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# On Discretization of the Evolution $p$-Bi-Laplace Equation

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**Abstract**—This article discusses a mixed finite element method combined with the backward-Euler method to study the hyperbolic $p-$bi-Laplace equation, where the existence and uniqueness of solution for the discretized problem are shown in Lebesgue and Sobolev spaces. A mixed formulation and an inf-sup condition are then given to prove the well-posedness of the scheme and optimal a priori error estimates for fully discrete schemes are extracted. Finally, a numerical example is given to confirm the theoretical results obtained.

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## 1. INTRODUCTION AND PRELIMINARIES

Let $\Omega$ be a bounded open domain of $\mathbb{R}^d$ with a Lipschitz-continuous boundary $\partial\Omega$. Fixing a final time $T > 0$, consider the following hyperbolic problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta(\operatorname{div}(|\Delta u|^{p-2}\nabla u)) = f(x,t) & \text{in } [0,T]\times\Omega, \\ u = \Psi, \ \nabla u = \nabla\Psi & \text{on } [0,T]\times\partial\Omega, \\ u(0,x) = u_0, \ \dfrac{\partial u}{\partial t}(0,x) = U_1 & \text{on } \Omega, \end{cases} \tag{1.1}$$

where $2 \le p < \infty$, $f(t)$ and $\Psi(t)$ are given functions in $L^q(\Omega)$ and $W^{2,\infty}(\Omega)$, respectively.

In recent years, substantial progress has been made in studying fourth order problems that are a nonlinear generalization of bi-Laplacian problems. The main drive to study (1.1) arises from the various applications in the field of elasticity that are used precisely in the modeling of traveling waves in suspension bridges (see [2, 11]).

High order PDEs with a constant exponent have been studied by several authors under various conditions on the data and by different methods, for example (see [1, 3, 10]). We also refer to some references interesting in the study of this type of equations with a variable exponent as in [13, 18].

The authors of [16] considered the following $p$-biharmonic elliptic problem:

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = f & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{in } \partial\Omega, \end{cases}$$

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where $\frac{\partial}{\partial n}$ is the outer normal derivative. By using a WEB-Spline mesh-free finite element method, the authors discussed the existence and uniqueness of a weak solution and then derived a discrete variational formulation for a $p$-biharmonic problem.

In [12], the authors proposed a discrete time method and uniform estimates for the following $p$-bi-Laplacian parabolic equation:

$$u_t + \Delta(|\Delta u|^{p-2}\Delta u) = |u|^q - \frac{1}{|\Omega|}\int\limits_{\Omega} |u| dx, \quad x \in \Omega,$$

where $\Omega \subset \mathbb{R}^N$, $p > 2$, and $\lambda > 0$. The authors established the existence and uniqueness of weak solutions.

The $C^1$ finite elements and the Argyris element [4] are among the similar approaches for such problems. But in three dimensions, we find obstacles in the design of $C^1$ finite elements difficult to implement. We mention other methods that can be applied to this class of problems: the discontinuous Galerkin methods which are a class of nonconforming finite element method and the $h$-$k$ $dG$ finite elements used for the 2-bi-Laplacian (see [7, 9]).

One of the options proposed to address our problem is to use mixed finite elements with respect to distance and the backward Euler method with respect to time.

The mixed finite elements are among the most popular methods used to study this family of problems. This method allows one to solve mixed problems where the unknowns are two functions. For more details about this method see [2, 5, 6]. Moreover, the convergence of this method is subject to inf-sup conditions taken from [8, 15].

The plan of the paper is as follows: In Section 1, we set some of the necessary and fundamental materials of the $p$-bi-Laplacian. Section 2 is devoted to extracting a semi-discretization scheme based on the backward Euler method and proving the existence and uniqueness of the solution for this scheme. In Section 3, we give a mixed formulation and an inf-sup condition to prove that the mixed approximation problem is well-posed. In Section 4, we deduce a fully discrete scheme and derive a priori error estimates with assistance of the Ritz projection operator and Galerkin orthogonality properties. Finally, we finish our work by a numerical experiment in Section 5.

We state some of the materials needed to prove our results.

For $1 \leq p < \infty$, we define the Lebesgue space $L^p(\Omega)$ as follows:

$$L^p(\Omega) = \left\{ u \,:\, \Omega \to \mathbb{R} \text{ measurable } \,:\, \int\limits_{\Omega} \|f(x)\|^p dx < \infty \right\}, \tag{1.2}$$

with the norm

$$\|f\|_{L^p} = \left( \int\limits_{\Omega} \|f(x)\|^p dx \right)^{1/p}. \tag{1.3}$$

**Definition 1.** For $1 \leq p < \infty$ and $m \in \mathbb{N}$, we define the Sobolev space $W^{m,p}(\Omega)$ as follows:

$$W^{m,p}(\Omega) = \Big\{ f \in L^p(\Omega)\,;\, D^\alpha f \in L^p(\Omega)\, \forall \alpha \in \mathbb{N} \text{ such that } |\alpha| \leq m \Big\}, \tag{1.4}$$

with the norm

$$\|f\|_{m,p} = \sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^p(\Omega)}. \tag{1.5}$$

**Definition 2.** We define the space $W^{2,p}_{\Psi}(\Omega)$ as follows:

$$W^{2,p}_{\Psi}(\Omega) = \Psi + W^{2,p}_{0}(\Omega) = \Big\{ f \in W^{2,p}_{\Psi}(\Omega)\,;\; f\backslash_{\partial\Omega} = \Psi \text{ and } \nabla f\backslash_{\partial\Omega} = \nabla\Psi \Big\}, \tag{1.6}$$

here

$$W^{2,p}_{0}(\Omega) = \overline{C^{\infty}_{0}(\Omega)}^{W^{2,p}(\Omega)}. \tag{1.7}$$

**Remark 1.**

**1**) Let $p, q \in [1, \infty)$, such that $q$ is the conjugate of $p$, which satisfies

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then for $u \in L^p(\Omega)$ we have

$$\Big\| \, |u|^{p-1} \Big\|_{L^q\Omega)} = \|u\|^{p-1}_{L^p(\Omega)}. \tag{1.8}$$

**2**) Let $p, q$, and $r \in [1, \infty)$, be such that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

For $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, we have

$$fg \in L^r(\Omega) \;\text{ and }\; \|uv\|_{L^r(\Omega)} \le \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}. \tag{1.9}$$

**Definition 3.** A function $u$ is said to be a weak solution of (1.1) if

$$\begin{cases} 1)\;\; u \in L^{\infty}\big([0,T], W^{2,p}_{0}(\Omega)\big) \cap W^{1,\infty}([0,T], L^2(\Omega)) \text{ such that} \\ \quad\;\; \forall v \in L^{\infty}\big([0,T], W^{2,p}_{0}(\Omega)\big) \cap W^{1,\infty}([0,T], L^2(\Omega)). \\ 2)\;\; \int\limits_0^T \Big(\dfrac{\partial^2 u}{\partial t^2}, v\Big) dt + \int\limits_0^T \big(|\Delta u|^{p-2}\Delta u, \Delta v\big) dt = \int\limits_0^T (f, v). \end{cases} \tag{1.10}$$

## 2. SEMI-DISCRETIZATION

We divide the time interval $[0, T]$ into $n$ subintervals of length $\tau = \frac{T}{n}$ and denote by $u^i$ the values of $u$ at $t_i = i\tau$, $i = 0, 1, \ldots, n$, and let

$$\delta u^i(x) = \frac{u^i - u^{i-1}}{\tau},$$

$$\delta^2 u^i(x) = \frac{\delta u^i(x) - \delta u^{i-1}(x)}{\tau}.$$

Let $u^{-1}$ be defined as $u^{-1}(x) = u^0(x) - \tau u^1(x)$.

For $i = 1, \ldots, n$, a recurrent approximation scheme is written as follows:

$$\begin{cases} \text{find } u^i \cong u(\cdot, t_i),\ i = 1, 2, \dots, n, \text{ such that} \\ \big(\delta^2 u^i, v\big) + \big(|\Delta u^i|^{p-2}\Delta u^i, \Delta v\big) = \big(f^i, v\big). \end{cases} \tag{2.1}$$

This implies

$$\begin{cases} \text{find } u^i \cong u(\cdot, t_i),\ i = 1, 2, \dots, n, \text{ such that} \\ \big(\delta u^i - \delta u^{i-1}, v\big) + \tau\big(|\Delta u^i|^{p-2}\Delta u^i, \Delta v\big) = \tau(f^i, v). \end{cases} \tag{2.2}$$

**Theorem 1.** *Let $f^i \in L^q(\Omega)$, the problem (2.2) admits a unique weak solution $u \in W^{2,p}_{\Psi}(\Omega)$ for $1 \leq i \leq n$.*

*Proof.* Let us define an operator

$$A : W_0^{2,p}(\Omega) \to (W_0^{2,p}(\Omega))^*$$

such that

$$Au^i = \delta u^i + \tau \Delta_p^2 u^i. \tag{2.3}$$

Here

$$\Delta_p^2 u^i = \Delta(|\Delta u|^{p-2}\Delta u). \tag{2.4}$$

We apply monotone operators theory to prove that $A$ is a semi-continuous, coercive, and monotone operator.

We introduce a functional $K$ on $W_0^{2,p}(\Omega)$ by

$$K(u^i) = \int_\Omega \left(\frac{1}{2}(\delta u^i)^2 + \frac{\tau}{p}|\Delta u^i|^p\right) dx, \tag{2.5}$$

$$\begin{aligned} (K'(u^i), v) &= \frac{d}{dt}\left\{A(u^i + tv)\right\}_{t=0} \\ &= \frac{d}{dt}\left\{\frac{1}{2}\int_\Omega (\delta u^i + tv)^2 dx + \frac{\tau}{p}\int_\Omega |\Delta(u^i + tv)|^p dx\right\}_{t=0} \\ &= \left\{\int_\Omega (\delta u^i + tv) v dx + \tau \int_\Omega |\Delta(u^i + tv)|^{p-1} \Delta v dx\right\}_{t=0} \\ &= \int_\Omega \delta u^i v dx + \tau \int_\Omega (|\Delta u^i|^{p-2}\Delta u^i)\Delta v dx \\ &= \int_\Omega \delta u^i v dx + \tau \int_\Omega \Delta(|\Delta u^i|^{p-2}\Delta u^i) v dx \\ &= (\delta u^i, v) + \tau(\Delta_p^2 u^i, v) = (Au^i, v)\ \ \forall \in W_0^{2,p}(\Omega). \end{aligned} \tag{2.6}$$

This implies that $K' = A$ and $K$ is differentiable in the Gateau sense, that is, semi-continuous.

By using an inequality in [14], for $p \in [1, \infty)$ and $a, b \in \mathbb{R}^n$, we have

$$|b|^p \geq |a|^p + p|a|^{p-2}a(b-a) + \frac{|b-a|^p}{2^{p-1}-1}, \tag{2.7}$$

and from the mean value theorem we have

$$\begin{aligned} \big(A(u^i) - A(v)\big) &= \big(\delta(u^i - v), u^i - v\big) + \tau\big(\Delta_p^2 u^i - \Delta_p^2 v, u^i - v\big) \\ &= \frac{1}{2}\delta\|u^i - v\|^2 + \tau(\Delta_p^2 u^i - \Delta_p^2 v, u^i - v) \\ &\geq C(\tau)\|u^i - v\|^2 + \tau(\Delta_p^2 u^i - \Delta_p^2 v, u^i - v) \\ &\geq \tau(\Delta_p^2 u^i - \Delta_p^2 v, u^i - v) \\ &= \tau\int_\Omega |\Delta u^i|^{p-2}\Delta u^i(\Delta u^i - \Delta v)dx - \tau\int_\Omega |\Delta v|^{p-2}\Delta v(\Delta u^i - \Delta v)dx \\ &= \frac{2}{p(2^{p-1}-1)}\int_\Omega |\Delta u^i - \Delta v|^p dx. \end{aligned} \tag{2.8}$$

Since the norm $\|\cdot\|_{W_0^{2,p}(\Omega)}$ is equivalent to the semi norm $\|\Delta(\cdot)\|_{L^p(\Omega)}$ over the space $W_0^{2,p}(\Omega)$ (by Calderon–Zygmund and Poincaré), we have

$$(A(u^i) - Av, u^i - v) \geq C(p)\|u^i - v\|^p_{W_0^{2,p}(\Omega)}. \tag{2.9}$$

This proves the monotonicity of $A$, then

$$(A(u^i), u^i) \geq C(p)\|u^i\|^p_{W_0^{2,p}(\Omega)}, \tag{2.10}$$

from which we conclude the coercivity of $A$.

By Hölder's inequality, we get

$$|(f^i, v)| = \left|\int_\Omega f^i v dx\right| \leq C\|f^i\|_q\|v\|_p, \tag{2.11}$$

and using $W_0^{2,p}(\Omega) \hookrightarrow L^q(\Omega)$, we obtain

$$|(f^i, v)| \leq C\|f^i\|_q\|v\|_{W_0^{2,p}(\Omega)}, \tag{2.12}$$

this implies that $f^i \in (W_0^{2,p}(\Omega))^*$. □

## 3. MIXED FORMULATION

Taking $X = W^{2,p}_{\Psi}(\Omega)$ and $Y = L^q(\Omega)$, let us choose an auxiliary variable

$$w^i = |\Delta u^i|^{p-1}\Delta u^i. \tag{3.1}$$

Depending on the following note: $\psi(z) = |z|^{p-2}z$ is such that the inverse is set as $\psi(z) = \mathrm{sgn}(z) \times |z|^{\frac{1}{p-1}}z = |z|^{q-2}z$, we can write the problem (1.1) as follows:

$$\begin{cases} -\Delta u^i = |w^i|^{q-2}w^i, \\ -\Delta w^i = f^i - \delta^2 u^i. \end{cases} \tag{3.2}$$

A mixed system can be written as

$$\begin{cases} a(w^i, v) + c(u^i, v) = 0 \quad \forall\, v \in X, \\ c(w^i, \eta) = L_Y(\eta) \quad \forall\, \eta \in Y. \end{cases} \tag{3.3}$$

Here

$$a(w^i, v) := \int\limits_{\Omega} |w^i|^{q-2}w^i v dx, \tag{3.4}$$

$$c(w^i, \eta) := \int\limits_{\Omega} -\Delta w^i \eta dx, \tag{3.5}$$

$$L_Y(\eta) := \int\limits_{\Omega} (f^i - \delta^2 u^i)\eta dx, \tag{3.6}$$

where $f^i = f(t_i, x)$.

**Proposition** (Inf-sup condition). For $u \in X$ we have

$$\gamma \le C \inf_{0\ne\eta\in Y} \sup_{0\ne u^i\in X} \frac{c(u^i, \eta)}{\|u^i\|_X \|\eta\|_Y}. \tag{3.7}$$

*Proof*. For $u^i \in W^{2,p}_0(\Omega)$ and taking $\eta = |\Delta u^i|^{p-2}\Delta u^i$, we have

$$\|\eta\|_{L^q(\Omega)} = \big\| \, |\Delta u^i|^{p-1} \big\|_{L^q(\Omega)} = \|\Delta u^i\|^{p-1}_{L^p(\Omega)} \tag{3.8}$$

and

$$c(u^i, \eta) = \|\Delta u^i\|^p_{L^p(\Omega)}, \tag{3.9}$$

therefore

$$c(u^i, \eta) = \|\Delta u^i\|^p_{L^p(\Omega)} \|\Delta u^i\|^{p-1}_{L^p(\Omega)} \|\Delta u^i\|_{L^p(\Omega)} = \|\Delta u^i\|_{L^p(\Omega)} \|\eta\|_{L^q(\Omega)}. \tag{3.10}$$

This implies

$$\|\Delta u^i\|_{L^p(\Omega)} \leq C \frac{c(u^i,\eta)}{\|\eta\|_{L^q(\Omega)}}. \tag{3.11}$$

Thus, we conclude that

$$\gamma \leq C \inf_{0\neq\eta\in Y} \sup_{0\neq u^i\in X} \frac{c(u^i,\eta)}{\|u^i\|_X\|\eta\|_Y}. \tag{3.12}$$

This completes the proof. □

## 4. FULL DISCRETIZATION

Let $\Upsilon_T$ be a triangulation made of triangles $T$ such that the intersection of two different elements is either a vertex, an edge, or empty.

$$\exists\, \mu > 0\, \frac{h_T}{\rho_T} \leq \mu \quad \forall\, T \in \Upsilon_h, \tag{4.1}$$

where $h_T$ is the diameter of $T$ and $\rho_T$ is the diameter of the largest ball contained inside $T$. We denote the edges by $e$ and define the jump operator for a function $v$ across an edge/face at point $x$ as

$$[v(x)]_e = \begin{cases} \lim v(x+\alpha\eta_e) - v(x+\alpha\eta_e) & \text{if } e \in \xi_h^{int}, \\ v(x) & \text{if } e \in \xi_h - \xi_h^{int}. \end{cases} \tag{4.2}$$

The mesh size $h$ is given by

$$h = \max_{T\in\Upsilon_T} h_T. \tag{4.3}$$

Let $\mathbb{P}^k(\Upsilon_h)$ define the space of piecewise polynomials of degree $k$ over a triangulation $\Upsilon_h$:

$$\mathbb{P}^k(\Upsilon_h) = \big\{\phi :\ \phi_{\backslash T} \in \mathbb{P}^k(T) \quad \forall\, T \in \Upsilon_h\big\}. \tag{4.4}$$

The following discrete finite spaces are given:

$$X^h = \mathbb{P}^k(\Upsilon_h) \cap C^0(\bar{\Omega}), \tag{4.5}$$

and

$$X^h_\Psi = \big\{\phi \in X^h\,;\ \phi_{\backslash\partial\Omega} = R\Psi\big\}, \tag{4.6}$$

here $R$ is the Ritz projection operator such that

$$\int_\Omega \nabla(Rv)\nabla\phi = \int_\Omega \nabla v\nabla\phi dx \ \ \forall\phi \in X^h \cap H^1_0(\Omega). \tag{4.7}$$

The discretized Laplace operator is defined as

$$(\Delta_h v)_{\backslash T} := (\Delta_h v)_{\backslash T} \quad \forall T \in \Upsilon. \tag{4.8}$$

The fully discrete scheme for (3.3) reads as follows: find a pair $(u_h^i, w_h^i) \in X_\Psi^h \times X^h$ such that

$$\begin{cases} a(w_h^i, v) + c_h(u_h^i, v) = 0, \\ c_h(w_h^i, \eta) = L(\eta) \quad \forall\, (v, \eta) \in X^h \times X_0^h. \end{cases} \tag{4.9}$$

from Green's formulation, we have

$$c_h(u_h, v) = \sum_{T \in \Upsilon_h} \int_T \nabla u_h \nabla v dx - \int_{\partial\Omega} \nabla\Psi \rho v dx = \int_\Omega \nabla u_h \nabla v dx - \int_{\partial\Omega} \nabla\Psi \rho v dx. \tag{4.10}$$

By substituting (4.10) in (4.9), the problem (3.3) can be written as follows:

$$\begin{cases} \displaystyle\int_\Omega |w_h^i|^{q-2} w_h^i v\, dx + \int_\Omega \nabla u_h^i \nabla v d\, x = \int_{\partial\Omega} \nabla\Psi \rho v\, dx, \\ \displaystyle\int_\Omega \nabla w_h^i \nabla\eta\, dx = \int_\Omega \big(f^i - \delta^2 u^i\big)\eta dx \quad \forall\, (v, \eta) \in X^h \times X_0^h. \end{cases} \tag{4.11}$$

**Lemma 1** [9]. *For $m \geq 2$ and $u \in W^{m+1,q}(\Omega)$, we have*

$$\|u - Ru\|_{L^q(\Omega)} + \|h(\nabla u - \nabla(Ru))\|_{L^q(\Omega)} + \left(\sum_{T \in \Upsilon} \|h^2(\Delta u - \Delta(Ru))\|_{L^q(T)}^q\right)^{1/q} \leq Ch^{m+1}|u|_{W^{m+1,q}(\Omega)}. \tag{4.12}$$

**Lemma 2** (Properties of $a(\cdot,\cdot)$, see [17, prop. 3.1). *For $w^i \in L^q(\Omega)$, $w_h^i, v_h \in X^h$ and $p \geq 2$, there exist positive constants $C_1$, $C_2$, and $C_3$ such that*

$$\frac{C_1}{2} \frac{\|w^i - w_h^i\|_{L^q(\Omega)}^2}{\|w^i\|_{L^q(\Omega)}^{2-q} + \|w_h^i\|_{L^q(\Omega)}} + \frac{C_2}{2} \int_\Omega \big||w^i|^{p-2} w^i - |w_h^i|^{p-2} w_h^i\big|\, |w^i - w_h^i| dx \leq a(w^i, w^i - w_h^i) - a(w_h^i, w^i - w_h^i). \tag{4.13}$$

$$a(w^i, w^i - v_h) - a(w_h^i, w^i - v_h) \leq C_3 \left(\int_\Omega \big||w^i|^{q-2} w^i - |w_h^i|^{q-2} w_h^i\big|\, |w^i - w_h^i| dx\right)^{\frac{1}{q}} \|w^i - v_h\|_{L^q(\Omega)}. \tag{4.14}$$

**Theorem 2.** *For $m \geq 2$, there exists $C \geq 0$ such that*

$$\|u^i - u_h^i\|_{W_h^{2,p}(\Omega)}^{p-1} + \|w^i - w_h^i\|_{L^q(\Omega)} \leq C\big(h^{\frac{q}{2}(m+1)}|w^i|_{W^{m+1,q}(\Omega)}^{\frac{q}{2}} + h^{m+1}|w^i|_{W^{m+1,q}(\Omega)} + h^{m-1}|u^i|_{W^{m+1,p}(\Omega)} + h^{m+1}|\delta^2 u^i|_{W^{m+1,q}(\Omega)}\big). \tag{4.15}$$

*Proof.* From the triangle inequality, we have

$$\|u^i - u_h^i\|_{W_h^{2,p}(\Omega)} \le \|Ru^i - u_h^i\|_{W_h^{2,p}(\Omega)} + \|u^i - Ru_h^i\|_{W_h^{2,p}(\Omega)}. \tag{4.16}$$

By the discrete inf-sup condition in Proposition, we obtain

$$\begin{aligned}\|Ru^i - u_h^i\|_{W_h^{2,p}(\Omega)} &\le \sup_{\eta\in X_0^h(\Omega);\,\eta\neq 0} \frac{c_h\big(Ru^i - u_h^i, \eta\big)}{\|\eta\|_{L_h^q(\Omega)}} \\ &\le \sup_{\eta\in X_0^h(\Omega);\,\eta\neq 0} \frac{a(w^i,\eta) - a(w_h^i,\eta)}{\|\eta\|_{L_h^q(\Omega)}} \\ &\le C_3 \frac{\left(\int\limits_\Omega \big|\,|w^i|^{p-2}w^i - |w_h^i|^{p-2}w_h^i\,\big|\,|w^i - w_h^i|\,dx\right)^{\frac{1}{p}} \|\eta\|_{L^q(\Omega)}}{\|\eta\|_{L_h^q(\Omega)}} \\ &\le C_3 C \left(\int\limits_\Omega \big|\,|w^i|^{p-2}w^i - |w_h^i|^{p-2}w_h^i\,\big|\,|w^i - w_h^i|\,dx\right)^{\frac{1}{p}}.\end{aligned} \tag{4.17}$$

Applying Lemma 2 and using Young's inequality, we have

$$\begin{aligned} C_2 \int\limits_\Omega \big|\,|w^i|^{p-2}w^i - |w_h^i|^{p-2}w_h^i\,\big|\,|w^i - w_h^i|dx &\le a(w^i, w^i - w_h^i) - a(w_h^i, w^i - w_h^i) \\ &\le \left(\int\limits_\Omega \big|\,|w^i|^{p-2}w^i - |w_h^i|^{p-2}w_h^i\,\big|\,|w^i - w_h^i|\,dx\right)^{\frac{1}{p}} \|w^i - w_h^i\|_{L^q(\Omega)} \\ &\le \frac{C_3^q}{q\epsilon^q}\|w^i - w_h^i\|^q_{L^q(\Omega)} + \frac{\epsilon^p}{p}\int\limits_\Omega \big|\,|w^i|^{p-2}w^i - |w_h^i|^{p-2}w_h^i\,\big|\,|w^i - w_h^i|\,dx. \end{aligned} \tag{4.18}$$

Choosing $\epsilon$ sufficiently small so that $\frac{\epsilon^p}{p} < 1$, we have

$$\int\limits_\Omega \big||w^i|^{p-2}w^i - |w_h^i|^{p-2}w_h^i\big|\,|w^i - w_h^i|dx \le C\|w^i - w_h^i\|^q_{L^q(\Omega)}. \tag{4.19}$$

Substituting (4.19) into (4.17), we obtain

$$\|Ru^i - u_h^i\|_{W_h^{2,p}(\Omega)} < C\|w^i - w_h^i\|^{\frac{q}{p}}_{L^q(\Omega)}. \tag{4.20}$$

Subtracting (4.9) from (3.3), we obtain

$$\begin{cases} a(w^i, v) - a(w_h^i, v) + c_h(u^i - u_h^i, v) = 0, \\ c_h(w^i - w_h^i, \eta) = 0. \end{cases} \tag{4.21}$$

From the semilinearity of $a(\cdot,\cdot)$, we conclude

$$
\begin{aligned}
a(w^i, w^i - w_h^i) - a(w_h^i, w^i - w_h^i) &= a(w^i, w^i - v) - a(w_h^i, w^i - v) \\
&\quad + a(w^i, v - w_h^i) - a(w_h^i, v - w_h^i) \\
&= \underbrace{a(w^i, w^i - v) - a(w_h^i, w^i - v)}_{I_1} + \underbrace{c_h(u^i - u_h^i, w_h^i - v)}_{I_2}.
\end{aligned} \tag{4.22}
$$

Applying Lemma 2, we get

$$
\frac{C_1}{2} \frac{\|w^i - w_h^i\|^2_{L^q(\Omega)}}{\|w^i\|^{2-q}_{L^q(\Omega)} + \|w_h^i\|^{2-q}_{L^q(\Omega)}} + \frac{C_2}{2} \int_\Omega \left| |w^i|^{p-2} w^i - |w_h^i|^{p-2} w_h^i \right| |w^i - w_h^i| dx \le I_1 + I_2. \tag{4.23}
$$

Now, by using $\epsilon$-Young's inequality and also Lemma 2, we find

$$
\begin{aligned}
I_1 &\le C_3 \left( \int_\Omega \left| |w^i|^{q-2} w^i - |w_h^i|^{q-2} w_h^i \right| |w^i - w_h^i| dx \right)^{\frac{1}{p}} \|w^i - v_h)\|_{L^q(\Omega)} \\
&\le \frac{\epsilon^p}{p} \int_\Omega \left| |w^i|^{q-2} w^i - |w_h^i|^{q-2} w_h^i \right| |w^i - w_h^i| dx + \frac{C_3^q}{\epsilon^q q} \|w^i - v_h)\|^q_{L^q(\Omega)}.
\end{aligned} \tag{4.24}
$$

Choosing $\epsilon$ such that $\frac{\epsilon^p}{p} = \frac{C_2}{2}$, we have

$$
I_1 \le \frac{C_2}{2} \int_\Omega \left| |w^i|^{q-2} w^i - |w_h^i|^{q-2} w_h^i \right| |w^i - w_h^i| dx + C(q) \|w^i - v_h)\|^q_{L^q(\Omega)}. \tag{4.25}
$$

On the other hand, we have

$$
\begin{aligned}
I_2 &= c_h(u^i - u_h^i, w_h^i - v) \\
&= c_h\big(u^i - u_h^i - R(u^i - u_h^i), w_h^i - v\big) + c_h\big(R(u^i - u_h^i), w_h^i - v\big) \\
&= c_h(u^i - Ru^i, w_h^i - v) + c_h(Ru^i - u_h^i, w_h^i - v) \\
&= c_h(u^i - Ru^i, w_h^i - v) + \int_\Omega \delta^2 (u^i - u_h^i)(Ru^i - u_h^i)\, dx.
\end{aligned} \tag{4.26}
$$

Further, using the continuity of $c_h$, we get

$$
\begin{aligned}
c_h\big(u^i - Ru^i, w_h^i - v\big) &\le C \, \|u^i - Ru^i\|_{W_h^{2,p}} \|w_h^i - v\|_{L^q(\Omega)} \\
&= \frac{C}{2\epsilon^2} \|u^i - Ru^i\|^2_{W_h^{2,p}} + \frac{C\epsilon^2}{2} \|w_h^i - v\|^2_{L^q(\Omega)} \\
&= \frac{C}{2\epsilon^2} \|u^i - Ru^i\|^2_{W_h^{2,p}} + \frac{C\epsilon^2}{2} \big( \|w_h^i - w^i\|^2_{L^q(\Omega)} + \|w^i - v\|^2_{L^q(\Omega)} \big),
\end{aligned} \tag{4.27}
$$

$$
\begin{aligned}
\int_\Omega \delta^2 (u^i - u_h^i)(Ru^i - u_h^i)\, dx &\le \left( \frac{1}{p} + \frac{1}{p} \right) \|\delta^2 (u^i - u_h^i)\|_{L^q(\Omega)} \, \|Ru^i - u_h^i\|_{L^q(\Omega)} \\
&\le C \, \|\delta^2 (u_h^i - u^i)\|_{L^q(\Omega)} \, \|Ru^i - u_h^i\|_{W_h^{2,p}(\Omega)} \\
&\le \frac{C}{2\epsilon} \|\delta^2 (u_h^i - u^i)\|^2_{L^q(\Omega)} + \frac{\epsilon}{2} \|Ru^i - u_h^i\|^2_{W_h^{2,p}(\Omega)}.
\end{aligned} \tag{4.28}
$$

By applying the triangle inequality and (4.20) to the right-hand side of (4.28), and also by substituting (4.25)–(4.28) into (4.23) and choosing $\epsilon$ small enough, we find

$$\begin{aligned} \|w_h^i - w^i\|^2_{L^q(\Omega)} \ \le\ C\Big(&\|w^i - v\|^q_{L^q(\Omega)} + \|Ru^i - u^i\|^2_{W_h^{2,p}(\Omega)} + \|w^i - v\|^2_{L^q(\Omega)} \\ &+\|\delta^2 u^i - R\delta^2(u^i)\|^2_{L^q(\Omega)}\Big). \end{aligned} \tag{4.29}$$

From the properties of the Ritz projection and Lemma 1, we get the estimation of $w^i - w_h^i$. To

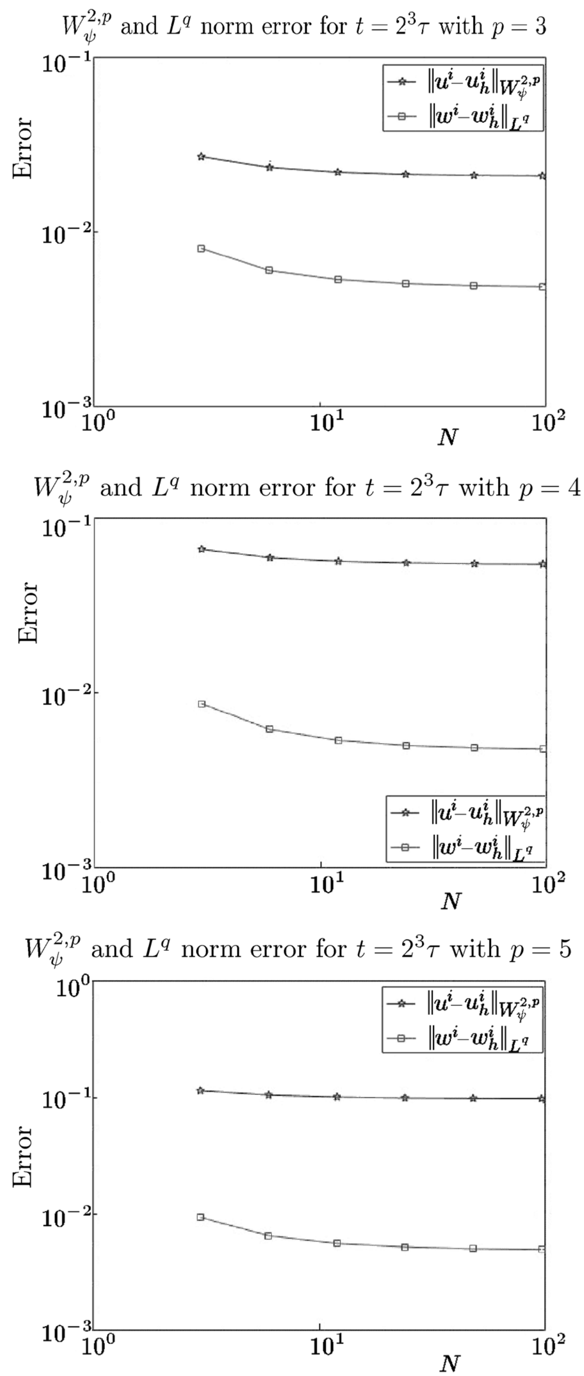


**Fig. 1.** The error results for $u$ and $w$ in log log-plot at $t = 2^3\tau$ for $p = 3, 4, 5$, respectively.

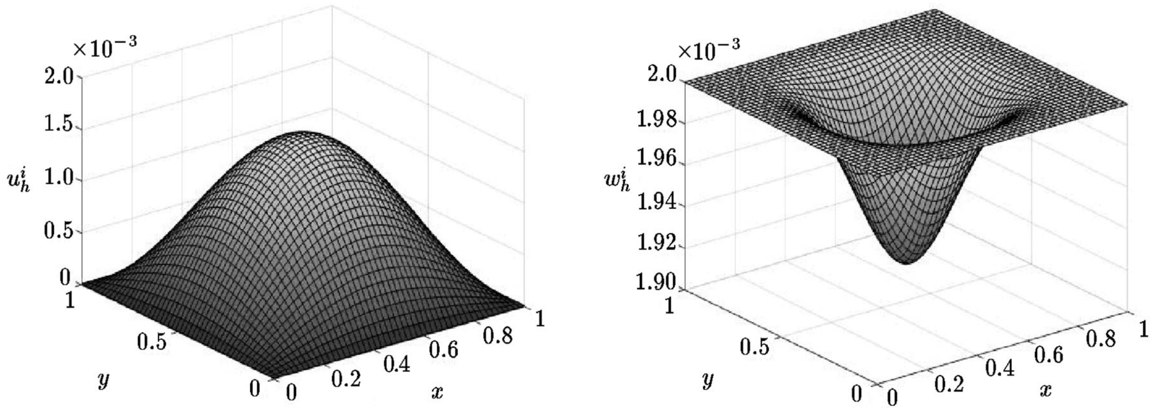


**Fig. 2.** The surface for $u_h^i$ and $w_h^i$ on $(0,1)\times(0,1)$ for the 4-bi-Laplacian.

estimate $u^i - u_h^i$, we substitute (4.29) into (4.20), taking into account inequality (4.16). This completes the proof. □

## 5. NUMERICAL EXPERIMENT

In this section, we turn our focus to a numerical experiment for different values of the power of $p$ that illustrates the accuracy and efficiency of the above-proposed method for a fully discrete scheme. First, we prescribe the computational domain as $\Omega = (0,1)\times(0,1)$ and the time interval as $(0,1)$. We use the Newton–Raphson method to solve the above-obtained nonlinear system, so we give initial values $w^0$, $w^1$, $u^0$, and $u^1$. The source function $f$ and the auxiliary variable $w$ are chosen according to the exact solution

$$u(x,t) = \frac{1}{\pi^2}\sin(\pi x)\sin(\pi y)\sin(t)^3.$$

In this experiment, the unknown function $u(x,y,t)$ and the auxiliary variable $w(x,y,t)$ are approximated by a linear polynomial, that is, $k=1$. For this test example we take the step length $h \in \{\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \frac{1}{48}, \frac{1}{96}\}$ and $p = 3,4,5$. Numerically, the errors are calculated at the final time level $t_i = 2^3\tau$ with $\tau = 2^5$.

In Fig. 1, we plot the error results for $u$ and $w$, and Fig. 2 represents the surface for $u_h^i$ and $w_h^i$ on $(0,1)\times(0,1)$.

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