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A Method for Obtaining Analytical Solutions to Boundary Value Problems by Defining Additional Boundary Conditions and Additional Sought-For Functions

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Abstract—A method of obtaining analytical solutions to thermal conduction problems is considered. It is based on a separation of thermal conduction into two stages of its time evolution using additional boundary conditions and additional sought-for functions in the heat balance integral method. Solving the partial differential equations is thus reduced to the integration of two ordinary differential equations for some additional sought-for functions. The first stage is characterized by fast convergence of an analytical solution to the exact one. For the second stage, an exact analytical solution is obtained. The additional boundary conditions for both stages are such that their fulfillment by the sought-for solution is equivalent to the fulfillment of the original equation at the boundary points and at the temperature perturbation front. It is shown that if the equation is valid at the boundary points, it is also valid inside the domain.

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1. INTRODUCTION

It is well known that the classical exact analytical methods can be used only for linear boundary value problems and that solutions obtained with them have poor convergence at small values of the time variable. To overcome these difficulties in heat conduction theory, methods using the concept of temperature perturbation front are employed. In these methods the heat conduction process is divided into two time stages: At the first stage, the temperature perturbation front gradually moves from the surface to the center, and at the second stage, the temperature varies over the entire bulk of the body [1–4].

A method for obtaining exact analytical solutions in the entire time range of the transient process is used at the second stage. It employs an additional sought-for function for the time evolution of the temperature at the center of a plate. Since the heat propagation speed is infinite, the temperature begins to change immediately after the boundary condition is applied to the surface of the plate.

Solutions for both stages are obtained by using additional boundary conditions so that their fulfillment by the sought-for solution is equivalent to the fulfillment of the equation at the boundary points and at the temperature perturbation front. Note that some methods of solving boundary value problems by using a differential equation at boundary points have been considered in [5–7]. Specifically, it is mathematically proved in paper [6] that an equation is valid inside the domain if it is valid at the boundaries. The method proposed in the present paper is distinctive in that a two-stage model of heat conduction is used, with a model based on a finite heat conduction speed (first stage) and a model based on an infinite heat conduction speed (second stage).

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2. MATHEMATICAL PROBLEM STATEMENT

Let us describe the main idea of the method by solving, as an example, the following heat conduction problem for an infinite plate with symmetric boundary conditions of the first kind:

$$\partial\Theta(\xi,\mathrm{Fo})/\partial\mathrm{Fo} = \partial^2\Theta(\xi,\mathrm{Fo})/\partial\xi^2 \quad (\mathrm{Fo} > 0,\ 0 < \xi < 1), \tag{1}$$

$$\Theta(\xi,0) = 0, \tag{2}$$

$$\Theta(0,\mathrm{Fo}) = 1, \tag{3}$$

$$\partial\Theta\,(1,\mathrm{Fo})\,/\partial\xi = 0, \tag{4}$$

where $\Theta = (T - T_0)/(T_\mathrm{w} - T_0)$ is the relative excess temperature; $\mathrm{Fo} = at/\delta^2$ is the Fourier number; x is the spatial coordinate; δ is the half-width of the plate; $\xi = x/\delta$, a nondimensional coordinate; a is the thermal diffusivity; t is the time; T_0 is the initial temperature; and T_w is the temperature of the wall.

Note that if symmetric boundary conditions are specified, the temperature field will also be symmetric. Therefore, only half thickness of the plate is usually considered if a condition of absence of heat exchange is specified at its center; such is the boundary condition (4).

Let us introduce a boundary moving in time (a temperature perturbation front $q_1(\mathrm{Fo})$). This boundary divides the initial domain $0 < \xi < 1$ into two subdomains: $0 < \xi < q_1(\mathrm{Fo})$ and $q_1(\mathrm{Fo}) < \xi < 1$, and the heat exchange process—into two time stages: $0 < \mathrm{Fo} \le \mathrm{Fo}_1$ and $\mathrm{Fo}_1 \le \mathrm{Fo} < \infty$. Here $q_1(\mathrm{Fo})$ is a function describing the time evolution of the interface along the coordinate ξ, and Fo_1 is the time in which the moving boundary reaches the center of the plate (Fig. 1). At the second stage, the temperature varies in the entire bulk of the body, $0 < \xi < 1$. An additional sought-for function $q_2(\mathrm{Fo}) = \Theta(1,\mathrm{Fo})$ describing the temperature evolution in time at the center of the plate is introduced.

3. FIRST STAGE OF THE HEAT CONDUCTION PROCESS

The mathematical problem statement for the first stage is as follows:

$$\partial\eta(\xi,\mathrm{Fo})/\partial\mathrm{Fo} = \partial^2\eta(\xi,\mathrm{Fo})/\partial\xi^2 \quad (0 < \mathrm{Fo} \le \mathrm{Fo}_1,\ 0 < \xi\ < q_1(\mathrm{Fo})), \tag{5}$$

$$\eta(0,\mathrm{Fo}) = 1, \tag{6}$$

$$\eta(q_1,\mathrm{Fo}) = 0, \tag{7}$$

$$\partial\eta(q_1,\mathrm{Fo})/\partial\xi = 0, \tag{8}$$

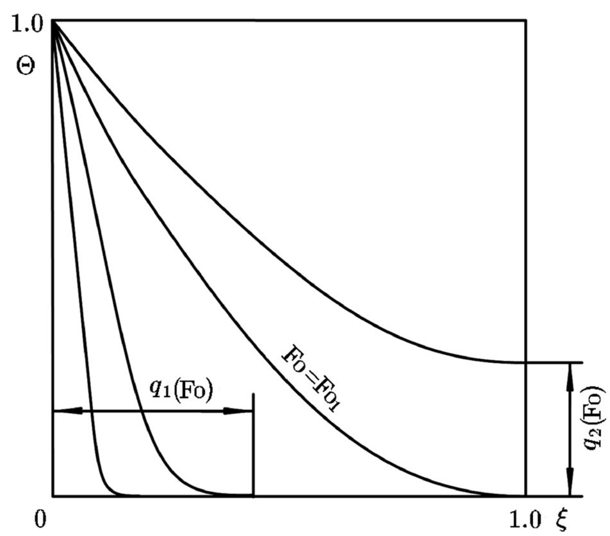

Fig. 1. Calculation scheme of heat exchange.

where relations (7) and (8) are matching conditions of the heated and unheated zones, respectively; $\eta = T/T_{\rm w}$.

Note that the problem (5)–(8) does not have an initial condition of the form (2) specified along the entire thickness of the plate, since this problem is not defined outside the temperature perturbation front. Note also that relations (7) and (8), which are specified at the temperature perturbation front $q_1(\mathrm{Fo})$, are matching conditions of the heated and unheated zones. According to relation (7), the temperature at the front is equal to the initial temperature, and it follows from (8) that the heat flux at the front is zero, although it moves along the coordinate ξ as a function of time. That is, the temperature perturbation front $q_1(\mathrm{Fo})$ is the heated layer boundary moving along the coordinate ξ as a function of time. The time law of this motion is determined by solving the problem (5)–(8) and taking $q_1(\mathrm{Fo})$ as an additional sought-for function.

A solution to the problem (5)–(8) is sought for in the form

$$\eta_n(\xi, \mathrm{Fo}) = 1 + \sum_{k=1}^{n} a_{kn}(q_{1n})\ \varphi_k(\xi), \tag{9}$$

where $a_{kn}(q_{1n})$ $(k = \overline{1,\, n})$ are unknown coefficients, and $\varphi_k(\xi) = \xi^{2k-1}$ are coordinate functions.

Let us consider obtaining the solution to the problem (5)–(8) in general form (for any number of approximations, n), and then use the method for a single approximation. We demand that relation (9) satisfy, instead of Eq. (5), an equation averaged over the heated layer thickness (heat balance integral):

$$\int\limits_0^{q_{1n}(\mathrm{Fo})} \frac{\partial \eta_n\,(\xi, \mathrm{Fo})}{\partial \mathrm{Fo}}\, d\xi = \int\limits_0^{q_{1n}(\mathrm{Fo})} \frac{\partial^2 \eta_n\,(\xi, \mathrm{Fo})}{\partial \xi^2}\, d\xi. \tag{10}$$

Relation (9) satisfies the boundary condition (6) at any approximation. To determine the unknown coefficients $a_{kn}(q_1)$, we use the main relations (7) and (8) and some additional boundary conditions. The physical meaning of these additional boundary conditions is that their fulfillment by the solution (9) at the boundary points $\xi = 0$ and $\xi = 1$ (for the first stage of the process at point $\xi = q_{1n}(\mathrm{Fo})$) is equivalent to the fulfillment of Eq. (5) at these points. A general formula for them (see [1]) is

$$\partial^i \eta_n(0, \mathrm{Fo})/\partial \xi^i = 0 \ \ (i = 2, 4, 6, \ldots). \tag{11}$$

Owing to the uneven powers of the variable ξ, relation (9) satisfies the conditions (11) at any approximation.

To obtain a more exact solution at a minimal number of approximations, some additional boundary conditions are specified at the moving boundary $\xi = q_{1n}(\mathrm{Fo})$ as well. The corresponding formula is

$$\partial^i \eta(q_{1n}, \mathrm{Fo})/\partial \xi^i = 0 \ \ (i = 2, 3, 4, \ldots). \tag{12}$$

Substituting (9) into the main and additional boundary conditions for the unknown coefficients $a_{kn}(q_{1n})$, we obtain a “chain” system of n algebraic linear equations. With $a_{kn}(q_{1n})$ determined by solving this system, relation (9) takes the form

$$\eta_n(\xi\,, \mathrm{Fo}) = 1 + \sum_{k=1}^{n} (-1)^k A_{kn} (\xi/q_{1n})^{2k-1}, \tag{13}$$

where A_{kn} are coefficients having specific numerical values as functions of the number of approximations n of the solution.

Substituting (13) into the heat balance integral (10), we obtain the following ordinary differential equation for the unknown function $q_{1n}(\mathrm{Fo})$:

$$q_{1n}(\mathrm{Fo})\, dq_{1n}(\mathrm{Fo}) = 2nd\mathrm{Fo}. \tag{14}$$

Integrating Eq. (14) with an initial condition $q_{1n}(0) = 0$, we find

$$q_{1n}(\mathrm{Fo}) = \sqrt{4n\mathrm{Fo}}. \tag{15}$$

The time of completion of the first stage of the process, Fo_{1n}, at $q_{1n}(\mathrm{Fo}) = 1$ and at any approximation is found by the formula $\mathrm{Fo}_{1n} = 1/(4n)$. Hence, as the number of approximations increases, the temperature perturbation front speed $dq_{1n}/d\mathrm{Fo} = 2n/q_{1n}$ increases, but the time (Fo_{1n}) when the front reaches the coordinate $\xi = 1$ decreases. In the limit, as $n \to \infty$ $\mathrm{Fo}_{1n} \to 0$. Hence, the solution (9) of the problem (5)–(8) as $n \to \infty$ confirms the fact that the heat propagation speed described by the parabolic equation (5) is infinite. However, it is assumed to be finite (which is a basis of the heat balance integral method) to divide the process into two stages with easy solutions for each of them.

Note that as the number of approximations of the integral method increases, the accuracy of the solutions also increases owing to increasing accuracy of the fulfillment of the problem relations (5)–(8). Thus, to obtain a solution at any approximation we have only to find the unknown coefficients A_{kn} from the main and additional boundary conditions. The differential equation for $q_{1n}(\mathrm{Fo})$, its solution, as well as the time of completion of the first stage, Fo_1, are found from the general formulas (14) and (15).

For instance, the coefficients A_{kn} in the third, fourth, and fifth approximations are, respectively, $A_{1,3} = 15/8$; $A_{2,3} = 5/4$; $A_{3,3} = 3/8$ (third approximation); $A_{1,4} = A_{2,4} = 35/16$; $A_{3,4} = 21/16$; $A_{4,4} = 5/16$ (fourth approximation); $A_{1,5} = 315/128$; $A_{2,5} = 105/32$; $A_{3,5} = 189/64$; $A_{4,5} = 45/32$; $A_{5,5} = 35/128$ (fifth approximation). The times of completion of the first stage of the process for these approximations are $\mathrm{Fo}_{1,3} = 0.08333$; $\mathrm{Fo}_{1,4} = 0.0625$; and $\mathrm{Fo}_{1,5} = 0.05$.

The results of calculations by formula (9) versus the exact solution [8] are presented in Figs. 2 and 3. Figure 4 shows temperature values for very small Fourier numbers ($10^{-8} \leq \mathrm{Fo} \leq 10^{-12}$).

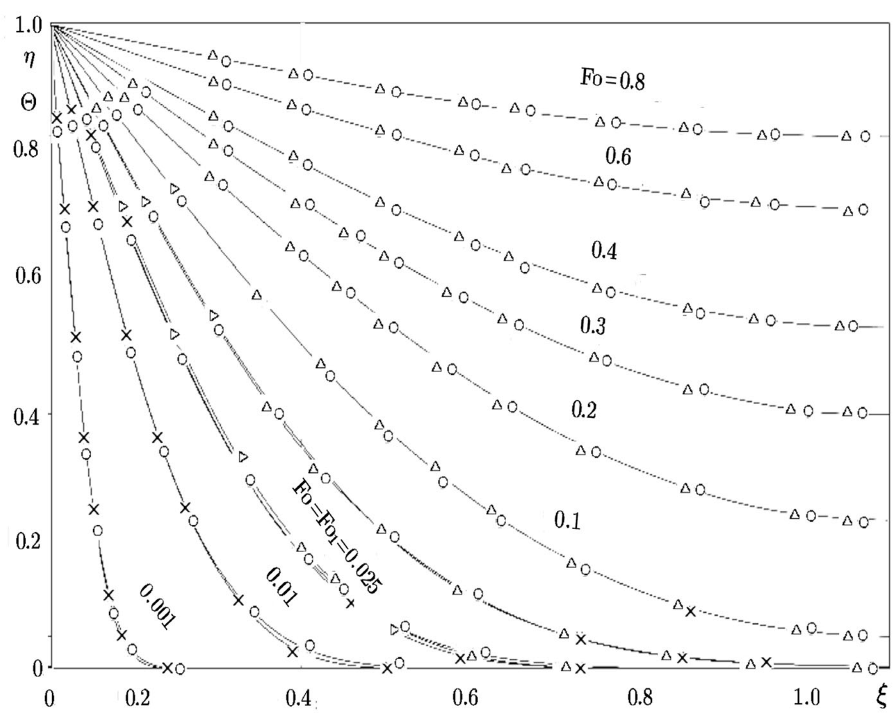

Fig. 2. Temperature distribution in the plate: ×—10th approximation of the first stage (by formula (9) at $n = 10$); △—4th approximation of the first stage (by formula (34) at $n = 4$); ○—exact solution from [8] and by formula (34) at $n = 100$.

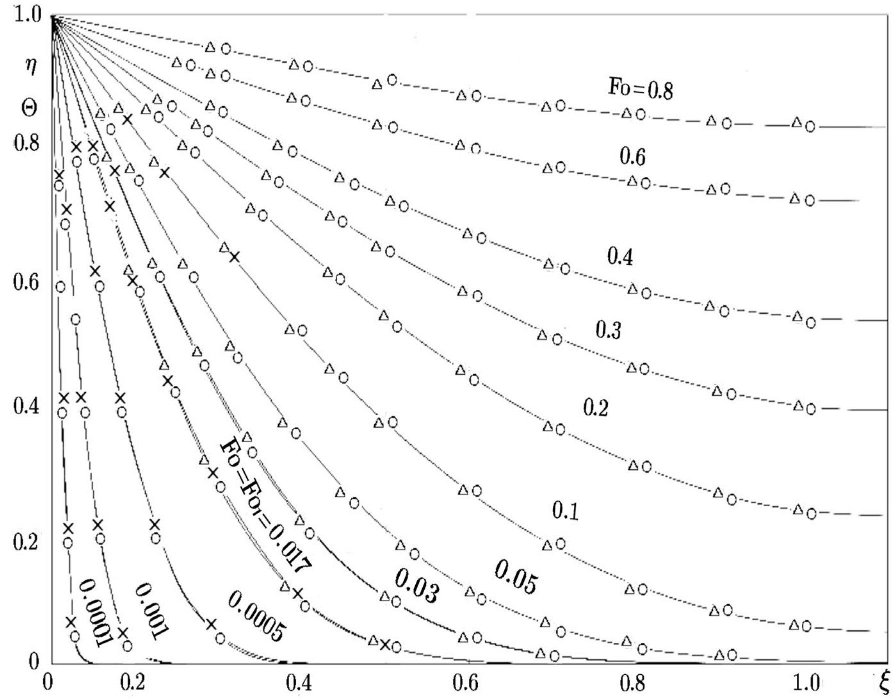

Fig. 3. Temperature distribution in the plate: ×—15th approximation of the first stage (by formula (9)); △—6th approximation of the second stage (by formula (34) at $n = 6$); ○—exact solution from [8] and by formula (34) at $n = 1000$.

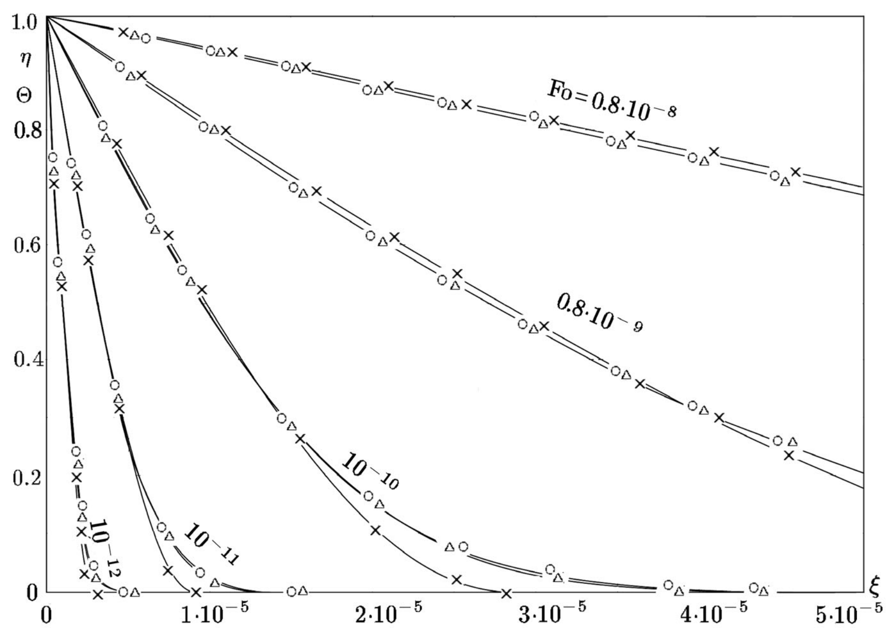

Fig. 4. Temperature distribution in the plate: ×—2nd approximation of the first stage; △—20th approximation of the first stage; ○—exact solution from [8] and by formula (34) at $n = 1000000$.

An analysis of the results makes it possible to conclude that the difference between the exact solution and the 20th approximation is less than 1%; these solutions practically coincide in the 30th approximation. It should be noted that it is difficult to obtain the exact solution by the formulas from [8] for such small Fourier numbers, since it requires a large number of terms in the series. Specifically, the calculations have shown that at $\mathrm{Fo} = 10^{-12}$ about $5 \cdot 10^5$ terms must be used for the exact solution to converge.

Let us demonstrate the convergence of the thus obtained solution to the exact one as $n \to \infty$ by proving, as an example, the following theorem:

Theorem 1. *If a function representing an infinite series with uneven powers of the spatial variable satisfies Eq.* (5) *at the boundary points, it also satisfies the equation inside the domain at an infinitely large number of terms of the series.*

Proof. To prove this, we demand that at $\xi = 0$ Eq. (5) be valid and that its kth derivatives with respect to the variable ξ be zero:

$$\frac{\partial^k}{\partial \xi^k}\left(\frac{\partial \eta(\xi, \mathrm{Fo})}{\partial \mathrm{Fo}}\right)_{\xi=0} = \frac{\partial^k}{\partial \xi^k}\left(\frac{\partial^2 \eta(\xi, \mathrm{Fo})}{\partial \xi^2}\right)_{\xi=0} \quad (k = \overline{0,\, n}). \tag{16}$$

Note that at $k = 0$, relation (16) is reduced to Eq. (5). At $k = (1, 2, 3, \ldots, n)$ relation (16) represents Eq. (5) with the derivatives of the kth order taken of its right- and left-hand sides. Thus, relation (16) is a system of n partial differential equations. Substituting (9) into (16), we find at $\xi = 0$

$$\frac{\partial a_1}{\partial \mathrm{Fo}} = 6a_2; \quad \frac{\partial a_2}{\partial \mathrm{Fo}} = 20a_3; \quad \frac{\partial a_3}{\partial \mathrm{Fo}} = 42a_4; \quad \frac{\partial a_4}{\partial \mathrm{Fo}} = 72a_5; \quad \ldots . \tag{17}$$

Multiplying the relations (17) by ξ^{2k-1} $(k = \overline{1,\, n})$ and adding together the expressions obtained yields

$$\frac{da_1}{d\mathrm{Fo}}\xi + \frac{da_2}{d\mathrm{Fo}}\xi^3 + \frac{da_3}{d\mathrm{Fo}}\xi^5 + \frac{da_4}{d\mathrm{Fo}}\xi^7 + \cdots = 6a_2\xi + 20a_3\xi^3 + 42a_4\xi^5 + 72a_5\xi^7 + \cdots . \tag{18}$$

Note that relation (18) is the result of substitution of the series (9) into Eq. (5). Thus, since the sought-for solution satisfies Eq. (5) at the boundary point $\xi = 0$, we find that at $k = \overline{1,\, n}$ and $n \to \infty$ Eq. (5) is satisfied everywhere in the domain under consideration. At even powers of k Eq. (5) is satisfied in the limit, that is, when its right- and left-hand sides are zero. □

In [6] it is proved that an equation is valid inside the domain under consideration if it is valid at the boundary for a solution consisting of even and uneven powers of a dimensional space variable for a half-space.

4. SECOND STAGE OF THE HEAT CONDUCTION PROCESS

At the second stage of the heat conduction process, which corresponds to the time $\mathrm{Fo}_1 \le \mathrm{Fo} < \infty$, the temperature varies over the entire thickness of the plate, and the concept of temperature perturbation front loses its meaning. As an additional sought-for function, we take a function that characterizes the temperature variation in time at the center of the plate. Since the heat propagation speed described by the solution to the parabolic heat conduction equation is infinite, which is confirmed by the solution obtained at the first stage of the process, the temperature at the center of the plate begins to change immediately after a boundary condition of the first kind is imposed at point $\xi = 0$. Therefore, at the second stage it is physically justified to take $\mathrm{Fo}_1 = 0$. In this case the mathematical statement of the problem is the same as that of the problem (1)–(4).

Let us introduce an additional sought-for function,

$$\Theta(1\,, \mathrm{Fo}) = q_2(\mathrm{Fo}), \tag{19}$$

to characterize the time variation of the temperature at the center of the plate, that is, at $\xi = 1$. Since the temperature at the center of the plate is a sought-for quantity of the problem (1)–(4), the use of this function does not change the problem, and it is only an auxiliary way to simplify the process of obtaining its solution.

An exact solution to the problem (1)–(4) is sought for in the form

$$\Theta_n(\xi\,,\mathrm{Fo}) = 1 + \sum_{k=1}^{n} b_{kn}(q_{2n})\,\Psi_k(\xi), \tag{20}$$

where $b_{kn}(q_{2n})$ $(k = \overline{1,\,n})$ are unknown coefficients and $\Psi_k(\xi) = \sin(r\pi\xi/2)$ are coordinate functions $(r = 2k - 1)$.

It is evident that relation (20), owing to the adopted system of coordinate functions at any n, satisfies the boundary conditions (3) and (4). The coefficients $b_{kn}(q_{2n})$ $(k = \overline{1,\,n})$ are determined from the auxiliary condition (19) and some additional boundary conditions to be found so that the differential equation (1) be valid at the boundary points $\xi = 0$ and $\xi = 1$. The additional boundary conditions at point $\xi = 0$ are found (as in the first stage of the process) by the general formula (11), that is, $\partial^i\Theta(0,\mathrm{Fo})/\partial\xi^i = 0$ $(i = 2, 4, 6, \ldots)$. Note that, owing to the adopted system of coordinate functions in relation (20), these are exactly satisfied in any approximation.

The general formulas for the additional boundary conditions as applied to point $\xi = 1$ have the following form [1]:

$$\partial^i\Theta(1,\mathrm{Fo})/\partial\xi^i = 0 \quad (i = 3, 5, 7, \ldots), \tag{21}$$

$$\partial^{2i}\Theta(1,\mathrm{Fo})/\partial\xi^{2i} = d^i q_2(\mathrm{Fo})/d\mathrm{Fo}^i \quad (i = 1, 2, 3, \ldots). \tag{22}$$

Note that the conditions (21) for relation (20) are valid in any approximation. The unknown coefficients $b_{kn}(q_{2n})$ $(k = \overline{1,\,n})$ will be found from the condition (19) and the additional boundary conditions obtained by the general formula (22).

As in the first stage of the process, we consider first a general algorithm of obtaining the solution, and then generalize the method for a specific approximation.

We demand that relation (20) satisfy, instead of Eq. (1), some equation averaged over the plate thickness (heat balance integral):

$$\int_0^1 \frac{\partial\Theta\,(\xi,\mathrm{Fo})}{\partial\mathrm{Fo}} d\xi = \int_0^1 \frac{\partial^2\Theta\,(\xi,\mathrm{Fo})}{\partial\xi^2}\, d\xi. \tag{23}$$

Substituting (20) into (23) gives an ordinary differential equation of the nth order for the unknown function $q_{2n}(\mathrm{Fo})$. The integration constants of this equation are found from the initial condition (2) by taking its residual and demanding that this function be orthogonal to all coordinate functions $\varphi_j\,(\xi) = \sin\,(j\pi\xi/2)$ $(j = r = 2k - 1)$. Since the sine function is orthogonal, the unknowns in the system of algebraic equations with respect to the integration constants are separated (each equation contains one unknown). As a result, a general formula for determining the integration constants in any approximation is obtained.

Consider, as an example, obtaining a solution to the problem (1)–(4) in the second approximation, by substituting (20) (taking only two terms of the series) into (19) and (22) (at $i = 1$). For the unknown coefficients $b_{1,2}(q_2)$ and $b_{2,2}(q_2)$, we obtain a system of two algebraic linear equations. Relation (20), with the coefficients found from its solution, takes the form

$$\Theta(\xi,\mathrm{Fo}) = 1 + \frac{1}{8\pi^2}\left[-\,(4q'_{2,2} + 9\pi^2(q_{2,2} - 1))\sin\left(\frac{\pi\xi}{2}\right) + (4q'_{2,2} + \pi^2(q_{2,2} - 1))\sin\left(\frac{3\pi\xi}{2}\right)\right], \tag{24}$$

where $q'_{2,2} = dq_{2,2}/d\mathrm{Fo}$.

Substituting (24) into (23) gives

$$\frac{4}{3\pi^3}q''_{2,2} + \frac{10}{3\pi}q'_{2,2} + \frac{3\pi}{4}q_{2,2} - \frac{3\pi}{4} = 0, \tag{25}$$

where $q''_{2,2} = d^2 q_{2,2}$.

Integrating Eq. (25), we have

$$q_{2,2}(\mathrm{Fo}) = C_{1,2}\exp(-\pi^2\mathrm{Fo}/4) + C_{2,2}\exp(-9\pi^2\mathrm{Fo}/4) + 1, \tag{26}$$

where $C_{1,2}$ and $C_{2,2}$ are integration constants.

Substituting (26) into (24) yields

$$\Theta_2(\xi\,,\mathrm{Fo}) = 1 + C_{1,2}\exp\left(-\frac{\pi^2}{4}\mathrm{Fo}\right)\sin\left(\frac{\pi}{2}\xi\right) + C_{2,2}\exp\left(-\frac{9\pi^2}{4}\mathrm{Fo}\right)\sin\left(\frac{3\pi}{2}\xi\right). \tag{27}$$

To determine the constants $C_{1,2}$ and $C_{2,2}$, we form the residual of the initial condition (2) and require that it be orthogonal to the coordinate functions $\sin(j\pi\xi/2)$ $(j = 1, 3)$:

$$\int\limits_0^1 \left(C_{1,2}\sin\left(\frac{\pi}{2}\xi\right) - C_{2,2}\sin\left(\frac{3\pi}{2}\xi\right)\right)\sin\left(\frac{j\pi}{2}\xi\right)d\xi = -\int\limits_0^1 \sin\left(\frac{j\pi}{2}\xi\right)d\xi \ \ (j = 1, 3). \tag{28}$$

Relation (28) for $C_{1,2}$ and $C_{2,2}$ is a system of two algebraic equations. Since the trigonometric functions are orthogonal, the unknowns in the system are separated so that each equation has only one unknown. A general form of these equations is

$$C_{k,2}\int\limits_0^1 \sin^2\left(\frac{r\pi}{2}\xi\right)d\xi = \int\limits_0^1 \sin\left(\frac{r\pi}{2}\xi\right)d\xi \quad (k = 1, 2;\ r = 2k-1). \tag{29}$$

Taking the integrals in (29), we find

$$C_{k,2} = 4(-1)^k/(r\pi) \quad (k = 1,\ 2;\ \ r = 2k-1). \tag{30}$$

Formula (27), with the thus found values of the integration constants, takes the form

$$\Theta_2(\xi\,,\mathrm{Fo}) = 1 - \sum_{k=1}^{2}\frac{4}{r\pi}\exp\left(-\frac{r^2\pi^2}{4}\mathrm{Fo}\right)\sin\left(\frac{r\pi}{2}\xi\right) \quad (r = 2k-1;\ n = 2). \tag{31}$$

In the third approximation, to find the unknown coefficients $b_{k3}(q_{2,3})$ $(k = 1, 2, 3)$ we use relation (19) and two additional boundary conditions obtained by the general formula (22) (at $i = 1, 2$). An ordinary differential equation for the unknown function $q_{2,3}(\mathrm{Fo})$ is found from the heat balance integral (23):

$$\frac{4}{15\pi^5}q'''_{2,3} + \frac{7}{3\pi^3}q''_{2,3} + \frac{259}{60\pi}q'_{2,3} + \frac{15}{16}\pi q_{2,3} - \frac{15}{16}\pi = 0, \tag{32}$$

where $q'''_{2,3} = d^3q_{2,3}/d\mathrm{Fo}^3$.

Integrating equation (32), we find

$$q_2(\mathrm{Fo}) = C_{1,3}\exp\left(-\frac{\pi^2}{4}\mathrm{Fo}\right) + C_{2,3}\exp\left(-\frac{9\pi^2}{4}\mathrm{Fo}\right) + C_{3,3}\exp\left(-\frac{25\pi^2}{4}\mathrm{Fo}\right) + 1. \tag{33}$$

Substituting (33) into (20), with the coefficients $C_{k,3}$ $(k = 1, 2, 3)$ found from the initial condition (2), gives

$$\Theta(\xi\,, \mathrm{Fo}) = 1 - \sum_{k=1}^{n} \frac{4}{r\pi} \exp\left(- \nu_k \mathrm{Fo}\right) \sin\left(\frac{r\pi\xi}{2}\right), \tag{34}$$

where $\nu_k = r^2\pi^2/4$ ($r = 2k-1$; $k = \overline{1,\,n}$); $n = 3$. The results of calculations in the second (by formula (31)) and third (by formula (34)) approximations versus the exact solution [8] are presented in Fig. 2. These make it possible to conclude that within the range of $0.02 \le \mathrm{Fo} < \infty$ the deviation from the exact solution in the second approximation (in comparison to that in the first approximation) decreased to 2%, and the solution in the third approximation practically coincides with the exact one. From the solution (34) one can see that the formula for the coefficients at the exponential function is the same as the formula for the similar coefficients (obtained by satisfying the initial condition) of the classical exact analytical solution, and the formula for the numbers ν_k is the same as that for the eigenvalues of this solution [8]. Hence, relation (34) as $n \to \infty$ is an exact analytical solution to the problem (1)–(4).

5. A THEORETICAL JUSTIFICATION OF OBTAINING AN EXACT ANALYTICAL SOLUTION

Let us prove, using the results obtained above, a theorem on obtaining exact analytical solutions to problems of transient heat conduction by the heat balance integral method (which is an approximate analytical method). We will use special systems of orthogonal coordinate functions exactly satisfying the boundary conditions of the boundary value problem under consideration and some additional boundary conditions.

Theorem 2. *If the heat balance integral method employs systems of orthogonal coordinate functions exactly satisfying the boundary conditions of the boundary value problem* (1)–(4) *in any approximation, an exact analytical solution in the form of an infinite series can be obtained.*

Proof. To prove this theorem, we present relation (20) in the following form:

$$\Theta(\xi\,, \mathrm{Fo}) = 1 - \sum_{k=1}^{\infty} f_k(\mathrm{Fo})\,\Psi_k(\xi), \tag{35}$$

where $f_k(\mathrm{Fo}) = b_k(q_2(\mathrm{Fo}))$ ($k = \overline{1,\,n}$) are unknown coefficients; $\Psi_k(\xi) = \sin(r\pi\xi/2)$ ($r = 2k-1$) are coordinate functions exactly satisfying the boundary conditions (3), (4) in any approximation.

Taking the residual of Eq. (1) and demanding that it be orthogonal to all coordinate functions $\Psi_k(\xi)$ ($k = \overline{1,\,\infty}$), we find

$$\int_0^1 \left[\sum_{k=1}^{\infty} \frac{\partial f_k}{\partial \mathrm{Fo}} \sin\left(\frac{r\pi}{2}\xi\right) + \left(\frac{r^2\pi^2}{4}\right) \int_0^1 \sum_{k=1}^{\infty} f_k \sin\left(\frac{r\pi}{2}\xi\right)\right] \sin\left(\frac{j\pi}{2}\xi\right) d\xi = 0 \tag{36}$$

$$(j = r = 2k-1).$$

Relation (36), taking into account the orthogonality of the sines, is reduced to

$$\frac{df_k}{d\mathrm{Fo}} + \frac{r^2\pi^2}{4} f_k = 0. \tag{37}$$

Integrating Eq. (37) and substituting the solution found into (35), we obtain

$$\Theta(\xi\,, \mathrm{Fo}) = 1 - \sum_{k=1}^{\infty} B_k e^{-\nu_k \mathrm{Fo}} \sin\left(\frac{r\pi\xi}{2}\right), \tag{38}$$

where $\nu_k = r^2\pi^2/4$ $(r = 2k-1)$; B_k are integration constants determined from the initial condition (2). For this, its residual is formed, and it is demanded that it be orthogonal to all coordinate functions $\Psi_k(\xi)$ $(k = \overline{1,\infty})$:

$$\int_0^1 \left(\sum_{k=1}^{\infty} B_k \sin\left(\frac{r\pi\xi}{2}\right) - 1\right) \sin\left(\frac{j\pi\xi}{2}\right) d\xi = 0 \quad (j = r = 2k-1). \tag{39}$$

Relation (39) for B_k is an infinite system of algebraic linear equations. Since the sines are orthogonal, the unknowns are separated, and the system is reduced to a single equation

$$\int_0^1 \left(B_k \sin^2\left(\frac{r\pi\xi}{2}\right) - \sin\left(\frac{r\pi\xi}{2}\right)\right) d\xi = 0, \tag{40}$$

and from the solution to this equation we have $B_k = 4/(r\pi)$ $(k = 1, \infty)$.

With the thus found values of B_k and ν_k, relation (38) reduces to (34) (as $n \to \infty$), which is an exact analytical solution to the problem (1)–(4), Q.E.D. Note that relation (34) satisfies the additional boundary conditions (11), (21), and (22). □

The above method is most efficient when used for boundary value problems whose solutions are difficult to obtain by classical analytical methods (nonlinear problems, problems with variable physical properties of the medium, time-variable boundary conditions, etc.). Solving these problems by dividing the thermal process into two time stages makes it much easier to obtain highly accurate approximate analytical solutions.

The results of calculations by formula (34) are presented in Figs. 2–4. For Fourier numbers Fo $=$ 0.001 to Fo $= 10^{-12}$ we used $n = 100$ to $n = 1000000$ terms of the series (34). Therefore, for small times it is recommended to use the solutions obtained for the first stage of the process.

6. CONCLUSIONS

1. By defining an additional sought-for function (the temperature perturbation front) and additional boundary conditions, solving the initial boundary value problem for a partial differential equation was reduced to integrating two ordinary differential equations for additional sought-for functions $q_1(\mathrm{Fo})$ and $q_2(\mathrm{Fo})$. It has been shown, by proving a corresponding theorem, that if the initial (partial differential) equation is satisfied at the boundary points of a boundary value problem, it is also satisfied inside the domain.

2. Considering the function $q_2(\mathrm{Fo}) = \Theta(1, \mathrm{Fo})$ at the second stage of the process is based on a property of the parabolic heat conduction equation based on an infinite heat propagation speed. According to this property, the temperature at the center of the plate begins to change immediately after a boundary condition of the first kind is applied to its surface. The use of this function in the heat balance integral method allows obtaining a solution that fully coincides with a classical exact analytical solution The method has been theoretically justified by proving a corresponding theorem.

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