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Two-Grid Methods for a New Mixed Finite Element Approximation of Semilinear Parabolic Integro-Differential Equations

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Abstract—In this paper, we present a two-grid scheme for a semilinear parabolic integro-differential equation using a new mixed finite element method. The gradient for the method belongs to the square integrable space instead of the classical $H(\mathrm{div};\Omega)$ space. The velocity and the pressure are approximated by the P_0^2–P_1 pair which satisfies the inf-sup condition. Firstly, we solve an original nonlinear problem on the coarse grid in our two-grid scheme. Then, to linearize the discretized equations, we use Newton iteration on the fine grid twice. It is shown that the algorithm can achieve asymptotically optimal approximation as long as the mesh sizes satisfy $h = \mathcal{O}(H^6|\ln H|^2)$. As a result, solving such a large class of nonlinear equations will not be much more difficult than the solution of one linearized equation. Finally, a numerical experiment is provided to verify theoretical results of the two-grid method.

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1. INTRODUCTION

Mixed finite element methods, which are used to approximate two different variables, have been found to be very important for solving the partial differential equations [1, 12, 22]. Especially, the second variable, which is usually related with some derivatives of the original variable, has its physical interest. For example, in the elasticity equations, where the stress can be introduced to be approximated at the same time as the displacement. In the recent years, Chen et al. [8, 23] developed a new mixed finite element scheme and used P_0^2–P_1 finite element pair to solve partial differential equations. The gradient of the primal variable for this method belongs to the square integrable space instead of the classical $H(\mathrm{div};\Omega)$ space.

The two-grid method was introduced by Xu [27, 28] as a discretization method for nonsymmetric, indefinite and nonlinear partial differential equations. The main idea is to use a coarse-grid space to produce a rough approximation of the solution of nonlinear problems, and then use it as the initial guess for one Newton-like iteration on the fine grid. After Xu's work, a two-grid method was further investigated by many authors (see, e.g., [2–6, 10, 26–29]). Dawson and Wheeler [10] analyzed a two-grid finite difference scheme for nonlinear parabolic equations. Wu and Allen [26] presented a two-step algorithm by using the two-grid idea for the semilinear reaction–diffusion equations with the expanded mixed finite element method. Based on this work, Chen et al. [5] proposed a three-step algorithm using the correction idea from reference [28]. Then, Chen et al. [6] presented a three-step two-grid algorithm and a four-step two-grid algorithm for semilinear reaction–diffusion problems by expanded mixed finite element method. Chen et al. [4] discussed a two-grid method for mixed finite element methods of fully nonlinear reaction–diffusion equations. Bi and Ginting [2] studied two-grid

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finite volume element method for linear and nonlinear elliptic problems. They also investigated two-grid discontinuous Galerkin method for quasi-linear elliptic problems in [3]. Xu and Zhou [29] presented a two-grid discretization scheme for eigenvalue problems. There are many other efficient methods such as multilevel algorithms for nonlinear elliptic equations and Ginzburg–Landau model, (see, e.g., [16, 17]). As far as we know there is no convergence analysis of two-grid method combined with mixed finite element method [8] for parabolic integro-differential equations in the literature.

Integro-differential equations can arise from many physical processes in which there is deficiency (the local characteristic) of the usual diffusion equations. Various numerical methods have been developed for solving these problems. Finite element approximation of linear or nonlinear integro-differential equations is extensively studied (see [7, 14, 18, 25] for standard finite element methods and [13, 19, 20, 24] for mixed finite element methods). In this paper, we consider the following semilinear parabolic integro-differential equations:

$$y_t - \mathrm{div}\boldsymbol{p} = f(y), \;\; x \in \Omega, \; t \in J, \tag{1.1}$$

$$\boldsymbol{p} = A\nabla y - \int\limits_0^t B(t,s)\nabla y(s)ds, \;\; x \in \Omega, \; t \in J, \tag{1.2}$$

$$y(x,t) = 0, \;\; x \in \partial\Omega, \; t \in J, \tag{1.3}$$

$$y(x,0) = y_0(x), \;\; x \in \Omega, \tag{1.4}$$

where $\Omega \subset \mathbf{R}^2$ is a convex polygonal domain with the boundary $\partial\Omega$, $J = (0,T]$, $f(y) = f(y,x,t)$ is a given real-valued function on Ω. We assume that the coefficient matrix $A = A(x) = (a_{ij}(x))_{2\times 2} \in W^{1,\infty}(\bar{\Omega}; \mathbf{R}^{2\times 2})$ is a symmetric 2×2 matrix and there are constants $c_1, c_2 > 0$ satisfying for any vector $\mathbf{X} \in \mathbf{R}^2$, $c_1\|\mathbf{X}\|^2_{\mathbf{R}^2} \le \mathbf{X}^t A\mathbf{X} \le c_2\|\mathbf{X}\|^2_{\mathbf{R}^2}$. Moreover, $B(t,s) = B(x,t,s)$ is also a 2×2 matrix. We also assume that

$$|f'(y)| + |f''(y)| \le M, \;\; y \in \mathbf{R}.$$

In this paper, we will combine the two-grid method with the new mixed finite element scheme [8] to solve the above-mentioned semilinear integro-differential parabolic equations based on the less regularity of flux. We first solve a nonlinear problem on the coarse-grid space, then we use the known coarse grid solution and a Taylor expansion to extrapolate the solution on the fine grid. On the fine grid we only need to solve a linear system.

The plan of this paper is as follows. In Section 2, we construct the fully discretized mixed finite element approximation of the problem (1.1)–(1.4). We shall derive the optimal a priori error estimates for all variables in Section 3. We will provide the two-grid algorithm and its error estimates in Section 4. In Section 5, we present a numerical example to verify the theoretical result. In Section 6, we conclude with a summary and possible extensions.

2. FULLY DISCRETIZED MIXED FINITE ELEMENT SCHEME

In this section, we will construct our new fully discretized mixed finite element approximation scheme of the problem (1.1)–(1.4).

We adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by

$$\|v\|^p_{m,p} = \sum_{|\alpha|\le m} \|D^\alpha v\|^p_{L^p(\Omega)},$$

a semi-norm $|\cdot|_{m,p}$ given by

$$|v|^p_{m,p} = \sum_{|\alpha| = m} \|D^\alpha v\|^p_{L^p(\Omega)}.$$

We set $W_0^{m,p}(\Omega) = \{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\}$. For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$.

We denote by $L^s(J; W^{m,p}(\Omega))$ the Banach space of all L^s integrable functions from J into $W^{m,p}(\Omega)$ with norm $\|v\|_{L^s(J;W^{m,p}(\Omega))} = \Big(\int\limits_0^T \|v\|^s_{W^{m,p}(\Omega)} dt\Big)^{1/s}$ for $s \in [1,\infty)$, and the standard modification for $s = \infty$. For simplicity of presentation, we denote $\|v\|_{L^s(J;W^{m,p}(\Omega))}$ by $\|v\|_{L^s(W^{m,p})}$. Similarly, one can define the spaces $H^1(J; W^{m,p}(\Omega))$ and $C^k(J; W^{m,p}(\Omega))$. In addition, C denotes a general positive constant independent of h and Δt, where h is the spatial mesh-size and Δt is time step.

Let

$$\boldsymbol{V} = (L^2(\Omega))^2 \text{ and } W = H_0^1(\Omega).$$

Set $M(t,s) = A^{-1}B(t,s)$, as in [8], we get the mixed variational form of (1.1), (1.2):

$$(A^{-1}\boldsymbol{p}, \boldsymbol{v}) + \int\limits_0^t (M(t,s)\nabla y(s), \boldsymbol{v})ds - (\nabla y, \boldsymbol{v}) = 0,\ \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{2.1}$$

$$(y_t, w) + (\boldsymbol{p}, \nabla w) = (f(y), w),\ \forall\, w \in W, \tag{2.2}$$

where $(\cdot,\cdot)$ is the inner product of $L^2(\Omega)$.

Let $b_1(t,s;\cdot,\cdot)$ and $b_2(t,s;\cdot,\cdot)$ be the bilinear forms defined on $L^2(\Omega) \times (H^1(\Omega))^2$ and $W \times \boldsymbol{V}$, respectively, by

$$b_1(t,s; v(s), \boldsymbol{v}) := (v(s), \mathrm{div}(\bar{M}(t,s)\boldsymbol{v})),\ \forall\, v(s) \in L^2(\Omega), \boldsymbol{v} \in (H^1(\Omega))^2,$$
$$b_2(t,s; v(s), \boldsymbol{v}) := (\tilde{M}(t,s)\nabla v(s), \boldsymbol{v}),\ \forall\, v(s) \in W, \boldsymbol{v} \in \boldsymbol{V},$$

where $\tilde{M}(t,s) = M(t,s)$ or $\tilde{M}(t,s) = M_t(t,s)$, $\bar{M}(t,s) = M^*(t,s)$ or $\bar{M}(t,s) = M_t^*(t,s)$, $M^*(t,s)$ be the transpose of $M(t,s)$ and $M_t^*(t,s)$ be the transpose of $M_t(t,s)$. We assume that the bilinear forms $b_1(t,s;\cdot,\cdot)$ and $b_2(t,s;\cdot,\cdot)$ are continuous, i.e.,

$$b_1(t,s; v(s), \boldsymbol{v}) \le \gamma \|\boldsymbol{v}\|_1 \|v(s)\|,\ \forall\, v(s) \in L^2(\Omega), \boldsymbol{v} \in (H^1(\Omega))^2, \tag{2.3}$$

$$b_2(t,s; v(s), \boldsymbol{v}) \le \gamma_1 \|\nabla v(s)\| \cdot \|\boldsymbol{v}\|,\ \forall\, v(s) \in W, \boldsymbol{v} \in \boldsymbol{V}, \tag{2.4}$$

with $\gamma, \gamma_1 \in R^+$.

Let $\mathcal{T}_h$ denote a regular triangulation of the polygonal domain Ω, h_τ denotes the diameter of τ and $h = \max h_\tau$. Let $\boldsymbol{V}_h \times W_h \subset \boldsymbol{V} \times W$ be defined by the following finite element pair P_0^2-P_1 [8, 23]:

$$\boldsymbol{V}_h = \{\boldsymbol{v}_h = (\boldsymbol{v}_{1h}, \boldsymbol{v}_{2h}) \in \boldsymbol{V} | \boldsymbol{v}_{1h}, \boldsymbol{v}_{2h} \in P_0(\tau),\ \forall\, \tau \in \mathcal{T}_h\},$$
$$W_h = \{w_h \in C^0(\Omega) \cap W | w_h \in P_1(\tau),\ \forall\, \tau \in \mathcal{T}_h\}.$$

Before the new mixed finite element scheme is given, we introduce three projection operators. Firstly, we define the standard elliptic projection [9] $P_h :\ W \to W_h$, which satisfies: for any $\phi \in W$

$$(\nabla(\phi - P_h\phi), \nabla w_h) = 0,\ \ \forall\, w_h \in W_h, \tag{2.5}$$

$$\|\phi - P_h\phi\|_s \le Ch^{2-s}\|\phi\|_2,\ s = 0,1,\ \forall\, \phi \in H^2(\Omega), \tag{2.6}$$

$$\|\phi - P_h\phi\|_{0,\infty} \le Ch|\ln h|\|\phi\|_{1,\infty},\ s = 0,1,\ \forall\, \phi \in W^{1,\infty}(\Omega). \tag{2.7}$$

Next, we define the standard L^2 projection [1] $\Pi_h : \boldsymbol{V} \to \boldsymbol{V}_h$, which satisfies: for any $\boldsymbol{q} \in \boldsymbol{V}$

$$(\boldsymbol{q} - \Pi_h \boldsymbol{q}, \boldsymbol{v}_h) = 0, \ \ \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.8}$$

$$\|\Pi_h \boldsymbol{q}\| \le C\|\boldsymbol{q}\|, \tag{2.9}$$

$$\|\boldsymbol{q} - \Pi_h \boldsymbol{q}\| \le Ch\|\boldsymbol{q}\|_1, \ \ \forall\, \boldsymbol{q} \in (H^1(\Omega))^2. \tag{2.10}$$

At last, we define new mixed elliptic projection $(R_h\boldsymbol{p}, R_h y) \in \boldsymbol{V}_h \times W_h$ by

$$(A^{-1}(\boldsymbol{p} - R_h\boldsymbol{p}), \boldsymbol{v}_h) + \int_0^t (M(t,s)\nabla(y - R_h y)(s), \boldsymbol{v}_h)ds - (\nabla(y - R_h y), \boldsymbol{v}_h) = 0, \ \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.11}$$

$$(\boldsymbol{p} - R_h\boldsymbol{p}, \nabla w_h) = 0, \ \forall\, w_h \in W_h. \tag{2.12}$$

We now consider the fully discrete mixed finite element scheme. Let $\Delta t > 0$, $N = T/\Delta t \in \mathbb{Z}$, and $t_n = n\Delta t$, $n \in \mathbb{Z}$. Also, let

$$\psi^n = \psi^n(x) = \psi(x, t_n), \quad dt\psi^n = \frac{\psi^n - \psi^{n-1}}{\Delta t}.$$

Then the fully discrete approximation scheme is to find $(\boldsymbol{p}_h^n, y_h^n) \in \boldsymbol{V}_h \times W_h$, $n = 1, 2, \ldots, N$, such that

$$(A^{-1}\boldsymbol{p}_h^n, \boldsymbol{v}_h) + \sum_{i=1}^{n} \Delta t(M(t_n, t_{i-1})\nabla y_h^{i-1}, \boldsymbol{v}_h) - (\nabla y_h^n, \boldsymbol{v}_h) = 0, \ \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.13}$$

$$(dt y_h^n, w_h) + (\boldsymbol{p}_h^n, \nabla w_h) = (f(y_h^n), w_h), \ \forall\, w_h \in W_h, \tag{2.14}$$

$$y_h^0 = R_h y_0(x). \tag{2.15}$$

For the proof of existence and uniqueness of the solution for the nonlinear algebraic system (2.13)–(2.15) see [1, 11].

3. OPTIMAL A PRIORI ERROR ESTIMATES

In this section, we will derive the optimal a priori error analysis for the problem (1.1)–(1.4). At first, we recall a result from Grisvard [15].

Lemma 3.1 [15]. *For every function $F \in L^2(\Omega)$, the solution ϕ of*

$$-\mathrm{div}(A\nabla\phi) = F \ \text{ in } \ \Omega, \ \phi|_{\partial\Omega} = 0, \tag{3.1}$$

belongs to $H_0^1(\Omega) \cap H^2(\Omega)$. Moreover, there exists a positive constant C such that

$$\|\phi\|_2 \le C\|F\|. \tag{3.2}$$

Next, we will discuss a priori error estimates between the exact solutions and their mixed elliptic projections in the following two lemmas.

Lemma 3.2. *Let $(R_h\boldsymbol{p}, R_h y) \in \boldsymbol{V}_h \times W_h$ be the new mixed elliptic projection defined in* (2.11), (2.12) *and $(\boldsymbol{p}, y)$ be the solution of* (2.1), (2.2)*, respectively. Then we have*

$$\|\boldsymbol{p} - R_h\boldsymbol{p}\| + \|\nabla(y - R_h y)\| \le Ch(\|y\|_2 + \|\boldsymbol{p}\|_1 + \|y\|_{L^2(H^2)}), \tag{3.3}$$

$$\|y - R_h y\| \le Ch^2(\|y\|_2 + \|\boldsymbol{p}\|_1 + \|y\|_{L^2(H^2)}). \tag{3.4}$$

Proof. Since $\nabla W_h \subset \boldsymbol{V}_h$, with the aid of (2.8), we subtract (2.11), (2.12) from (2.1), (2.2) to get

$$(A^{-1}(\Pi_h \boldsymbol{p} - R_h \boldsymbol{p}), \boldsymbol{v}_h) - (\nabla(P_h y - R_h y), \boldsymbol{v}_h) = -\int_0^t (M(t,s)\nabla(y - P_h y)(s), \boldsymbol{v}_h)ds$$

$$-\int_0^t (M(t,s)\nabla(P_h y - R_h y)(s), \boldsymbol{v}_h)ds + (\nabla(y - P_h y), \boldsymbol{v}_h) \tag{3.5}$$

$$-(A^{-1}(\boldsymbol{p} - R_h \boldsymbol{p}), \boldsymbol{v}_h), \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h,$$

$$(\Pi_h \boldsymbol{p} - R_h \boldsymbol{p}, \nabla w_h) = 0, \forall\, w_h \in W_h. \tag{3.6}$$

Choosing $\boldsymbol{v}_h = \Pi_h \boldsymbol{p} - R_h \boldsymbol{p}$ and $w_h = P_h y - R_h y$ in (3.5) and (3.6), respectively, then, adding the two resulting equations, using (2.3), (2.6), and (2.10), Cauchy inequality and the assumption on A, we have

$$\|\Pi_h \boldsymbol{p} - R_h \boldsymbol{p}\| \leq Ch(\|y\|_{L^2(H^2)} + \|y\|_2 + \|\boldsymbol{p}\|_1) + C\int_0^t \|\nabla(P_h y - R_h y)(s)\|ds. \tag{3.7}$$

Setting $\boldsymbol{v}_h = \nabla(P_h y - R_h y)$ in (3.5), by use of (2.3), (2.6), and (2.10), Cauchy inequality and the assumption on A again, we find that

$$\|\nabla(P_h y - R_h y)\| \leq Ch(\|y\|_{L^2(H^2)} + \|y\|_2 + \|\boldsymbol{p}\|_1)$$

$$+C\|\Pi_h \boldsymbol{p} - R_h \boldsymbol{p}\| + C\int_0^t \|\nabla(P_h y - R_h y)(s)\|ds. \tag{3.8}$$

It follows from (3.7), (3.8), and Gronwall's inequality that

$$\|\Pi_h \boldsymbol{p} - R_h \boldsymbol{p}\| + \|\nabla(P_h y - R_h y)\| \leq Ch(\|y\|_{L^2(H^2)} + \|y\|_2 + \|\boldsymbol{p}\|_1). \tag{3.9}$$

Then, (3.3) can be derived by (2.6), (2.10), (3.9) and the triangle inequality.

Let ϕ be the solution of (3.1) with $F = y - R_h y$, using (2.1)–(2.4), (2.6), (2.10)–(2.12), Cauchy inequality and the assumption on A, we see that

$$\begin{aligned}
\|y - R_h y\|^2 &= (y - R_h y, -\mathrm{div}(A\nabla\phi)) = (A\nabla\phi, \nabla(y - R_h y)) \\
&= (\nabla(y - R_h y), A\nabla\phi - \Pi_h(A\nabla\phi)) + (A^{-1}(\boldsymbol{p} - R_h \boldsymbol{p}), \Pi_h(A\nabla\phi)) \\
&\quad + \int_0^t (M(t,s)\nabla(y - R_h y)(s), \Pi_h(A\nabla\phi))ds \\
&= (\nabla(y - R_h y), A\nabla\phi - \Pi_h(A\nabla\phi)) + (A^{-1}(\boldsymbol{p} - R_h \boldsymbol{p}), \Pi_h(A\nabla\phi) - A\nabla\phi) \\
&\quad +(\boldsymbol{p} - R_h \boldsymbol{p}, \nabla(\phi - P_h\phi)) - \int_0^t ((y - R_h y)(s), \mathrm{div}(M^*(t,s)A\nabla\phi))ds \\
&\quad + \int_0^t (M(t,s)\nabla(y - R_h y)(s), \Pi_h(A\nabla\phi) - A\nabla\phi)ds
\end{aligned} \tag{3.10}$$

$$\leq Ch\|\nabla(y-R_hy)\|\cdot\|\phi\|_2+Ch\int\limits_0^t\|\nabla(y-R_hy)(s)\|ds\|\phi\|_2$$
$$+Ch\|\boldsymbol{p}-R_h\boldsymbol{p}\|\cdot\|\phi\|_2+C\int\limits_0^t\|(y-R_hy)(s)\|ds\|\phi\|_2.$$

Combining (3.10), (3.2), (3.3) with Gronwall's inequality, we derive (3.4). Thus, we complete the proof of lemma. □

Lemma 3.3. *Let* $(R_h\boldsymbol{p}, R_hy)\in \boldsymbol{V}_h\times W_h$ *be the new mixed elliptic projection defined in* (2.11), (2.12) *and* $(\boldsymbol{p}, y)$ *be the solution of* (2.1), (2.2), *respectively. Then we have*

$$\|(\boldsymbol{p}-R_h\boldsymbol{p})_t\|+\|\nabla(y-R_hy)_t\|\leq Ch(\|y\|_2+\|\boldsymbol{p}\|_1+\|y_t\|_2+\|\boldsymbol{p}_t\|_1+\|y\|_{L^2(H^2)}),\tag{3.11}$$

$$\|(y-R_hy)_t\|\leq Ch^2(\|y\|_2+\|\boldsymbol{p}\|_1+\|y_t\|_2+\|\boldsymbol{p}_t\|_1+\|y\|_{L^2(H^2)}).\tag{3.12}$$

Proof. Differentiating Eqs. (3.5), (3.6) with respect to t, we get

$$\begin{aligned}(A^{-1}(\Pi_h\boldsymbol{p}-R_h\boldsymbol{p})_t,\boldsymbol{v}_h)-(\nabla(P_hy-R_hy)_t,\boldsymbol{v}_h)=-\int\limits_0^t(M_t(t,s)\nabla(y-P_hy)(s),\boldsymbol{v}_h)ds\\ -(M(t,t)\nabla(y-R_hy),\boldsymbol{v}_h)+(\nabla(y-P_hy)_t,\boldsymbol{v}_h)\\ -(A^{-1}(\Pi_h\boldsymbol{p}-R_h\boldsymbol{p}),\boldsymbol{v}_h),\forall\,\boldsymbol{v}_h\in\boldsymbol{V}_h,\end{aligned}\tag{3.13}$$

$$((\Pi_h\boldsymbol{p}-R_h\boldsymbol{p})_t,\nabla w_h)=0,\forall\,w_h\in W_h.\tag{3.14}$$

Similar to (3.7) and (3.8), it is easy to see that

$$\|(\Pi_h\boldsymbol{p}-R_h\boldsymbol{p})_t\|\leq Ch(\|y_t\|_2+\|\boldsymbol{p}_t\|_1)+C\|\nabla(y-R_hy)\|+C\|\nabla(y-R_hy)\|_{L^2(L^2)}\tag{3.15}$$

and

$$\begin{aligned}\|\nabla(P_hy-R_hy)_t\|\leq Ch(\|y_t\|_2+\|\boldsymbol{p}_t\|_1)+C\|\nabla(y-R_hy)\|_{L^2(L^2)}\\ +C\|(\Pi_h\boldsymbol{p}-R_h\boldsymbol{p})_t\|+C\|\nabla(y-R_hy)\|.\end{aligned}\tag{3.16}$$

Using (2.6), (2.10), (3.3), (3.15), (3.16), and the triangle inequality, we derive (3.11).

Let ϕ be the solution of (3.1) with $F=(y-R_hy)_t$, similar to (3.10), we conclude that

$$\begin{aligned}\|(y-R_hy)_t\|^2 &= ((y-R_hy)_t,-\mathrm{div}(A\nabla\phi))=(A\nabla\phi,\nabla(y-R_hy)_t)\\ &= (\nabla(y-R_hy)_t,A\nabla\phi-\Pi_h(A\nabla\phi))+(M(t,t)\nabla(y-R_hy),\Pi_h(A\nabla\phi))\\ &\quad+\int\limits_0^t(M_t(t,s)\nabla(y-R_hy)(s),\Pi_h(A\nabla\phi))ds+(A^{-1}(\boldsymbol{p}-R_h\boldsymbol{p})_t,\Pi_h(A\nabla\phi))\\ &= (\nabla(y-R_hy)_t,A\nabla\phi-\Pi_h(A\nabla\phi))+(A^{-1}(\boldsymbol{p}-R_h\boldsymbol{p})_t,\Pi_h(A\nabla\phi)-A\nabla\phi)\\ &\quad+\int\limits_0^t(M_t(t,s)\nabla(y-R_hy)(s),\Pi_h(A\nabla\phi)-A\nabla\phi)ds\\ &\quad+(M(t,t)\nabla(y-R_hy),\Pi_h(A\nabla\phi)-A\nabla\phi)+((\boldsymbol{p}-R_h\boldsymbol{p})_t,\nabla(\phi-P_h\phi))\end{aligned}\tag{3.17}$$

$$
\begin{aligned}
&-\int_0^t ((y-R_hy)(s), \operatorname{div}(M_t^*(t,s)A\nabla\phi))ds - (y-R_hy, \operatorname{div}(M^*(t,t)A\nabla\phi)) \\
\leq\ & Ch(\|\nabla(y-R_hy)_t\| + \|(\boldsymbol{p}-R_h\boldsymbol{p})_t\| + \|\nabla(y-R_hy)\|)\|\phi\|_2 \\
&+Ch\|\nabla(y-R_hy)\|_{L^2(L^2)}\|\phi\|_2 + C(\|y-R_hy\|_{L^2(L^2)} + \|y-R_hy\|)\|\phi\|_2.
\end{aligned}
$$

Using (3.2)–(3.4), (3.11), and (3.17), we have (3.12). The proof is completed. □

Now, we will discuss the optimal a priori error estimates between the exact solutions and their numerical solutions in the following two theorems.

Theorem 3.1. *Let* $(\boldsymbol{p}_h^n, y_h^n) \in \boldsymbol{V}_h \times W_h$ *be the solution of* (2.13), (2.14) *and* $(\boldsymbol{p}, y)$ *be the solution of* (2.1), (2.2), *respectively. Assume that the exact solution* $(\boldsymbol{p}, y)$ *has enough regularities for our purpose. Then, for* Δt *small enough and* $1 \leq n \leq N$, *we have*

$$
\|y^n - y_h^n\| \leq C(\Delta t + h^2). \tag{3.18}
$$

Proof. For convenience, let

$$
\boldsymbol{p}^n - \boldsymbol{p}_h^n = (\boldsymbol{p}^n - R_h\boldsymbol{p}^n) + (R_h\boldsymbol{p}^n - \boldsymbol{p}_h^n) =: \eta_{\boldsymbol{p}}^n + \xi_{\boldsymbol{p}}^n,
$$

$$
y^n - y_h^n = (y^n - R_hy^n) + (R_hy^n - y_h^n) =: \eta_y^n + \xi_y^n.
$$

Using (2.13), (2.14), (2.1), (2.2) and (2.11), (2.12), we have the following error equations:

$$
(A^{-1}\xi_{\boldsymbol{p}}^n, \boldsymbol{v}_h) + \int_0^{t_n} (M(t_n,s)\nabla R_hy(s), \boldsymbol{v}_h)ds - \sum_{i=1}^n \Delta t(M(t_n,t_{i-1})\nabla y_h^{i-1}, \boldsymbol{v}_h) - (\nabla\xi_y^n, \boldsymbol{v}_h) = 0, \tag{3.19}
$$

$$
(dt\xi_y^n, w_h) + (\xi_{\boldsymbol{p}}^n, \nabla w_h) = (f(y^n) - f(y_h^n), w_h) + (dty^n - y_t^n, w_h) - (dt\eta_y^n, w_h), \tag{3.20}
$$

for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $w_h \in W_h$.
Setting $\boldsymbol{v}_h = \nabla\xi_y^n$ in (3.19), we have

$$
\begin{aligned}
\|\nabla\xi_y^n\|^2 = &\int_0^{t_n} (M(t_n,s)\nabla R_hy(s), \nabla\xi_y^n)ds - \sum_{i=1}^n \Delta t(M(t_n,t_{i-1})\nabla R_hy^{i-1}, \nabla\xi_y^n) \\
&+(A^{-1}\xi_{\boldsymbol{p}}^n, \nabla\xi_y^n) + \sum_{i=1}^n \Delta t(M(t_n,t_{i-1})\nabla\xi_y^{i-1}, \nabla\xi_y^n).
\end{aligned} \tag{3.21}
$$

Noticed that

$$
\begin{aligned}
&\left\| \int_0^{t_n} M(t_n,s)\nabla R_hy(s)ds - \sum_{i=1}^n \Delta t M(t_n,t_{i-1})\nabla R_hy^{i-1} \right\|^2 \\
&\leq C(\Delta t)^2 \int_0^{t_n} (\|\nabla R_hy_t(s)\|^2 + \|\nabla R_hy(s)\|^2)ds.
\end{aligned} \tag{3.22}
$$

Using Cauchy inequality, (3.21), (3.22), (2.4), and the assumption on A, we obtain

$$\|\nabla\xi_y^n\|^2 \le C(\Delta t)^2 \int_0^{t_n} (\|\nabla R_h y_t(s)\|^2 + \|\nabla R_h y(s)\|^2) ds + \|\xi_{\boldsymbol{p}}^n\|^2 + C\sum_{i=1}^{n} \|\nabla \xi_y^{i-1}\|^2 \Delta t. \tag{3.23}$$

Apply the discrete Gronwall's inequality to (3.23), we can see that

$$\|\nabla\xi_y^n\| \le C\|\xi_{\boldsymbol{p}}^n\| + C\Delta t(\|\nabla R_h y_t\|_{L^2(L^2)} + \|\nabla R_h y\|_{L^2(L^2)}), \tag{3.24}$$

where

$$\begin{aligned}\|\nabla R_h y_t\|_{L^2(L^2)} + \|\nabla R_h y\|_{L^2(L^2)} \le \|\nabla(y_t - R_h y_t)\|_{L^2(L^2)} + \|\nabla(y - R_h y)\|_{L^2(L^2)} \\ + \|\nabla y_t\|_{L^2(L^2)} + \|\nabla y\|_{L^2(L^2)}.\end{aligned} \tag{3.25}$$

Choosing $\boldsymbol{v}_h = \xi_{\boldsymbol{p}}^n$ in (3.19) and $w_h = \xi_y^n$ in (3.20), respectively, adding the two resulting equations we have

$$\begin{aligned}(dt\xi_y^n, \xi_y^n) + (A^{-1}\xi_{\boldsymbol{p}}^n, \xi_{\boldsymbol{p}}^n) = & -\sum_{i=1}^{n} \Delta t(M(t_n, t_{i-1})\nabla \xi_y^{i-1}, \xi_{\boldsymbol{p}}^n) \\ & + \sum_{i=1}^{n} \Delta t(M(t_n, t_{i-1})\nabla R_h y^{i-1}, \xi_{\boldsymbol{p}}^n) \\ & - \int_0^{t_n} (M(t_n, s)\nabla R_h y(s), \xi_{\boldsymbol{p}}^n) ds \\ & + (f(y^n) - f(y_h^n), \xi_y^n) + (dty^n - y_t^n, \xi_y^n) - (dt\eta_y^n, \xi_y^n).\end{aligned} \tag{3.26}$$

It is easy to see that

$$(dt\xi_y^n, \xi_y^n) \ge \frac{1}{2}(\|\xi_y^n\|^2 - \|\xi_y^{n-1}\|^2). \tag{3.27}$$

Multiplying Δt and summing over n from 1 to l ($1 \le l \le N$) at both sides of (3.26), using (3.27), the assumption on A, and $\xi_y^0 = 0$, we have

$$\begin{aligned}& \|\xi_y^l\|^2 + \sum_{n=1}^{l} \|\xi_{\boldsymbol{p}}^n\|^2 \Delta t \\ \le\, & C\sum_{n=1}^{l} \Delta t \left(\sum_{i=1}^{n} \Delta t M(t_n, t_{i-1})\nabla R_h y^{i-1} - \int_0^{t_n} M(t_n, s)\nabla R_h y(s) ds, \xi_{\boldsymbol{p}}^n \right) \\ & - C\sum_{n=1}^{l} \Delta t \sum_{i=1}^{n} \Delta t(M(t_n, t_{i-1})\nabla \xi_y^{i-1}, \xi_{\boldsymbol{p}}^n) + C\sum_{n=1}^{l} \Delta t(f(y^n) - f(y_h^n), \xi_y^n) \\ & + C\sum_{n=1}^{l} \Delta t(dty^n - y_t^n, \xi_y^n) - C\sum_{n=1}^{l} \Delta t(dt\eta_y^n, \xi_y^n) = \sum_{i=1}^{5} I_i.\end{aligned} \tag{3.28}$$

Now we estimate the right-hand terms of (3.28), for I_1, using Cauchy inequality and (3.22), we have

$$|I_1| \le C(\Delta t)^2(\|\nabla R_h y\|^2_{L^2(L^2)} + \|\nabla R_h y_t\|^2_{L^2(L^2)}) + \frac{1}{3}\sum_{n=1}^{l}\|\xi_{\boldsymbol{p}}^n\|^2\Delta t. \tag{3.29}$$

For I_2, using (2.4), Cauchy inequality and (3.24), we see that

$$\begin{aligned}|I_2| \le C\sum_{n=1}^{l}\Delta t\sum_{i=1}^{n}\|\xi_{\boldsymbol{p}}^{i-1}\|^2\Delta t + \frac{1}{3}\sum_{n=1}^{l}\|\xi_{\boldsymbol{p}}^n\|^2\Delta t \\ +C(\Delta t)^2(\|\nabla R_h y\|^2_{L^2(L^2)} + \|\nabla R_h y_t\|^2_{L^2(L^2)}).\end{aligned} \tag{3.30}$$

For I_3, by mean value inequality, Cauchy inequality and the assumption on f, we conclude that

$$\begin{aligned}|I_3| \;&=\; C\sum_{n=1}^{l}\Delta t(f'(\hat{y}^n)(y^n - y_h^n), \xi_y^n) \\ &\le C\sum_{n=1}^{l}(\|\xi_y^n\|^2 + \|\eta_y^n\|^2)\Delta t + C\sum_{n=1}^{l}\|\xi_y^n\|^2\Delta t \\ &\le C\sum_{n=1}^{l}\|\eta_y^n\|^2\Delta t + C\sum_{n=1}^{l}\|\xi_y^n\|^2\Delta t,\end{aligned} \tag{3.31}$$

where $\hat{y}^n$ is located between y^n and y_h^n.

For I_4, from the results given in [21], we have

$$|I_4| \le C(\Delta t)^2\|y_{tt}\|^2_{L^2(L^2)} + C\sum_{n=1}^{l}\|\xi_y^n\|^2\Delta t. \tag{3.32}$$

For I_5, it follows from Cauchy inequality that

$$|I_5| \le C\sum_{n=1}^{l}\int_{t_{n-1}}^{t_n}\|(\eta_y)_t\|^2 dt + C\sum_{n=1}^{l}\|\xi_y^n\|^2\Delta t \le C\|(\eta_y)_t\|^2_{L^2(L^2)} + C\sum_{n=1}^{l}\|\xi_y^n\|^2\Delta t. \tag{3.33}$$

Now, for sufficiently small Δt, combining (3.28)–(3.33) with the discrete Gronwall's inequality, we conclude that

$$\begin{aligned}\|\xi_y^l\|^2 + \sum_{n=1}^{l}\|\xi_{\boldsymbol{p}}^n\|^2\Delta t \;\le\; C(\Delta t)^2(\|\nabla R_h y_t\|^2_{L^2(L^2)} + \|\nabla R_h y\|^2_{L^2(L^2)}) + C\sum_{n=1}^{N}\|\eta_y^n\|^2\Delta t \\ +C(\Delta t)^2\|y_{tt}\|^2_{L^2(L^2)} + C\|(\eta_y)_t\|^2_{L^2(L^2)}.\end{aligned} \tag{3.34}$$

Thus, (3.18) follows from (3.34) and Lemmas 3.2, 3.3. □

Theorem 3.2. *Let $(\boldsymbol{p}_h^n, y_h^n) \in \boldsymbol{V}_h \times W_h$ be the solution of (2.13), (2.14) and $(\boldsymbol{p}, y)$ be the solution of (2.1), (2.2), respectively. Assume that the exact solution $(\boldsymbol{p}, y)$ has enough regularities for our purpose. Then, for Δt small enough and $1 \le n \le N$, we have*

$$\|\nabla(y^n - y_h^n)\| + \|\boldsymbol{p}^n - \boldsymbol{p}_h^n\| \le C(\Delta t + h). \tag{3.35}$$

Proof. Take the difference in time of (3.19) to obtain

$$
\begin{aligned}
&(A^{-1}dt\xi_{\boldsymbol{p}}^n, \boldsymbol{v}_h) - (\nabla dt\xi_y^n, \boldsymbol{v}_h)\\
&= \sum_{i=1}^{n-1}(M(t_{n-1}, t_{i-1})\nabla\xi_y^{i-1}, \boldsymbol{v}_h) - \sum_{i=1}^{n}(M(t_n, t_{i-1})\nabla\xi_y^{i-1}, \boldsymbol{v}_h)\\
&+\frac{1}{\Delta t}\Bigg[\sum_{i=1}^{n}\Delta t(M(t_n, t_{i-1})\nabla R_h y^{i-1}, \boldsymbol{v}_h) - \int_0^{t_n}(M(t_n, s)\nabla R_h y(s), \boldsymbol{v}_h)ds\\
&-\sum_{i=1}^{n-1}\Delta t(M(t_{n-1}, t_{i-1})\nabla R_h y^{i-1}, \boldsymbol{v}_h) + \int_0^{t_{n-1}}(M(t_{n-1}, s)\nabla R_h y(s), \boldsymbol{v}_h)ds\Bigg].
\end{aligned}
\tag{3.36}
$$

Choosing $\boldsymbol{v}_h = \xi_{\boldsymbol{p}}^n$ in (3.36) and $w_h = dt\xi_y^n$ in (3.20), respectively. Adding the two resulting equations and using the inequality

$$
(A^{-1}dt\xi_{\boldsymbol{p}}^n, \xi_{\boldsymbol{p}}^n) \geq \frac{1}{2\Delta t}(\|A^{-\frac{1}{2}}\xi_{\boldsymbol{p}}^n\|^2 - \|A^{-\frac{1}{2}}\xi_{\boldsymbol{p}}^{n-1}\|^2),
\tag{3.37}
$$

we have

$$
\begin{aligned}
&\|dt\xi_y^n\|^2 + \frac{1}{2\Delta t}(\|A^{-\frac{1}{2}}\xi_{\boldsymbol{p}}^n\|^2 - \|A^{-\frac{1}{2}}\xi_{\boldsymbol{p}}^{n-1}\|^2)\\
&\leq \left[\sum_{i=1}^{n-1}(M(t_{n-1}, t_{i-1})\nabla\xi_y^{i-1}, \xi_{\boldsymbol{p}}^n) - \sum_{i=1}^{n}(M(t_n, t_{i-1})\nabla\xi_y^{i-1}, \xi_{\boldsymbol{p}}^n)\right]\\
&+\frac{1}{\Delta t}\Bigg[\sum_{i=1}^{n}\Delta t(M(t_n, t_{i-1})\nabla R_h y^{i-1}, \xi_{\boldsymbol{p}}^n) - \int_0^{t_n}(M(t_n, s)\nabla R_h y(s), \xi_{\boldsymbol{p}}^n)ds\\
&-\sum_{i=1}^{n-1}\Delta t(M(t_{n-1}, t_{i-1})\nabla R_h y^{i-1}, \xi_{\boldsymbol{p}}^n) + \int_0^{t_{n-1}}(M(t_{n-1}, s)\nabla R_h y(s), \xi_{\boldsymbol{p}}^n)ds\Bigg]\\
&+(f(y^n) - f(y_h^n), dt\xi_y^n) + (dty^n - y_t^n, dt\xi_y^n) - (dt\eta_y^n, dt\xi_y^n).
\end{aligned}
\tag{3.38}
$$

It follows from (3.19) that $\xi_{\boldsymbol{p}}^0 = 0$. Multiplying Δt and summing over n from 1 to l $(1 \leq l \leq n)$ at both sides of (3.38), using $\xi_{\boldsymbol{p}}^0 = 0$ and the assumption on A, it is easy see that

$$
\begin{aligned}
&\|\xi_{\boldsymbol{p}}^l\|^2 + \sum_{n=1}^{l}\|dt\xi_y^n\|^2\Delta t\\
&\leq C\sum_{n=1}^{l}\Delta t\left[\sum_{i=1}^{n-1}(M(t_n, t_{i-1})\nabla\xi_y^{i-1}, \xi_{\boldsymbol{p}}^n) - \sum_{i=1}^{n}(M(t_{n-1}, t_{i-1})\nabla\xi_y^{i-1}, \xi_{\boldsymbol{p}}^n)\right]\\
&+C\sum_{n=1}^{l}\Bigg[\sum_{i=1}^{n}\Delta t(M(t_n, t_{i-1})\nabla R_h y^{i-1}, \xi_{\boldsymbol{p}}^n) - \int_0^{t_n}(M(t_n, s)\nabla R_h y(s), \xi_{\boldsymbol{p}}^n)ds\\
&-\sum_{i=1}^{n-1}\Delta t(M(t_{n-1}, t_{i-1})\nabla R_h y^{i-1}, \xi_{\boldsymbol{p}}^n) + \int_0^{t_{n-1}}(M(t_{n-1}, s)\nabla R_h y(s), \xi_{\boldsymbol{p}}^n)ds\Bigg]\\
&+\sum_{n=1}^{l}\Delta t(f(y^n) - f(y_h^n), dt\xi_y^n) + \sum_{n=1}^{l}\Delta t(dty^n - y_t^n, dt\xi_y^n)\\
&-\sum_{n=1}^{l}\Delta t(dt\eta_y^n, dt\xi_y^n) = \sum_{i=1}^{5}Q_i.
\end{aligned}
\tag{3.39}
$$

Now, we estimate the right-hand terms of (3.39). For Q_1, using Cauchy inequality and (2.4), we see that

$$\begin{aligned} Q_1 = {} & C\sum_{n=1}^{l}\Delta t\left[\sum_{i=1}^{n-1}((M(t_{n-1},t_{i-1})-M(t_n,t_{i-1}))\nabla\xi_y^{i-1},\xi_{\boldsymbol{p}}^n)-(M(t_n,t_{n-1})\nabla\xi_y^{n-1},\xi_{\boldsymbol{p}}^n)\right] \\ & \le C\sum_{n=1}^{l}\Delta t\sum_{i=1}^{n-1}\|\nabla\xi_y^{i-1}\|^2\Delta t + C\sum_{n=1}^{l}\|\xi_{\boldsymbol{p}}^n\|^2\Delta t + C\sum_{n=1}^{l}\|\nabla\xi_y^{n-1}\|^2\Delta t, \end{aligned} \tag{3.40}$$

where we used

$$M(t_{n-1},s)-M(t_n,s)=\Delta t M_t(t_{n^*},s),\ t_{n-1}<t_{n^*}<t_n. \tag{3.41}$$

For Q_2, using (3.41), (2.4), and Cauchy inequality, we have

$$\begin{aligned} Q_2 \ = \ & C\sum_{n=1}^{l}\left[(\Delta t M(t_n,t_{n-1})\nabla R_h y^{n-1},\xi_{\boldsymbol{p}}^n)-\int_{t_{n-1}}^{t_n}(M(t_n,s)\nabla R_h y(s),\xi_{\boldsymbol{p}}^n)ds\right. \\ & +\sum_{i=1}^{n-1}\Delta t((M(t_n,t_{i-1})-M(t_{n-1},t_{i-1}))\nabla R_h y^{i-1},\xi_{\boldsymbol{p}}^n) \\ & \left.-\int_0^{t_{n-1}}((M(t_n,s)-M(t_{n-1},s))\nabla R_h y(s),\xi_{\boldsymbol{p}}^n)ds\right] \\ & = C\sum_{n=1}^{l}\Delta t(M(t_n,t_{n-1})\nabla R_h y^{n-1}-M(t_n,s_n)\nabla R_h y(s_n),\xi_{\boldsymbol{p}}^n) \\ & +C\sum_{n=1}^{l}\Delta t\left[\sum_{i=1}^{n-1}\Delta t(M_t(t_{n^*},t_{i-1})\nabla R_h y^{i-1},\xi_{\boldsymbol{p}}^n)\right. \\ & \left.-\int_0^{t_{n-1}}(M_t(t_{n^*},s)\nabla R_h y(s),\xi_{\boldsymbol{p}}^n)ds\right] \\ \le \ & C(\Delta t)^2(\|\nabla R_h y\|_{L^2(L^2)}^2+\|\nabla R_h y_t\|_{L^2(L^2)}^2)+C\sum_{n=1}^{l}\|\xi_{\boldsymbol{p}}^n\|^2\Delta t, \end{aligned} \tag{3.42}$$

where we also used

$$\int_{t_{n-1}}^{t_n} M(t_n,s)\nabla R_h y(s)ds=\Delta t M(t_n,s_n)\nabla R_h y(s_n),\ t_{n-1}<s_n<t_n$$

and

$$\begin{aligned} & \left\|\sum_{i=1}^{n-1}\Delta t M_t(t_{n^*},t_{i-1})\nabla R_h y^{i-1}-\int_0^{t_{n-1}}M_t(t_{n^*},s)\nabla R_h y(s)ds\right\| \\ & \qquad \le C\Delta t(\|\nabla R_h y\|_{L^2(L^2)}+\|\nabla R_h y_t\|_{L^2(L^2)}). \end{aligned}$$

Similarly to (3.31)–(3.33), we estimate Q_3–Q_5 as

$$Q_3 \le C\sum_{n=1}^{l}(\|\xi_y^n\|^2 + \|\eta_y^n\|^2)\Delta t + \frac{1}{4}\sum_{n=1}^{l}\|dt\xi_y^n\|^2\Delta t, \tag{3.43}$$

$$Q_4 \le C(\Delta t)^2\|y_{tt}\|^2_{L^2(L^2)} + \frac{1}{4}\sum_{n=1}^{l}\|dt\xi_y^n\|^2\Delta t, \tag{3.44}$$

$$Q_5 \le C\|(\eta_y)_t\|^2_{L^2(L^2)} + \frac{1}{4}\sum_{n=1}^{l}\|dt\xi_y^n\|^2\Delta t. \tag{3.45}$$

Now, for sufficiently small Δt, using (3.24), (3.34), (3.39), (3.40), (3.42)–(3.45), Lemmas 3.2, 3.3, and the discrete Gronwall's inequality, we complete the proof of theorem. □

4. A PRIORI ERROR ESTIMATES OF TWO-GRID ALGORITHM

In this section, we will present the main algorithm of the paper. The fundamental ingredient of the algorithm is another mixed finite element space $\boldsymbol{V}_H \times W_H \subset \boldsymbol{V}_h \times W_h$ defined on a coarser mesh. The algorithm has the following three steps:

Step **1**. On the coarse grid $\mathcal{T}_H$, compute $(\boldsymbol{p}_H^n, y_H^n) \in \boldsymbol{V}_H \times W_H$ to satisfy the following original nonlinear system:

$$(A^{-1}\boldsymbol{p}_H^n, \boldsymbol{v}_H) + \sum_{i=1}^{n}\Delta t(M(t_n, t_{i-1})\nabla y_H^{i-1}, \boldsymbol{v}_H) - (\nabla y_H^n, \boldsymbol{v}_H) = 0,\ \forall\, \boldsymbol{v}_H \in \boldsymbol{V}_H, \tag{4.1}$$

$$(dty_H^n, w_H) + (\boldsymbol{p}_H^n, \nabla w_H) = (f(y_H^n), w_H),\ \forall\, w_H \in W_H, \tag{4.2}$$

$$y_H^0 = R_H y_0, \tag{4.3}$$

where R_H is defined in the same way as R_h defined by (2.11), (2.12).

Step **2**. On the fine grid $\mathcal{T}_h$, compute $(\tilde{\boldsymbol{p}}_h^n, \tilde{y}_h^n) \in \boldsymbol{V}_h \times W_h$ to satisfy the following linear system:

$$(A^{-1}\tilde{\boldsymbol{p}}_h^n, \boldsymbol{v}_h) + \sum_{i=1}^{n}\Delta t(M(t_n, t_{i-1})\nabla \tilde{y}_h^{i-1}, \boldsymbol{v}_h) - (\nabla \tilde{y}_h^n, \boldsymbol{v}_h) = 0,\ \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{4.4}$$

$$(dt\tilde{y}_h^n, w_h) + (\tilde{\boldsymbol{p}}_h^n, \nabla w_h) = (f(y_H^n) + f'(y_H^n)(\tilde{y}_h^n - y_H^n), w_h),\ \forall\, w_h \in W_h, \tag{4.5}$$

$$\tilde{y}_h^0 = R_h y_0. \tag{4.6}$$

Step **3**. On the fine grid $\mathcal{T}_h$, compute $(\bar{\boldsymbol{p}}_h^n, \bar{y}_h^n) \in \boldsymbol{V}_h \times W_h$ to satisfy the following linear system:

$$(A^{-1}\bar{\boldsymbol{p}}_h^n, \boldsymbol{v}_h) + \sum_{i=1}^{n}\Delta t(M(t_n, t_{i-1})\nabla \bar{y}_h^{i-1}, \boldsymbol{v}_h) - (\nabla \bar{y}_h^n, \boldsymbol{v}_h) = 0,\ \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{4.7}$$

$$(dt\bar{y}_h^n, w_h) + (\bar{\boldsymbol{p}}_h^n, \nabla w_h) = (f(\tilde{y}_h^n) + f'(\tilde{y}_h^n)(\bar{y}_h^n - \tilde{y}_h^n), w_h),\ \forall\, w_h \in W_h, \tag{4.8}$$

$$\bar{y}_h^0 = R_h y_0. \tag{4.9}$$

Now, we will discuss the error estimates of the above two-grid algorithm.

Theorem 4.1. *Let* $(\tilde{\boldsymbol{p}}_h^n, \tilde{y}_h^n) \in \boldsymbol{V}_h \times W_h$ *be the solution of* (4.1)–(4.6) *and* $(\boldsymbol{p}, y)$ *be the solution of* (2.1), (2.2), *respectively. Assume that the exact solution* $(\boldsymbol{p}, y)$ *has enough regularities for our purpose. Then, for* Δt *small enough and* $1 \le n \le N$, *we have*

$$\|\nabla(y^n - \tilde{y}_h^n)\| + \|\boldsymbol{p}^n - \tilde{\boldsymbol{p}}_h^n\| \le C(\Delta t + h + H^3|\ln H|). \tag{4.10}$$

Proof. From (4.4), (4.5) and (2.1), (2.2), we have the following error equations:

$$\begin{aligned}(A^{-1}(R_h\boldsymbol{p}^n - \tilde{\boldsymbol{p}}_h^n), \boldsymbol{v}_h) + \int_0^{t_n} (M(t_n,s)\nabla R_h y(s), \boldsymbol{v}_h)ds - \sum_{i=1}^{n} \Delta t(M(t_n,t_{i-1})\nabla \tilde{y}_h^{i-1}, \boldsymbol{v}_h)\\ -(\nabla(R_h y^n - \tilde{y}_h^n), \boldsymbol{v}_h) = 0,\ \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h,\end{aligned} \tag{4.11}$$

$$\begin{aligned}(dt(R_h y^n - \tilde{y}_h^n), w_h) + (R_h\boldsymbol{p}^n - \tilde{\boldsymbol{p}}_h^n, \nabla w_h) = (f(y^n) - f(y_H^n) + f'(y_H^n)(\tilde{y}_h^n - y_H^n), w_h)\\ +(dty^n - y_t^n, w_h) - (dt(y^n - R_h y^n), w_h),\ \forall\, w_h \in W_h.\end{aligned} \tag{4.12}$$

Notice that a Taylor expansion about y_H^n yields

$$f(y^n) = f(y_H^n) + f'(y_H^n)(y^n - y_H^n) + \frac{1}{2}f''(\widetilde{y})(y^n - y_H^n)^2 \tag{4.13}$$

for some function $\widetilde{y}$. Then

$$\begin{aligned}f(y^n) - f(y_H^n) - f'(y_H^n)(\tilde{y}_h^n - y_H^n) &= f'(y_H^n)(y^n - \tilde{y}_h^n) + \tfrac{1}{2}f''(\widetilde{y})(y^n - y_H^n)^2\\ &= f'(y_H^n)(y^n - R_h y^n + R_h y^n - \tilde{y}_h^n) + \tfrac{1}{2}f''(\widetilde{y})(y^n - y_H^n)^2.\end{aligned} \tag{4.14}$$

Thus, except the theoretical analysis in Theorems 3.1 and 3.2, we only need to estimate the error $\|(y^n - y_H^n)^2\|^2$. Since

$$\begin{aligned}\|(y^n - y_H^n)^2\|^2 &\le \|y^n - y_H^n\|_{0,\infty}^2 \|y^n - y_H^n\|^2\\ &\le (\|y^n - P_H y^n\|_{0,\infty} + \|P_H y^n - R_H y^n\|_{0,\infty}\\ &\quad +\|R_H y^n - y_H^n\|_{0,\infty})^2 \|y^n - y_H^n\|^2,\end{aligned} \tag{4.15}$$

where P_H is defined in the same way as P_h defined by (2.5). By (2.6), (2.7), (3.4) and (3.18), and the inverse estimate, we get

$$\begin{aligned}\|(y^n - y_H^n)^2\|^2 &\le C(H|\ln H| + H^{-1}H^2 + H^{-1}(\Delta t + H^2))^2(\Delta t + H^2)^2\\ &\le C(H|\ln H|\Delta t + H^3|\ln H| + H^{-1}(\Delta t)^2 + 2H\Delta t + H^3)^2.\end{aligned} \tag{4.16}$$

We choose H and Δt such that $H^{-1}\Delta t\ < C$, then we have

$$\|(y^n - y_H^n)^2\|^2 \le C(\Delta t + H^3 \ln|H|)^2. \tag{4.17}$$

Thus, we complete the proof of theorem. □

Theorem 4.2. *Let* $(\bar{\boldsymbol{p}}_h^n, \bar{y}_h^n) \in \boldsymbol{V}_h \times W_h$ *be the solution of* (4.1)–(4.9) *and* $(\boldsymbol{p}, y)$ *be the solution of* (2.1), (2.2)*, respectively. Assume that the exact solution* $(\boldsymbol{p}, y)$ *has enough regularities for our purpose. Then, for* Δt *small enough and* $1 \le n \le N$*, we have*

$$\|\nabla(y^n - \bar{y}_h^n)\| + \|\boldsymbol{p}^n - \bar{\boldsymbol{p}}_h^n\| \le C(\Delta t + h + H^6|\ln H|^2). \tag{4.18}$$

Proof. Similar to (4.11), (4.12), we have the following error equations:

$$\begin{gathered}(A^{-1}(R_h\boldsymbol{p}^n - \bar{\boldsymbol{p}}_h^n), \boldsymbol{v}_h) + \int_0^{t_n} (M(t_n, s)\nabla R_h y(s), \boldsymbol{v}_h)ds - \sum_{i=1}^{n} \Delta t(M(t_n, t_{i-1})\nabla \bar{y}_h^{i-1}, \boldsymbol{v}_h) \\ -(\nabla(R_h y^n - \bar{y}_h^n), \boldsymbol{v}_h) = 0,\ \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h,\end{gathered} \tag{4.19}$$

$$\begin{gathered}(dt(R_h y^n - \bar{y}_h^n), w_h) + (R_h\boldsymbol{p}^n - \bar{\boldsymbol{p}}_h^n, \nabla w_h) = (f(y^n) - f(\tilde{y}_h^n) + f'(\tilde{y}_h^n)(\bar{y}_h^n - \tilde{y}_h^n), w_h) \\ +(dty^n - y_t^n, w_h) - (dt(y^n - R_h y^n), w_h),\ \forall\, w_h \in W_h.\end{gathered} \tag{4.20}$$

Now, a Taylor expansion about $\tilde{y}_h^n$ yields

$$\begin{gathered}f(y^n) - f(\tilde{y}_h^n) - f'(\tilde{y}_h^n)(\bar{y}_h^n - \tilde{y}_h^n) \\ = f(\tilde{y}_h^n) + f'(\tilde{y}_h^n)(y^n - \tilde{y}_h^n) + \tfrac{1}{2}f''(\theta)(y^n - \tilde{y}_h^n)^2 - f(\tilde{y}_h^n) - f'(\tilde{y}_h^n)(\bar{y}_h^n - \tilde{y}_h^n) \\ = f'(\tilde{y}_h^n)(y_h^n - \bar{y}_h^n) + \tfrac{1}{2}f''(\theta)(y^n - \tilde{y}_h^n)^2\end{gathered} \tag{4.21}$$

for some function θ. As in Theorem 4.1, we need to estimate the error $\|(y^n - \tilde{y}_h^n)^2\|^2$. Using the embedding $\|v\|_{0,4} \le C\|v\|_1$, the Poincare's inequality $\|v\| \le C\|\nabla v\|$ and (4.10), we have

$$\|(y^n - \tilde{y}_h^n)^2\|^2 = \|y^n - \tilde{y}_h^n\|_{0,4}^4 \le C\|\nabla(y^n - \tilde{y}_h^n)\|^4 \le C(\Delta t + h + H^3|\ln H|)^4. \tag{4.22}$$

Now, using (4.22) and the same analysis as Theorems 3.1 and 3.2, we complete the proof. □

5. NUMERICAL EXPERIMENTS

In this section, we are going to validate the priori error estimates for two-grid discretization method for nonlinear parabolic integro-differential equations. To simplify the calculation, we shall consider the two-step two-grid scheme (4.1)–(4.6) instead of the scheme (4.1)–(4.9), and choose $h = H^2$ in the following numerical example.

We consider the following semilinear parabolic integral differential equation:

$$\begin{gathered}y_t - \operatorname{div}(A(t)\nabla y) + \int_0^t \operatorname{div}(B(t,s)\nabla y)ds = y^3 + g(x,t),\ x \in \Omega,\ t \in J, \\ y(x,t) = 0,\ x \in \partial\Omega,\ t \in J, \\ y(x,0) = y_0(x),\ x \in \Omega,\end{gathered}$$

with $\Omega = [0,1]^2$ and $J = [0,1]$. For simplicity, we let $A(t) = \boldsymbol{I}$, $B(t,s) = \boldsymbol{I}$ with $\boldsymbol{I}$ be the identity matrix in numerical implementation. We choose $g(x,t)$ in a way such that the exact solution is

$$y(x,t) = \sin(\pi t)\sin(\pi x_1)\sin(\pi x_2).$$

Then, the explicit formulation of $g(x,t)$ is

$$g(x,t) = (\pi\cos(\pi t) + 2\pi^2\sin(\pi t)(\cos(\pi t)+2) + 2\pi(\cos(\pi t)-1))\sin(\pi x_1)\sin(\pi x_2) - (\sin(\pi t)\sin(\pi x_1)\sin(\pi x_2))^3.$$

First, in Tables 1, 2, we show the numerical errors of $\|\nabla(y^n - y_h^n)\|$, $\|\boldsymbol{p}^n - \boldsymbol{p}_h^n\|$, $\|\nabla(y^n - \tilde{y}_h^n)\|$ and $\|\boldsymbol{p}^n - \tilde{\boldsymbol{p}}_h^n\|$ solved by mixed finite element method (MFEM) and two-grid method, respectively.

We can easily see that the two-grid method and the mixed finite element method have the same convergence order. These numerical results coincide with the theoretical analysis. Second, comparing the computing time of two methods in Tables 3, 4, we find that the computing time for the two-grid

Table 1. The results of $\|\nabla(y^n - y_h^n)\|$ and $\|\boldsymbol{p}^n - \boldsymbol{p}_h^n\|$ by using MFEM ($n = N/2$)

$h = \Delta t$	$\|\nabla(y^n - y_h^n)\|$	Rate	$\|\boldsymbol{p}^n - \boldsymbol{p}_h^n\|$	Rate
1/16	2.5505e-1	—	1.5227e-1	—
1/36	1.1434e-1	0.99	6.7827e-2	1.00
1/64	6.4482e-2	1.00	3.8239e-2	1.00
1/100	4.1313e-2	1.00	2.4479e-2	1.00

Table 2. The results of $\|\nabla(y^n - \tilde{y}_h^n)\|$ and $\|\boldsymbol{p}^n - \tilde{\boldsymbol{p}}_h^n\|$ by using two-grid method ($n = N/2$)

H	$h = \Delta t$	$\|\nabla(y^n - \tilde{y}_h^n)\|$	Rate	$\|\boldsymbol{p}^n - \tilde{\boldsymbol{p}}_h^n\|$	Rate
1/4	1/16	2.5645e-1	—	1.5282e-1	—
1/6	1/36	1.1473e-1	0.99	6.8078e-2	1.00
1/8	1/64	6.4617e-2	1.00	3.8291e-2	1.00
1/10	1/100	4.1370e-2	1.00	2.4501e-2	1.00

Table 3. The computing time of MFEM ($h = 1/16$) and two-grid method ($H = 1/4$, $h = 1/16$)

Time level	Computing time (MFEM)	Computing time (two-grid)
4	65 s	52 s
8	52 s	37 s
12	74 s	42 s
16	56 s	34 s

Table 4. The computing time of MFEM ($h = 1/36$) and two-grid method ($H = 1/6$, $h = 1/36$)

Time level	Computing time (MFEM)	Computing time (two-grid)
6	893 s	688 s
12	741 s	347 s
18	729 s	264 s
24	804 s	287 s
30	1311 s	401 s
36	1043 s	354 s

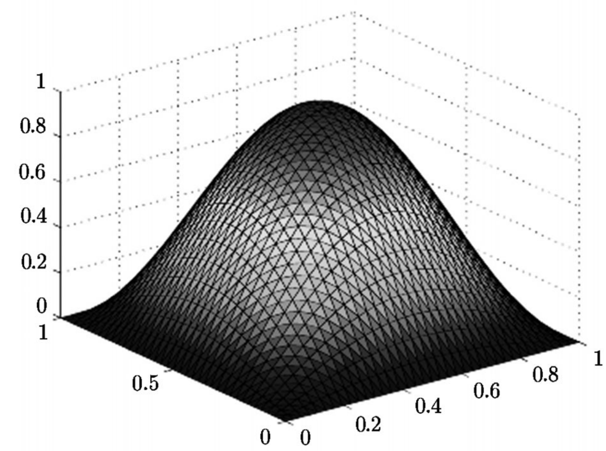

Fig. 1. The profile of the exact solution y on a 36×36 triangle mesh at $t = 0.5$.

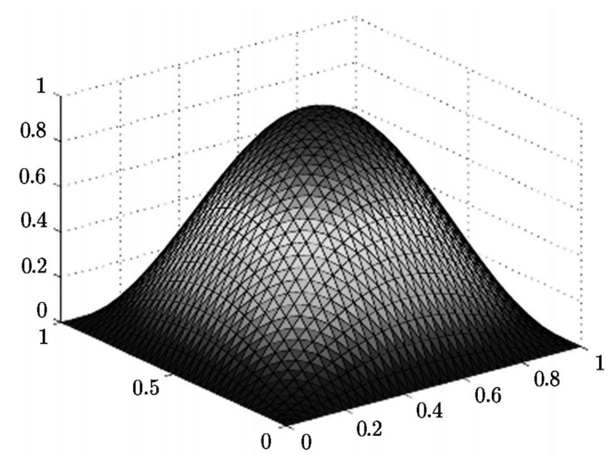

Fig. 2. The profile of the mixed finite element method solution of y on a 36×36 triangle mesh at $t = 0.5$.

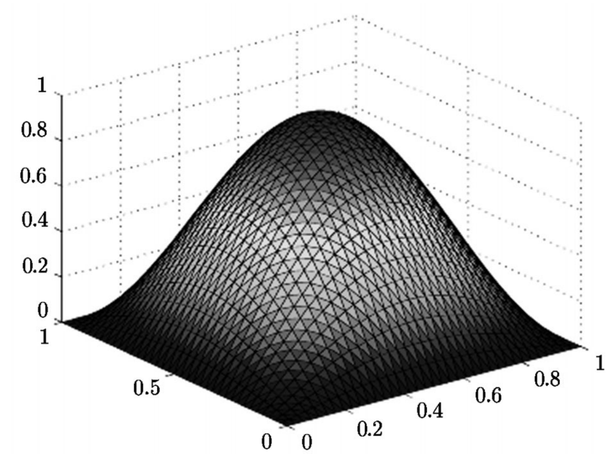

Fig. 3. The profile of the two-grid solution of y on a 36×36 triangle mesh at $t = 0.5$.

method is significantly less than that for MFEM. Finally, in Figs. 1–3, we plot the graphs of the exact solution y, the mixed finite element solution of y and the two-grid solution of y on a 36×36 triangle mesh at $t = 0.5$, respectively.

6. CONCLUSIONS

In this paper, we present a two-grid algorithm for semilinear parabolic integro-differential equations discretized by a new mixed finite element method. The gradient for the method belongs to the square integrable space instead of the classical $H(\mathrm{div};\Omega)$ space. The key ingredient of the two-grid method in this paper is that we use two Newton iterations on the fine grid. We show that when the coarse grid and the fine grid satisfy $h = \mathcal{O}(H^6|\ln H|^2)$, the two-grid algorithm can achieve the same accuracy of the mixed finite element solution. In our future work, we will consider the more complicated two-grid algorithms for (1.1)–(1.4) and give some numerical experiments for these algorithms.

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