{1-|\lambda|C}\right)\int_a^t \gamma\,(s)\,ds. \tag{6}$$

Furthermore, according to inequalities (4), (5), and (6), we confirm that there exists a positive parameter L which fulfils:

$$\gamma\,(t) \leq L\int_a^t \gamma\,(s)\,ds,$$

where L is given by:

$$L=\left(2\theta+\overline{\theta}+\frac{\theta(|\lambda|A+|\lambda|B+1)+\overline{\theta}(|\lambda|B+2)+\widetilde{\theta}}{1-|\lambda|C}\right).$$

Thanks to the lemma, we obtain $\gamma(t)=0$, which implies that Eq. (1) has a unique solution in the space $C^2(\mathcal{I})$. □

4. NUMERICAL STUDY

In the previous sections, under the assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_3)$, we have shown that Eq. (1) has a unique solution in $C^2(\mathcal{I})$. As a matter of fact, this solution cannot be found exactly. For this reason, one must approach this solution by considering some numerical methods. In this section, we will use the Nyström method described in [9], which enables us to obtain an approximate solution of Eq. (1). First, we start by recalling Nyström's method. For $N\in\mathbb{N}$, and by considering the discretization step $h=\frac{b-a}{N}$, we define an equidistant subdivision of interval $\mathcal{I}$ as follows:

$$s_j=a+jh,\quad 0\leq j\leq N,$$

then, the Nyström method is a technique seeks the approximate solution of an integral equation by replacing the integral with a chosen quadrature formula such as

$$\int\limits_a^b \xi(s)ds \simeq h\sum_{j=0}^{N}\omega_j\xi(s_j),$$

where ω_i are real weights such that: $\max\limits_{0\leq j\leq N} \mid \omega_j \mid \leq \varpi < \infty$.

Now, by collocating Eqs. (1), (2), and (3) at the following grid points $t_i=a+ih,\ 0\leq i\leq N$, then by applying the Nyström method, we obtain the following algebraic system:

for $i=0$: (initial values)

$$U_0=f(a),\quad V_0=f'(a),\quad W_0=f''(a)+\lambda\varphi(a,a,U_0,V_0,W_0); \tag{7}$$

for $1\leq i\leq N$

$$U_i=f(t_i)+h\sum_{j=0}^{i}\omega_j g(t_i-t_j)\varphi(t_i,t_j,U_j,V_j,W_j), \tag{8}$$

$$V_i=f'(t_i)+h\sum_{j=0}^{i}\omega_j\big(\partial_t g(t_i-t_j)\varphi(t_i,t_j,U_j,V_j,W_j)+g(t_i-t_j)\partial_t\varphi(t_i,t_j,U_j,V_j,W_j)\big), \tag{9}$$

$$W_i = f''(t_i) + \lambda\varphi(t_i, t_i, U_i, V_i, W_i) + h\sum_{j=0}^{i} 2\omega_j \partial_t g(t_i - t_j)\partial_t \varphi(t_i, t_j, U_j, V_j, W_j)$$

$$+h\sum_{j=0}^{i}\omega_j\left(\partial_t^2 g(t_i - t_j)\varphi(t_i, t_j, U_j, V_j, W_j) + g(t_i - t_j)\partial_t^2\varphi(t_i, t_j, U_j, V_j, W_j)\right), \qquad (10)$$

where U_i, V_i, and W_i represent the approximate values at the grid points of $u(t_i)$, $u'(t_i)$, and $u''(t_i)$, respectively.

Finally, we can see that the arising system is nonlinear. So, in practice, we must use a computing environment like MATLAB software, in order to get the roots of this system. Which means that we have found the approximate solution of Eq. (1).

On the other hand, an important question remains: Are the previous assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ sufficient to ensure the existence and uniqueness of solution of the system (7)–(10)? This is what we will see in the next subsection.

4.1. System Study

In general, the hypotheses that confirm the existence and uniqueness of the solution of an equation in infinite dimensional space, do not remain the same hypotheses in a finite dimensional space. Therefrom, in the next theorem we add the necessary conditions in order to ensure that the arising system (7)–(10) has a unique solution.

Theorem 3. *Let* $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ *be verified. In addition, we assume that*

$$|\lambda| C < 1, \quad |\lambda| A < 1, \quad |\lambda| B < 1,$$

and for all sufficiently small h*, then the system* (7)–(10) *has a unique solution.*

Proof. First, it is obvious that Eq. (7) has a unique solution W_0 in view of the condition $|\lambda| C < 1$.

Now, consider the Euclidean space $\mathbb{R}^3$ having the following standard norm:

$$\forall \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \in \mathbb{R}^3, \qquad \left\| \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \right\|_1 = |X| + |Y| + |Z|.$$

For technical reasons, we define the application $\Psi_i : \mathbb{R}^3 \to \mathbb{R}^3$, for all $1 \leq i \leq N$, by the following

$$\Psi_i \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \Upsilon_1(X,Y,Z) \\ \Upsilon_2(X,Y,Z) \\ \Upsilon_3(X,Y,Z) \end{pmatrix},$$

where

$$\begin{aligned} \Upsilon_1(X,Y,Z) &= f(t_i) + h\omega_i g(t_i - t_i)\varphi(t_i, t_i, X, Y, Z) + h\sum_{j=0}^{i-1}\omega_j g(t_i - t_j)\varphi(t_i, t_j, U_j, V_J, W_j) \\ &= f(t_i) + h\sum_{j=0}^{i-1}\omega_j g(t_i - t_j)\varphi(t_i, t_j, U_j, V_j, W_j) = \vartheta_i^1. \end{aligned}$$

$$\begin{aligned}\Upsilon_2(X,Y,Z) &= f'(t_i) + h\omega_i \partial_t g(t_i - t_i)\varphi(t_i,t_i,X,Y,Z) + h\omega_i g(t_i - t_i)\partial_t \varphi(t_i,t_i,X,Y,Z)\\ &+ h\sum_{j=0}^{i-1} \omega_j \left(\partial_t g(t_i - t_j)\varphi(t_i,t_j,U_j,V_j,W_j) + g(t_i - t_j)\partial_t\varphi(t_i,t_j,U_j,V_j,W_j)\right)\\ &= f'(t_i) + \lambda\omega_i h\varphi(t_i,t_i,X,Y,Z) + \vartheta_i^2\end{aligned}$$

with

$$\vartheta_i^2 = h\sum_{j=0}^{i-1} \omega_j \left(\partial_t g(t_i - t_j)\varphi(t_i,t_j,U_j,V_j,W_j) + g(t_i - t_j)\partial_t\varphi(t_i,t_j,U_j,V_j,W_j)\right).$$

And

$$\begin{aligned}\Upsilon_3(X,Y,Z) &= f''(t_i) + \lambda\varphi(t_i,t_i,X,Y,Z) + 2h\omega_i\partial_t g(t_i - t_i)\partial_t\varphi(t_i,t_i,X,Y,Z)\\ &\quad + h\omega_i\partial_t^2 g(t_i - t_i)\varphi(t_i,t_i,X,Y,Z) + h\omega_i g(t_i - t_i)\partial_t^2\varphi(t_i,t_i,X,Y,Z) + \vartheta_i^3,\\ &= f''(t_i) + \lambda\varphi(t_i,t_i,X,Y,Z) + 2h\lambda\omega_i\partial_t\varphi(t_i,t_i,X,Y,Z)\\ &\quad + h\omega_i\partial_t^2 g(0)\varphi(t_i,t_i,X,Y,Z) + \vartheta_i^3,\end{aligned}$$

where

$$\begin{aligned}\vartheta_i^3 &= h\sum_{j=0}^{i-1}\omega_j\left(\partial_t^2 g(t_i - t_j)\varphi(t_i,t_j,U_j,V_j,W_j) + g(t_i - t_j)\partial_t^2\varphi(t_i,t_j,U_j,V_j,W_j)\right)\\ &\quad + h\sum_{j=0}^{i-1} 2\omega_j\partial_t g(t_i - t_j)\partial_t\varphi(t_i,t_j,U_j,V_j,W_j).\end{aligned}$$

Therefore, we can see that

$$\Psi_i\begin{pmatrix}X_1\\ Y_1\\ Z_1\end{pmatrix} - \Psi_i\begin{pmatrix}X_2\\ Y_2\\ Z_2\end{pmatrix} = \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix},$$

where β_1, β_2, and β_3 are given by

$$\begin{aligned}\beta_1 &= 0,\\ \beta_2 &= h\lambda\omega_i\left(\varphi(t_i,t_i,X_1,Y_1,Z_1) - \varphi(t_i,t_i,X_2,Y_2,Z_2)\right),\\ \beta_3 &= \lambda\left(\varphi(t_i,t_i,X_1,Y_1,Z_1) - \varphi(t_i,t_i,X_2,Y_2,Z_2)\right)\\ &\quad + 2\lambda h\omega_i\left(\partial_t\varphi(t_i,t_i,X_1,Y_1,Z_1) - \partial_t\varphi(t_i,t_i,X_2,Y_2,Z_2)\right)\\ &\quad + h\omega_i\partial_t^2 g(0)\left(\varphi(t_i,t_i,X_1,Y_1,Z_1) - \varphi(t_i,t_i,X_2,Y_2,Z_2)\right).\end{aligned}$$

As a result, using assumption $(\mathcal{A}_3)$, and by taking $\varrho = \mid \partial_t^2 g(0) \mid$, we obtain

$$\begin{aligned}\mid \beta_2 \mid &\le h \mid \lambda \mid \varpi\left(A \mid X_1 - X_2 \mid + B \mid Y_1 - Y_2 \mid + C \mid Z_1 - Z_2 \mid\right),\\ \mid \beta_3 \mid &\le \mid \lambda \mid \left(A \mid X_1 - X_2 \mid + B \mid Y_1 - Y_2 \mid + C \mid Z_1 - Z_2 \mid\right)\\ &\quad + 2 \mid \lambda \mid h\varpi\left(\bar{A} \mid X_1 - X_2 \mid + \bar{B} \mid Y_1 - Y_2 \mid + \bar{C} \mid Z_1 - Z_2 \mid\right)\\ &\quad + h\varpi\varrho\left(A \mid X_1 - X_2 \mid + B \mid Y_1 - Y_2 \mid + C \mid Z_1 - Z_2 \mid\right).\end{aligned}$$

Thus,

$$| \beta_1 | + | \beta_2 | + | \beta_3 | \leq \eta_1 | X_1 - X_2 | + \eta_2 | Y_1 - Y_2 | + \eta_3 | Z_1 - Z_2 |,$$

where

$$\begin{aligned}
\eta_1 &= h | \lambda | \varpi A + | \lambda | A + 2 | \lambda | h\varpi \bar{A} + h\varpi\varrho A, \\
\eta_2 &= h | \lambda | \varpi B + | \lambda | B + 2 | \lambda | h\varpi \bar{B} + h\varpi\varrho B, \\
\eta_3 &= h | \lambda | \varpi C + | \lambda | C + 2 | \lambda | h\varpi \bar{C} + h\varpi\varrho C.
\end{aligned}$$

If we denote $\eta = \max(\eta_1, \eta_2, \eta_3)$, we find:

$$\left\| \Psi_i \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} - \Psi_i \begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \end{pmatrix} \right\|_1 \leq \eta \left\| \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} - \begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \end{pmatrix} \right\|_1 .$$

For all sufficiently small h, and during conditions $| \lambda | C < 1$, $| \lambda | A < 1$, and $| \lambda | B < 1$, we get $0 < \eta < 1$. So, we conclude that Ψ_i is a contraction from $\mathbb{R}^3$ into itself. Consequently, the Banach fixed point theorem confirms us that system (8)–(10) has a unique solution. □

5. ILLUSTRATIVE EXAMPLES

In this section, we discuss two main examples, in order to validate the accuracy and practicality of the adduced results in this work.

Example 1. Consider the first equation:

$$u(t) = \int_0^t \ln(1 + t - s) \frac{t((s+1)^4 + 2s^2 + 4s + 2)}{(1 + u(s) + u'(s) + u''(s))^2} ds + f(t), \ \ t \in [0, 1],$$

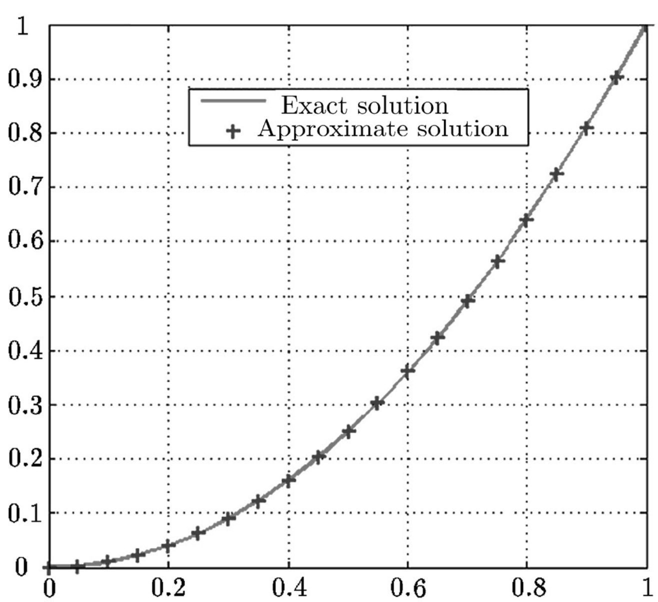

Fig. 1. Plot of exact and numerical solution of Example 1.

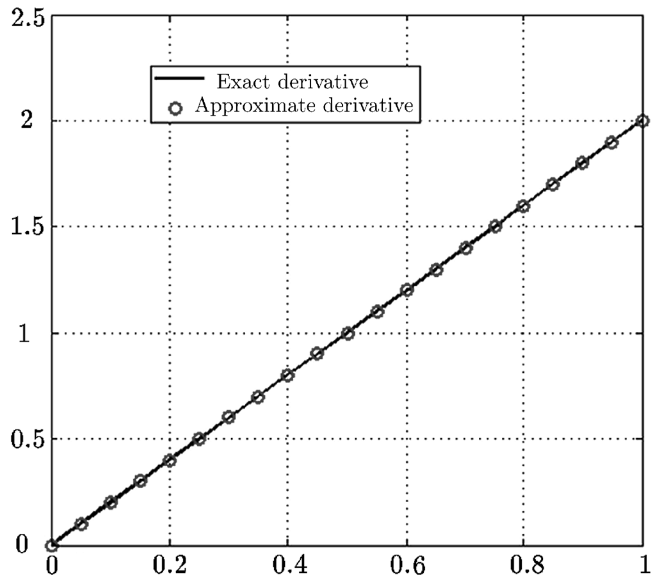

Fig. 2. Plot of exact and numerical derivative of solution of Example 1.

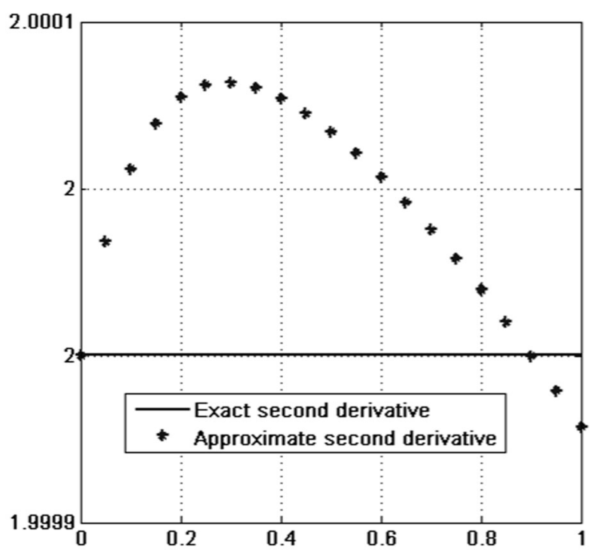

Fig. 3. Plot of exact and numerical second derivative of solution of Example 1.

if we take $f(t) = 2t^2 - (t^2 + t)\ln(t+1)$, we get the exact solution $u(t) = t^2$.
Example 2. Consider the second equation:

$$u(t) = \int\limits_0^t \left(\frac{t-s}{5}\right) \cos\left(s + t - 12\cos(4s)e^s + 13\sin(4s)e^s + u(s) + u'(s) + u''(s)\right) ds + f(t),$$

if we take

$$f(t) = \sin(4t)\exp(t) - 0.2(\cos(t) - 2\cos^2(t) - t\sin(t) + 1), \quad t \in [0,1],$$

we get the exact solution $u(t) = \sin(4t)\exp(t)$.

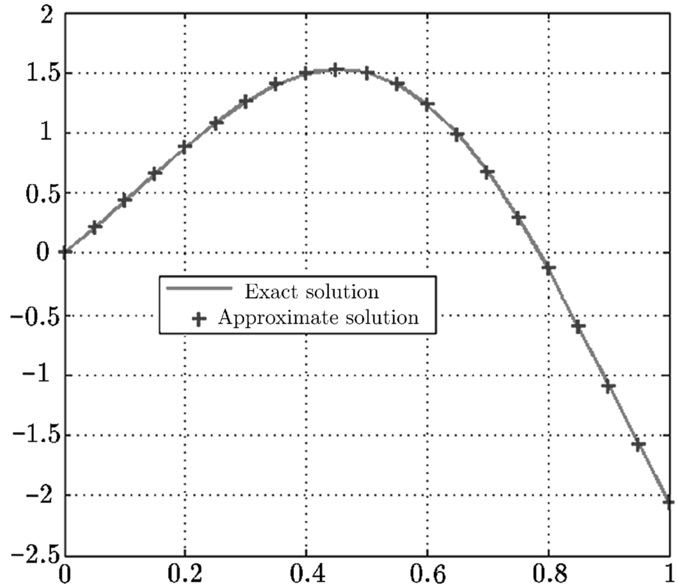

Fig. 4. Plot of exact and numerical solution of Example 2.

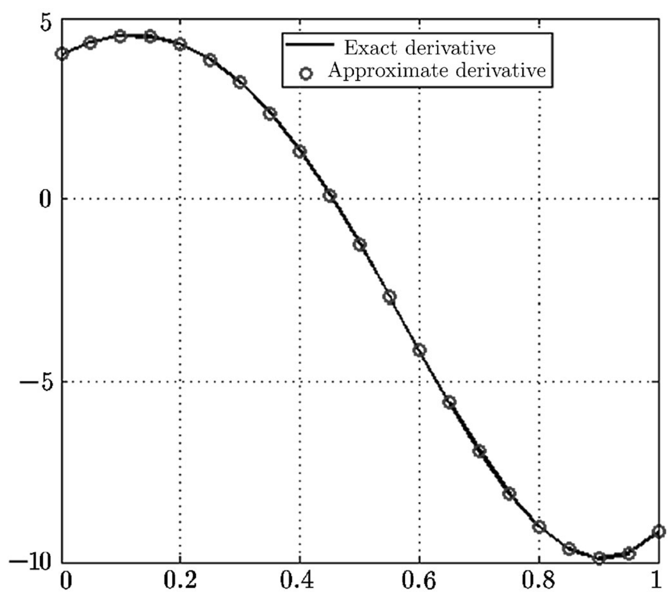

Fig. 5. Plot of exact and numerical derivative of solution of Example 2.

First, we can see that the kernels $g(t,s)$ and $\varphi(t,s,x,y,z)$ of Example 1 satisfy the assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_3)$. Moreover, we have $g(t-s) = \ln(1+t-s)$, so $g(0) = \ln(1) = 0$ and $\partial_t g(0) = \lambda = 1$, as well as the Lipschitz constants A, B, and C of the kernel φ verifying $A = B = C = \frac{1}{4}$, then we conclude that the necessary conditions proposed above $\mid \lambda \mid A < 1$, $\mid \lambda \mid B < 1$, and $\mid \lambda \mid C < 1$ are also fulfilled. Regarding to the second example, the kernels $g(t,s)$, and $\varphi(t,s,x,y,z)$ also satisfying the assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_3)$. The function $g(t-s) = \frac{t-s}{5}$, gives $g(0) = 0$ and $\partial_t g(0) = \frac{1}{5}$, and the Lipschitz constants A, B, and C verify $A = B = C = 1$. These confirm us that the conditions $\mid \lambda \mid A < 1$, $\mid \lambda \mid B < 1$, and $\mid \lambda \mid C < 1$ are fulfilled. Consequently, each of the two examples has a unique solution. Going now to approach their solutions by considering the system (7)–(10). Note that in all simulations, we have chosen the trapezoidal technique as a quadrature rule, and we have

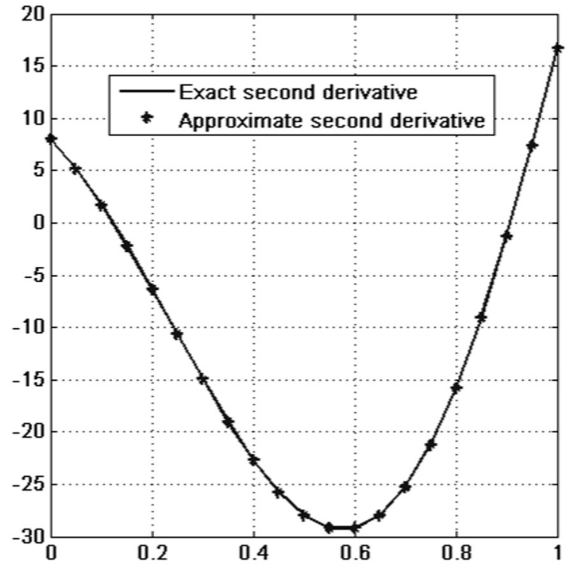

Fig. 6. Plot of exact and numerical second derivative of solution of Example 2.

Table 1. Errors analysis of the present method for Example 1

Error	N=10	N=100	N=250	N=500	N=1000
E_1	4.37E–4	4.38E–6	7.01E–7	1.75E–7	4.38E–8
E_2	1.87E–4	1.86E–6	2.99E–7	7.47E–8	1.87E–8
E_3	3.27E–4	3.26E–6	5.22E–7	1.30E–7	3.33E–8

Table 2. Errors analysis of the present method for Example 2

Error	N=10	N=100	N=250	N=500	N=1000
E_1	2.90E–4	2.90E–6	4.64E–7	1.16E–7	2.90E–8
E_2	7.51E–5	7.51E–7	1.20E–7	3.00E–8	7.51E–9
E_3	2.95E–5	2.97E–7	4.81E–8	1.26E–8	3.76E–9

used the Picard method as an iterative scheme. For comparison, we need to introduce the following error functions:

$$E_1 = \max_{0 \le i \le N} \mid u(t_i) - U_i \mid, \;\; E_2 = \max_{0 \le i \le N} \mid u'(t_i) - V_i \mid, \;\; E_3 = \max_{0 \le i \le N} \mid u''(t_i) - W_i \mid,$$

and by using a different values of N, we provide some tables and graphical illustrations.

In Figs. 1–6, a plot of the exact and approximate solutions of Examples 1 and 2, with their derivatives is displayed, which appear to be almost identical with only $N = 20$. Moreover, Tables 1 and 2, show us that the error functions E_1, E_2, and E_3 close to zero when N increases, which means that the approximate solutions and its derivatives converge to the exact solutions and its derivatives, respectively. So, these simulation results confirm the accuracy and performance of our work.

CONCLUSIONS

In this paper, we have suggested a class of nonlinear integro-differential Volterra equation with a convolution kernel. First, we have discussed the necessary and sufficient conditions to guarantee the existence and uniqueness of solution of the proposed equation. Then, we have constructed a numerical process based on the Nyström method to obtain an approximate solution of this equation. As well as, we have also supported our results by some illustrative examples.

CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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