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# Mixed Methods for Optimal Control Problems

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**Abstract**—In this paper, we investigate a posteriori error estimates of a mixed finite element method for elliptic optimal control problems with an integral constraint. The gradient for our method belongs to the square integrable space instead of the classical $H(\mathrm{div};\Omega)$ space. The state and co-state are approximated by the $P_0^2$-$P_1$ (velocity—pressure) pair and the control variable is approximated by piecewise constant functions. Using duality argument method and energy method, we derive the residual a posteriori error estimates for all variables.

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## 1. INTRODUCTION

Optimal control problems governed by partial differential equations have been widely studied and applied in the science and engineering numerical simulation. Many numerical methods have been developed to solve these optimal control problems, among them, the standard finite element approximation of optimal control problems has been extensively studied in the literature. It is impossible to even give a very brief review here. For the studies about convergence and superconvergence of finite element approximations for optimal control problems, see [1, 6, 9, 15, 16, 25—29] for the standard finite element method, see [4, 5, 7] for Raviart—Thomas mixed finite element method, and see [11] for splitting positive definite mixed finite element method. A systematic introduction of finite element methods for PDEs and optimal control problems can be found in, for example, [3, 18].

Adaptive finite element approximation is among the most important means to boost the accuracy and efficiency of finite element discretizations. It ensures a higher density of nodes in a certain area of the given domain, where the solution is more difficult to approximate. At the heart of any adaptive finite element method is an a posteriori error estimator or indicator. In recent years, the adaptive finite element method has been extensively investigated in optimal control [2, 13, 14, 19—24, 32, 33]. Sharp a posteriori error estimators of finite element method for a class of distributed elliptic optimal control problems are derived in [19]. The recovery type a posteriori error estimates of finite element approximation are obtained for elliptic optimal control problems [20]. In [22], Li and Yan investigated a posteriori error estimates of finite element method for an elliptic boundary control problem. They considered a posteriori error estimates for optimal control problems governed by Stokes equations [23]. They also discussed a posteriori error estimates of fully discrete finite element method for parabolic optimal control problems, the backward Euler method and the discontinuous Galerkin method were used for time discretization in [24] and [21], respectively. In [2], the authors analyzed finite element Galerkin discretizations for a class of constrained optimal control problems that are governed by Fredholm integral or integro-differential equations. The analysis in that paper focused on the derivation of a priori error estimates and a posteriori error estimators for the approximation schemes. In [32], the authors derived equivalent a posteriori error estimators with lower and upper bounds of finite element approximation of a constrained optimal control problem governed by a parabolic integro-differential equation. In [14], Hou developed a mixed discontinuous finite element method for linear parabolic optimal control problems, and derived a priori and a posteriori error estimates.

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In recent years, Chen et al. [8] developed a new mixed finite element scheme and used a $P_0^2$-$P_1$ finite element pair for solving partial differential equations. The gradient of the primal variable for this method belongs to the square integrable space instead of the classical $H(\mathrm{div};\Omega)$ space. Using this method, we can derive two approximations for the gradient of the primal variable $y$, one is the numerical approximation solution $\boldsymbol{p}_h$, the other is the derivative of the approximation solution $y_h$.

The goal of this paper is to derive a posteriori error estimates of a new mixed finite element approximation for elliptic control problems. We are interested in the following linear optimal control problems for the state variables $\boldsymbol{p}$, $y$, and the control $u$ with an integral constraint:

$$\min_{u\in U_{ad}}\left\{\frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\frac{1}{2}\|y-y_d\|^2+\frac{\nu}{2}\|u\|^2\right\} \tag{1.1}$$

subject to the state equation

$$-\mathrm{div}(A(x)\nabla y)=f+u,\ \ x\in\Omega, \tag{1.2}$$

which can be written in the form of the first-order system

$$\mathrm{div}\boldsymbol{p}=f+u,\ \ \boldsymbol{p}=-A\nabla y,\ \ x\in\Omega, \tag{1.3}$$

and the boundary condition

$$y=0,\ \ x\in\partial\Omega, \tag{1.4}$$

where $\Omega$ is a polygonal domain. $U_{ad}$ denotes the admissible set of the control variable, defined by

$$U_{ad}=\left\{u\in L^2(\Omega):\int\limits_{\Omega}udx\geq 0\right\}.$$

We assume that $y_d\in H^1(\Omega)$, $\boldsymbol{p}_d\in(H^1(\Omega))^2$ and $\nu$ is a fixed positive number. The coefficient $A(x)=(a_{ij}(x))$ is a symmetric matrix function with $a_{ij}(x)\in W^{1,\infty}(\Omega)$, which satisfies the ellipticity condition

$$a_*|\xi|^2\leq\sum_{i,j=1}^{2}a_{ij}(x)\xi_i\xi_j\leq a^*|\xi|^2,\ \ \forall\,(\xi,x)\in\mathbb{R}^2\times\bar{\Omega},\ \ 0<a_*<a^*.$$

The plan of this paper is as follows. In Section 2, we construct our new mixed finite element approximation scheme for the optimal control problem (1.1)–(1.4) and give its equivalent optimality conditions. The main results of this paper are stated in Section 3. In this section, by using the duality argument method and the energy method, we derive the residual a posteriori error estimates for all the variables. In Section 4, we briefly summarize the results obtained and some possible future extensions.

In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on $\Omega$ with a norm $\|\cdot\|_{m,p}$ given by

$$\|v\|_{m,p}^p=\sum_{|\alpha|\leq m}\|D^\alpha v\|_{L^p(\Omega)}^p,$$

and a semi-norm $|\cdot|_{m,p}$ given by

$$|v|_{m,p}^p=\sum_{|\alpha|=m}\|D^\alpha v\|_{L^p(\Omega)}^p.$$

We set $W_0^{m,p}(\Omega)=\{v\in W^{m,p}(\Omega):v|_{\partial\Omega}=0\}$. For $p=2$, we denote $H^m(\Omega)=W^{m,2}(\Omega)$, $H_0^m(\Omega)=W_0^{m,2}(\Omega)$, and $\|\cdot\|_m=\|\cdot\|_{m,2}$, $\|\cdot\|=\|\cdot\|_{0,2}$. In addition, $C$ denotes a general positive constant independent of $h$, where $h$ is the spatial mesh-size for the control and state discretization.

## 2. MIXED METHODS FOR OPTIMAL CONTROL PROBLEMS

In this section, we shall construct our mixed finite element approximation scheme of the control problem (1.1)–(1.4).

Let

$$\boldsymbol{V} = (L^2(\Omega))^2 \text{ and } W = H_0^1(\Omega).$$

As in [8], for (1.3), we get the following mixed variational form:

$$\begin{aligned} -(\boldsymbol{p}, \nabla w) &= (f+u, w),\ \forall\, w \in W, \\ (A^{-1}\boldsymbol{p}, \boldsymbol{v}) + (\nabla y, \boldsymbol{v}) &= 0,\ \forall\, \boldsymbol{v} \in \boldsymbol{V}, \end{aligned}$$

where $(\cdot,\cdot)$ is the inner product of $L^2(\Omega)$.

Now, we recast (1.1)–(1.4) as the following weak form: find $(\boldsymbol{p}, y, u) \in \boldsymbol{V} \times W \times U_{ad}$ such that

$$\min_{u \in U_{ad}} \left\{ \frac{1}{2}\|\boldsymbol{p} - \boldsymbol{p}_d\|^2 + \frac{1}{2}\|y - y_d\|^2 + \frac{\nu}{2}\|u\|^2 \right\} \tag{2.1}$$

$$-(\boldsymbol{p}, \nabla w) = (f+u, w),\ \forall\, w \in W, \tag{2.2}$$

$$(A^{-1}\boldsymbol{p}, \boldsymbol{v}) + (\nabla y, \boldsymbol{v}) = 0,\ \forall\, \boldsymbol{v} \in \boldsymbol{V}. \tag{2.3}$$

Since the objective functional is convex, it then follows from [18] that the optimal control problem (2.1)–(2.3) has a unique solution $(\boldsymbol{p}, y, u)$, and that a triplet $(\boldsymbol{p}, y, u)$ is the solution of (2.1)–(2.3) if and only if there is a co-state $(\boldsymbol{q}, z) \in \boldsymbol{V} \times W$ such that $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ satisfies the following optimality conditions:

$$-(\boldsymbol{p}, \nabla w) = (f+u, w),\ \forall\, w \in W, \tag{2.4}$$

$$(A^{-1}\boldsymbol{p}, \boldsymbol{v}) + (\nabla y, \boldsymbol{v}) = 0,\ \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{2.5}$$

$$(\boldsymbol{q}, \nabla w) = (y - y_d, w),\ \forall\, w \in W, \tag{2.6}$$

$$(A^{-1}\boldsymbol{q}, \boldsymbol{v}) - (\nabla z, \boldsymbol{v}) = (\boldsymbol{p} - \boldsymbol{p}_d, \boldsymbol{v}),\ \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{2.7}$$

$$(\nu u + z, \tilde{u} - u) \geq 0,\ \forall\, \tilde{u} \in U_{ad}. \tag{2.8}$$

In [10], the expression of the control variable is given. Here, we adopt the same method to derive the following operator:

$$u = \frac{\max\{0, \bar{z}\} - z}{\nu}, \tag{2.9}$$

where $\bar{z} = \int_\Omega z / \int_\Omega 1$ denotes the integral average on $\Omega$ of the function $z$.

Let $\mathcal{T}_h$ denote a regular triangulation of the domain $\Omega$, $h_\tau$ denotes the diameter of $\tau$ and $h = \max h_\tau$. Let $\boldsymbol{V}_h \times W_h \subset \boldsymbol{V} \times W$ be defined by the following finite element pair $P_0^2$-$P_1$ [8, 31]:

$$\begin{aligned} \boldsymbol{V}_h &= \{\boldsymbol{v}_h = (\boldsymbol{v}_{1h}, \boldsymbol{v}_{2h}) \in \boldsymbol{V} | \boldsymbol{v}_{1h}, \boldsymbol{v}_{2h} \in P_0(\tau),\ \forall\, \tau \in \mathcal{T}_h\}, \\ W_h &= \{w_h \in C^0(\Omega) \cap W | w_h \in P_1(\tau),\ \forall\, \tau \in \mathcal{T}_h\}. \end{aligned}$$

And the approximated space of control is given by

$$U_h := \{\tilde{u}_h \in U_{ad} : \forall\, \tau \in \mathcal{T}_h,\ \tilde{u}_h|_\tau = \text{const}\}.$$

Before the new mixed finite element scheme is given, we introduce three projection operators. Firstly, we define the standard elliptic projection [3] $P_h:\ W \to W_h$, which satisfies: for any $\phi \in W$

$$(A\nabla(\phi - P_h\phi), \nabla w_h) = 0, \ \ \forall\, w_h \in W_h, \tag{2.10}$$

$$\|\phi - P_h\phi\|_s \le Ch^{2-s}\|\phi\|_2, \ \ \forall\, \phi \in H^s(\Omega), \ s = 0, 1. \tag{2.11}$$

Next, we define the standard $L^2$ projection $\Pi_h:\ \boldsymbol{V} \to \boldsymbol{V}_h$, which satisfies: for any $\boldsymbol{q} \in \boldsymbol{V}$

$$(\boldsymbol{q} - \Pi_h\boldsymbol{q}, \boldsymbol{v}_h) = 0, \ \ \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.12}$$

$$\|\Pi_h\boldsymbol{q}\| \le C\|\boldsymbol{q}\|, \tag{2.13}$$

$$\|\boldsymbol{q} - \Pi_h\boldsymbol{q}\| \le Ch\|\boldsymbol{q}\|_1, \ \ \forall\, \boldsymbol{q} \in (H^1(\Omega))^2. \tag{2.14}$$

At last, we define the standard $L^2$-orthogonal projection $Q_h : U_{ad} \to U_h$, which satisfies: for any $u \in U_{ad}$

$$(u - Q_h u, \tilde{u}_h) = 0, \ \ \forall\, \tilde{u}_h \in U_h. \tag{2.15}$$

We have the following approximation property:

$$\|u - Q_h u\|_{-s,r} \le Ch^{1+s}|u|_{1,r}, \ \ \forall\, u \in W^{1,r}(\Omega), \ s = 0, 1. \tag{2.16}$$

Then the new mixed finite element discretization of (2.1)–(2.3) is as follows: find $(\boldsymbol{p}_h, y_h, u_h) \in \boldsymbol{V}_h \times W_h \times U_h$ such that

$$\min_{u_h \in U_h} \left\{ \frac{1}{2}\|\boldsymbol{p}_h - \boldsymbol{p}_d\|^2 + \frac{1}{2}\|y_h - y_d\|^2 + \frac{\nu}{2}\|u_h\|^2 \right\} \tag{2.17}$$

$$-(\boldsymbol{p}_h, \nabla w_h) = (f + u_h, w_h), \ \forall\, w_h \in W_h, \tag{2.18}$$

$$(A^{-1}\boldsymbol{p}_h, \boldsymbol{v}_h) + (\nabla y_h, \boldsymbol{v}_h) = 0, \ \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{2.19}$$

As in the continuous case, the above optimal control problem has a unique solution $(\boldsymbol{p}_h, y_h, u_h)$, and a triplet $(\boldsymbol{p}_h, y_h, u_h)$ is the solution of (2.17)–(2.19) if and only if there is a co-state $(\boldsymbol{q}_h, z_h) \in \boldsymbol{V}_h \times W_h$ such that $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h)$ satisfies the following optimality conditions:

$$-(\boldsymbol{p}_h, \nabla w_h) = (f + u_h, w_h), \quad \forall\, w_h \in W_h, \tag{2.20}$$

$$(A^{-1}\boldsymbol{p}_h, \boldsymbol{v}_h) + (\nabla y_h, \boldsymbol{v}_h) = 0, \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.21}$$

$$(\boldsymbol{q}_h, \nabla w_h) = (y_h - y_d, w_h), \quad \forall\, w_h \in W_h, \tag{2.22}$$

$$(A^{-1}\boldsymbol{q}_h, \boldsymbol{v}_h) - (\nabla z_h, \boldsymbol{v}_h) = (\boldsymbol{p}_h - \boldsymbol{p}_d, \boldsymbol{v}_h), \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.23}$$

$$(\nu u_h + z_h, \tilde{u}_h - u_h) \ge 0, \quad \forall\, \tilde{u}_h \in U_h. \tag{2.24}$$

For the variational inequality (2.24), we have the following conclusion.

**Lemma 2.1** [10]. *Assume that $z_h$ is known in the variational inequality* (2.24). *The solution of the variational inequality* (2.24) *is*

$$u_h = Q_h\left(-\frac{z_h}{\nu} + \max\left\{0, \frac{\overline{z_h}}{\nu}\right\}\right), \ \ \overline{z_h} = \frac{\int_\Omega z_h}{\int_\Omega 1}.$$

In the rest of the paper, we shall use some intermediate variables, we define the state solution $(\boldsymbol{p}(u_h), y(u_h), \boldsymbol{q}(u_h), z(u_h) \in (\boldsymbol{V} \times W)^2$ that satisfies

$$-\big(\boldsymbol{p}(u_h), \nabla w\big) = (f + u_h, w), \quad \forall\, w \in W, \tag{2.25}$$

$$\big(A^{-1}\boldsymbol{p}(u_h), \boldsymbol{v}\big) + \big(\nabla y(u_h), \boldsymbol{v}\big) = 0, \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{2.26}$$

$$\big(\boldsymbol{q}(u_h), \nabla w\big) = \big(y(u_h) - y_d, w\big), \quad \forall\, w \in W, \tag{2.27}$$

$$\big(A^{-1}\boldsymbol{q}(u_h), \boldsymbol{v}\big) - \big(\nabla z(u_h), \boldsymbol{v}\big) = \big(\boldsymbol{p}(u_h) - \boldsymbol{p}_d, \boldsymbol{v}\big), \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}. \tag{2.28}$$

## 3. A POSTERIORI ERROR ESTIMATES

In this section, we will discuss the residual type a posteriori error estimates for the optimal control problems. In order to derive our estimators, we need the following three important lemmas.

**Lemma 3.1** [3]. *Let $\pi_h$ be the standard Lagrange interpolation operator. For $m = 0$ or $1$ and $q > \frac{1}{2}$,*

$$|v - \pi_h v|_{W^{m,q}(\Omega)} \leq Ch^{2-m}|v|_{W^{2,q}(\Omega)}. \tag{3.1}$$

**Lemma 3.2.** *Let $\hat{\pi}_h$ be the average interpolation operator defined in* [30]. *For $m = 0$ or $1$ and $1 \leq q \leq \infty$,*

$$|v - \hat{\pi}_h v|_{W^{m,q}(\tau)} \leq \sum_{\bar{\tau}' \cap \bar{\tau} \neq \emptyset} Ch_\tau^{l-m}|v|_{W^{1,q}(\tau')}, \ \ \forall\, v \in W^{1,q}(\Omega). \tag{3.2}$$

**Lemma 3.3** [17]. *For $v \in W^{1,q}(\Omega)$ and $1 \leq q < \infty$,*

$$\|v\|_{W^{m,q}(\partial\tau)} \leq C\left(h_\tau^{-\frac{1}{q}}\|v\|_{W^{0,q}(\tau)} + h_\tau^{l-\frac{1}{q}}|v|_{W^{1,q}(\tau)}\right). \tag{3.3}$$

Using the stability estimates, we have the following lemma.

**Lemma 3.4.** *Let $(\boldsymbol{p}, y, \boldsymbol{q}, z)$ and $(\boldsymbol{p}(u_h), y(u_h), \boldsymbol{q}(u_h), z(u_h))$ be the solutions of* (2.4)–(2.7) *and* (2.25)–(2.28)*, respectively. Then we have*

$$\|y - y(u_h)\| + \|\nabla(y - y(u_h))\| + \|\boldsymbol{p} - \boldsymbol{p}(u_h)\| \leq C\|u - u_h\|, \tag{3.4}$$

$$\|z - z(u_h)\| + \|\nabla(z - z(u_h))\| + \|\boldsymbol{q} - \boldsymbol{q}(u_h)\| \leq C\|u - u_h\|. \tag{3.5}$$

As in [10, Lemma 3.2], we can prove that

**Lemma 3.5.** *Let $u$ and $u_h$ be the solutions of* (2.4)–(2.8) *and* (2.20)–(2.24)*, respectively. Then we have*

$$\|u - u_h\|^2 \leq C\eta_0^2 + C\|z(u_h) - z_h\|^2, \tag{3.6}$$

*where*

$$\eta_0^2 = \sum_{\tau \in \mathcal{T}_h} \|z_h - Q_h z_h\|^2_{L^2(\tau)}.$$

Now, we shall derive our main results.

**Theorem 3.1.** *Let $(u, y, \boldsymbol{p}, z, \boldsymbol{q})$ and $(u_h, y_h, \boldsymbol{p}_h, z_h, \boldsymbol{q}_h)$ be the solutions of* (2.4)–(2.8) *and* (2.20)–(2.24)*, respectively. Then we have*

$$\|u-u_h\|^2+\|\nabla(y-y_h)\|^2+\|\boldsymbol{p}-\boldsymbol{p}_h\|^2+\|\nabla(z-z_h)\|^2+\|\boldsymbol{q}-\boldsymbol{q}_h\|^2\le C\sum_{i=0}^{2}\eta_i^2, \tag{3.7}$$

*where $\eta_0$ is defined in Lemma 3.5, and*

$$\eta_1^2=\sum_{\tau\in\mathcal{T}_h}h_\tau^2\|f+u_h\|^2_{L^2(\tau)}+\sum_{l\in\partial\mathcal{T}_h}\int_l h_l[\boldsymbol{p}_h\cdot\mathbf{n}]^2+\sum_{\tau\in\mathcal{T}_h}\|A^{-1}\boldsymbol{p}_h+\nabla y_h\|^2_{L^2(\tau)},$$

$$\eta_2^2=\sum_{\tau\in\mathcal{T}_h}h_\tau^2\|y_h-y_d\|^2_{L^2(\tau)}+\sum_{l\in\partial\mathcal{T}_h}\int_l h_l[\boldsymbol{q}_h\cdot\mathbf{n}]^2+\sum_{\tau\in\mathcal{T}_h}\|\boldsymbol{p}_h-\boldsymbol{p}_d-A^{-1}\boldsymbol{q}_h+\nabla z_h\|^2_{L^2(\tau)},$$

*where $l$ is an edge of an element $\tau$, $[\boldsymbol{v}_h\cdot\mathbf{n}]_l$ is the normal derivative jumps over the interior edge $l$, defined by*

$$[\boldsymbol{v}_h\cdot\mathbf{n}]_l=[\boldsymbol{v}_h|_{\tau_l^1}-\boldsymbol{v}_h|_{\tau_l^2}]\cdot\mathbf{n},$$

*where $\mathbf{n}$ is the unit normal vector on $l=\tau_l^1\cap\tau_l^2$ outwards $\tau_l^1$, $h_l$ is the maximum diameter of the edge $l$.*

*Proof.* For the sake of simplicity, let

$$e_y=y(u_h)-y_h,\ e_{\boldsymbol{p}}=\boldsymbol{p}(u_h)-\boldsymbol{p}_h,$$
$$e_z=z(u_h)-z_h,\ e_{\boldsymbol{q}}=\boldsymbol{q}(u_h)-\boldsymbol{q}_h.$$

From Eqs. (2.25)–(2.28) and (2.20)–(2.23), we can easily obtain the following error equations:

$$-(e_{\boldsymbol{p}},\nabla w_h)=0,\qquad\forall\, w_h\in W_h, \tag{3.8}$$
$$(A^{-1}e_{\boldsymbol{p}},\boldsymbol{v}_h)+(\nabla e_y,\boldsymbol{v}_h)=0,\qquad\forall\,\boldsymbol{v}_h\in\boldsymbol{V}_h, \tag{3.9}$$
$$(e_{\boldsymbol{q}},\nabla w_h)=(e_y,w_h),\qquad\forall\, w_h\in W_h, \tag{3.10}$$
$$(A^{-1}e_{\boldsymbol{q}},\boldsymbol{v}_h)-(\nabla e_z,\boldsymbol{v}_h)=(e_{\boldsymbol{p}},\boldsymbol{v}_h),\qquad\forall\,\boldsymbol{v}_h\in\boldsymbol{V}_h. \tag{3.11}$$

It follows from the assumption on $A$, (3.8), (3.9), (2.20), (2.21), (2.25), (2.26), (3.2), (3.3), and Cauchy inequality that

$$\begin{aligned}C\|e_{\boldsymbol{p}}\|^2&\le(A^{-1}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),\boldsymbol{p}(u_h)-\boldsymbol{p}_h)\\&=(A^{-1}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),\boldsymbol{p}(u_h))-(A^{-1}\boldsymbol{p}(u_h),\boldsymbol{p}_h)+(A^{-1}\boldsymbol{p}_h,\boldsymbol{p}_h)\\&=(\nabla y(u_h),\boldsymbol{p}(u_h))-(A^{-1}\boldsymbol{p}(u_h),\boldsymbol{p}_h)+(\nabla y(u_h),\boldsymbol{p}_h)-(\nabla y_h,\boldsymbol{p}(u_h))\\&=-(f+u_h,e_y)-(A^{-1}\boldsymbol{p}_h+\nabla y_h,e_{\boldsymbol{p}})-(\boldsymbol{p}_h,\nabla e_y)\\&=-(f+u_h,e_y-\hat{\pi}_h e_y)-(A^{-1}\boldsymbol{p}_h+\nabla y_h,e_{\boldsymbol{p}})-(\boldsymbol{p}_h,\nabla(e_y-\hat{\pi}_h e_y))\\&=-\sum_{\tau\in\mathcal{T}_h}\int_\tau(f+u_h)(e_y-\hat{\pi}_h e_y)-(A^{-1}\boldsymbol{p}_h+\nabla y_h,e_{\boldsymbol{p}})-\sum_{l\in\partial\mathcal{T}_h}\int_l[\boldsymbol{p}_h\cdot\mathbf{n}](e_y-\hat{\pi}_h e_y)\\&\le C\|\eta_1\|^2+\epsilon\|e_y\|_1^2+\frac{C}{2}\|e_{\boldsymbol{p}}\|^2.\end{aligned} \tag{3.12}$$

Moreover, using Poincare's inequality and (2.26), it easy to see that

$$\begin{aligned}\|e_y\|_1^2 \leq & C\|\nabla e_y\|^2 = C\| - A^{-1}e_{\boldsymbol{p}} - (A^{-1}\boldsymbol{p}_h + \nabla y_h)\|^2 \\ \leq & C\|A^{-1}\|_{0,\infty}\|e_{\boldsymbol{p}}\|^2 + C\|A^{-1}\boldsymbol{p}_h + \nabla y_h\|^2. \end{aligned} \tag{3.13}$$

For sufficiently small $\epsilon$, using (3.12) and (3.13), we have

$$\|e_y\|_1^2 + \|e_{\boldsymbol{p}}\|^2 \leq C\|\eta_1\|^2. \tag{3.14}$$

Similar to (3.12) and (3.13), we have

$$\begin{aligned} C\|e_{\boldsymbol{q}}\|^2 \leq & (A^{-1}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), \boldsymbol{q}(u_h) - \boldsymbol{q}_h) \\ = & \ (A^{-1}\boldsymbol{q}(u_h), \boldsymbol{q}(u_h)) - (A^{-1}\boldsymbol{q}_h, \boldsymbol{q}(u_h)) - (\nabla e_z, \boldsymbol{q}_h) - (e_{\boldsymbol{p}}, \boldsymbol{q}_h) \\ = & \ (y_h - y_d, e_z) - (\boldsymbol{q}_h, \nabla e_z) + (e_{\boldsymbol{p}}, e_{\boldsymbol{q}}) + (e_y, e_z) \\ & + (\boldsymbol{p}_h - \boldsymbol{p}_d + \nabla z_h - A^{-1}\boldsymbol{q}_h, e_{\boldsymbol{q}}) \\ = & \ (y_h - y_d, e_z - \hat{\pi}_h e_z) - (\boldsymbol{q}_h, \nabla(e_z - \hat{\pi}_h e_z)) + (e_{\boldsymbol{p}}, e_{\boldsymbol{q}}) + (e_y, e_z) \\ & + (\boldsymbol{p}_h - \boldsymbol{p}_d + \nabla z_h - A^{-1}\boldsymbol{q}_h, e_{\boldsymbol{q}}) \\ = & \ \sum_{\tau \in \mathcal{T}_h} \int_\tau (y_h - y_d)(e_z - \hat{\pi}_h e_z) - \sum_{l \in \partial\mathcal{T}_h} \int_l [\boldsymbol{q}_h \cdot \mathbf{n}](e_z - \hat{\pi}_h e_z) \\ & + (\boldsymbol{p}_h - \boldsymbol{p}_d + \nabla z_h - A^{-1}\boldsymbol{q}_h, e_{\boldsymbol{q}}) + (e_{\boldsymbol{p}}, e_{\boldsymbol{q}}) + (e_y, e_z) \\ \leq & \ C\|\eta_2\|^2 + \epsilon\|e_z\|_1^2 + \frac{C}{2}\|e_{\boldsymbol{q}}\|^2 + C\|e_{\boldsymbol{p}}\|^2 + C\|e_y\|^2 \end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\|e_z\|_1^2 \leq & C\|\nabla e_z\|^2 = C\|A^{-1}\boldsymbol{q}(u_h) - \boldsymbol{p}(u_h) + \boldsymbol{p}_d - \nabla z_h)\|^2 \\ \leq & C\|A^{-1}\|_{0,\infty}\|e_{\boldsymbol{q}}\|^2 + C\|\boldsymbol{p}_h - \boldsymbol{p}_d + \nabla z_h - A^{-1}\boldsymbol{q}_h\|^2 + C\|e_{\boldsymbol{p}}\|^2. \end{aligned} \tag{3.16}$$

For sufficiently small $\epsilon$, using (3.15) and (3.16), we have

$$\|e_z\|_1^2 + \|e_{\boldsymbol{q}}\|^2 \leq C(\|\eta_1\|^2 + \|e_y\|^2 + \|e_{\boldsymbol{p}}\|^2). \tag{3.17}$$

Now, combining (3.14), (3.17), Lemmas 3.4 and 3.5, we complete the proof. □

Next, we recall a result from Grisvard [12].

**Lemma 3.6** [12]. *For every function $F \in L^2(\Omega)$, the solution $\phi$ of*

$$-\mathrm{div}(A\nabla\phi) = F \quad \text{in } \Omega,\ \phi|_{\partial\Omega} = 0, \tag{3.18}$$

*belongs to $H_0^1(\Omega) \cap H^2(\Omega)$. Moreover, there exists a positive constant $C$ such that*

$$\|\phi\|_2 \leq C\|F\|. \tag{3.19}$$

**Theorem 3.2.** *Let $(u, y, \boldsymbol{p}, z, \boldsymbol{q})$ and $(u_h, y_h, \boldsymbol{p}_h, z_h, \boldsymbol{q}_h)$ be the solutions of (2.4)–(2.8) and (2.20)–(2.24), respectively. Then we have*

$$\|u - u_h\|^2 + \|y - y_h\|^2 + \|z - z_h\|^2 \leq C\left(\eta_0^2 + \sum_{i=1}^2 \hat{\eta}_i^2\right), \tag{3.20}$$

*where $\eta_0$ is defined in Lemma 3.5 and*

$$\hat{\eta}_1^2 = \sum_{\tau\in\mathcal{T}_h} h_\tau^4\|f+u_h\|^2_{L^2(\tau)} + \sum_{l\in\partial\mathcal{T}_h}\int_l h_l^3[\boldsymbol{p}_h\cdot\mathbf{n}]^2 + \sum_{\tau\in\mathcal{T}_h} h_\tau^2\|A^{-1}\boldsymbol{p}_h+\nabla y_h\|^2_{L^2(\tau)},$$

$$\hat{\eta}_2^2 = \sum_{\tau\in\mathcal{T}_h} h_\tau^4\|y_h-y_d\|^2_{L^2(\tau)} + \sum_{l\in\partial\mathcal{T}_h}\int_l h_l^3[\boldsymbol{q}_h\cdot\mathbf{n}]^2 + \sum_{\tau\in\mathcal{T}_h} h_\tau^2\|\boldsymbol{p}_h-\boldsymbol{p}_d-A^{-1}\boldsymbol{q}_h+\nabla z_h\|^2_{L^2(\tau)}.$$

*Proof.* First, let $\phi\in H^2(\Omega)\cap H_0^1(\Omega)$ be the solution of (3.18) with $F=y(u_h)-y_h$. We can see that

$$\begin{aligned}\|e_y\|^2 &= (A\nabla\phi,\nabla e_y)\\ &= (A\nabla\phi,\nabla y(u_h)) - (A\nabla\phi,\nabla y_h+A^{-1}\boldsymbol{p}_h) + (\boldsymbol{p}_h,\nabla\phi)\\ &= (\nabla y_h+A^{-1}\boldsymbol{p}_h,\Pi_h(A\nabla\phi)-A\nabla\phi) + (\boldsymbol{p}_h,\nabla\phi) + (f+u_h,\phi)\\ &= (\nabla y_h+A^{-1}\boldsymbol{p}_h,\Pi_h(A\nabla\phi)-A\nabla\phi) + (\boldsymbol{p}_h,\nabla(\phi-\pi_h\phi)) + (f+u_h,\phi-\pi_h\phi)\\ &\le C\|\phi\|_2\hat{\eta}_1, \end{aligned}\tag{3.21}$$

where we used (3.8), (3.9), (2.20), (2.21), (2.25), (2.26), (3.1), (3.3), (2.14), and Cauchy inequality.

Second, let $\phi\in H^2(\Omega)\cap H_0^1(\Omega)$ be the solution of (3.18) with $F=z(u_h)-z_h$. Similar to (3.21), we have

$$\begin{aligned}\|e_z\|^2 &= (A\nabla\phi,\nabla e_z)\\ &= (\boldsymbol{p}_h-\boldsymbol{p}_d-A^{-1}\boldsymbol{q}_h+\nabla z_h,\Pi_h(A\nabla\phi)-A\nabla\phi) - (\boldsymbol{q}_h,\nabla\phi)\\ &\quad + (y_h-y_d,\phi) + (e_y,\phi) - (e_{\boldsymbol{p}},A\nabla\phi)\\ &= (y_h-y_d,\phi-\pi_h\phi) - (\boldsymbol{q}_h,\nabla(\phi-\pi_h\phi)) - (\nabla y_h+A^{-1}\boldsymbol{p}_h,\Pi_h(A^2\nabla\phi)-A^2\nabla\phi)\\ &\quad + (e_y,\phi) + (e_y,\operatorname{div}(A^2\nabla\phi)) + (\boldsymbol{p}_h-\boldsymbol{p}_d-A^{-1}\boldsymbol{q}_h+\nabla z_h,\Pi_h(A\nabla\phi)-A\nabla\phi)\\ &\le C\|\phi\|_2(\hat{\eta}_1+\hat{\eta}_2+\|e_y\|). \end{aligned}\tag{3.22}$$

Using (3.19), (3.21), (3.22), and Lemmas 3.4 and 3.5, we complete the proof of the theorem. □

## 4. CONCLUSIONS

In this paper, we discussed a posteriori error estimates of a new mixed finite element method for a linear elliptic optimal control problem (1.1)–(1.4). Notice that the gradient of the primal variable for this method belongs to the square integrable space instead of the classical $H(\text{div};\Omega)$ space. Using this method, we can derive two approximations for the gradient of the primal variable $y$, one is the numerical approximation solution $\boldsymbol{p}_h$, the other is the derivative of the approximation solution $y_h$. Our a posteriori error estimates for linear elliptic optimal control problems by the mixed finite element method seem to be new. In our future work, we will investigate a priori and posteriori error estimates for parabolic optimal control problems.

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## REFERENCES


1. Bonnans, J.F. and Casas, E., An Extension of Pontryagin's Principle for State-Constrained Optimal Control of Semilinear Elliptic Equations and Variational Inequalities, *SIAM J. Contr. Optim.*, 1995, vol. 33, pp. 274–298.
2. Brunner, H. and Yan, N., Finite Element Methods for Optimal Control Problems Governed by Integral Equations and Integro-Differential Equations, *Num. Math.*, 2005, vol. 101, pp. 1–27.
3. Ciarlet, P.G., *The Finite Element Method for Elliptic Problems*, Amsterdam: North-Holland, 1978.
4. Chen, Y., Superconvergence of Mixed Finite Element Methods for Optimal Control Problems, *Math. Comp.*, 2008, vol. 77, pp. 1269–1291.
5. Chen, Y., Superconvergence of Quadratic Optimal Control Problems by Triangular Mixed Finite Element Methods, *Int. J. Num. Meth. Eng.*, 2008, vol. 75, no. 8, pp. 881–898.
6. Chen, Y. and Dai, Y., Superconvergence for Optimal Control Problems Governed by Semi-Linear Elliptic Equations, *J. Sci. Comput.*, 2009, vol. 39, pp. 206–221.
7. Chen, Y., Huang, Y., Liu, W.B., and Yan, N., Error Estimates and Superconvergence of Mixed Finite Element Methods for Convex Optimal Control Problems, *J. Sci. Comput.*, 2010, vol. 42, no. 3, pp. 382–403.
8. Chen, S.C. and Chen, H.R., New Mixed Element Schemes for a Second-Order Elliptic Problem, *Math. Num. Sinica*, 2010, vol. 32, no. 2, pp. 213–218.
9. Gunzburger, M.D. and Hou, L.S., Finite-Dimensional Approximation of a Class of Constrained Nonlinear Optimal Control Problems, *SIAM J. Contr. Optim.*, 1996, vol. 34, pp. 1001–1043.
10. Ge, L., Liu, W.B., and Yang, D.P., Adaptive Finite Element Approximation for a Constrained Optimal Control Problem via Multi-Meshes, *J. Sci. Comput.*, 2009, vol. 41, no. 2, pp. 238–255.
11. Guo, H., Fu, H., and Zhang, J., A Splitting Positive Definite Mixed Finite Element Method for Elliptic Optimal Control Problem, *Appl. Math. Comp.*, 2013, vol. 219, pp. 11178–11190.
12. Grisvard, P., *Elliptic Problems in Nonsmooth Domains*, London: Pitman, 1985.
13. Gong, W. and Yan, N., Adaptive Finite Element Method for Elliptic Optimal Control Problems: Convergence and Optimality, *Num. Math.*, 2017, vol. 135, no. 4, pp. 1121–1170.
14. Hou, T. and Chen, Y., Mixed Discontinuous Galerkin Time-Stepping Method for Linear Parabolic Optimal Control Problems, *J. Comput. Math.*, 2015, vol. 33, no. 2, pp. 158–178.
15. Hou, L. and Turner, J.C., Analysis and Finite Element Approximation of an Optimal Control Problem in Electrochemistry with Current Density Controls, *Num. Math.*, 1995, vol. 71, pp. 289–315.
16. Knowles, G., Finite Element Approximation of Parabolic Time Optimal Control Problems, *SIAM J. Contr. Optim.*, 1982, vol. 20, pp. 414–427.
17. Kufner, A., John, O., and Fucik, S., *Function Spaces*, Leiden: Nordhoff, 1977.
18. Lions, J.L., *Optimal Control of Systems Governed by Partial Differential Equations*, Berlin: Springer-Verlag, 1971.
19. Li, R., Liu, W.B., Ma, H., and Tang, T., Adaptive Finite Element Approximation of Elliptic Optimal Control Problems, *SIAM J. Contr. Optim.*, 2002, vol. 41, pp. 1321–1349.
20. Li, R., Liu, W.B., and Yan, N., A Posteriori Error Estimates of Recovery Type for Distributed Convex Optimal Control Problems, *J. Sci. Comput.*, 2002, vol. 41, no. 5, pp. 1321–1349.
21. Liu, W., Ma, H., Tang, T., and Yan, N., A Posteriori Error Estimates for Discontinuous Galerkin Time-Stepping Method for Optimal Control Problems Governed by Parabolic Equations, *SIAM J. Num. An.*, 2004, vol. 42, pp. 1032–1061.
22. Liu, W. and Yan, N., A Posteriori Error Estimates for Convex Boundary Control Problems, *SIAM J. Num. An.*, 2001, vol. 39, pp. 73–99.
23. Liu, W. and Yan, N., A Posteriori Error Estimates for Control Problems Governed by Stokes Equations, *SIAM J. Num. An.*, 2003, vol. 40, pp. 1850–1869.
24. Liu, W. and Yan, N., A Posteriori Error Estimates for Optimal Control Problems Governed by Parabolic Equations, *Num. Math.*, 2003, vol. 93, pp. 497–521.
25. Meyer, C. and Rösch, A., Superconvergence Properties of Optimal Control Problems, *SIAM J. Contr. Optim.*, 2004, vol. 43, no. 3, pp. 970–985.
26. Meyer, C. and Rösch, A., $L^\infty$-Estimates for Approximated Optimal Control Problems, *SIAM J. Contr. Optim.*, 2005, vol. 44, no. 5, pp. 1636–1649.
27. Meidner, D. and Vexler, B., A Priori Error Estimates for Space-Time Finite Element Discretization of Parabolic Optimal Control Problems. Part I: Problems without Control Constraints, *SIAM J. Contr. Optim.*, 2008, vol. 47, no. 3, pp. 1150–1177.
28. Meidner, D. and Vexler, B., A Priori Error Estimates for Space-Time Finite Element Discretization of Parabolic Optimal Control Problems. Part II: Problems with Control Constraints, *SIAM J. Contr. Optim.*, 2008, vol. 47, no. 3, pp. 1301–1329.

29. McKnight, R.S. and Bosarge, W.E., The Ritz–Galerkin Procedure for Parabolic Control Problems, *SIAM J. Contr. Optim.*, 1973, vol. 11, no. 3, pp. 510–542.
30. Scott, L.R. and Zhang, S., Finite Element Interpolation of Nonsmooth Functions Satisfying Boundary Conditions, *Math. Comp.*, 1990, vol. 54, pp. 483–493.
31. Shi, F., Yu, J.P., and Li, K.T., A New Stabilized Mixed Finite-Element Method for Poisson Equation Based on Two Local Gauss Integrations for Linear Element Pair, *Int. J. Comput. Math.*, 2011, vol. 88, pp. 2293–2305.
32. Shen, W., Ge, L., Yang, D., and Liu, W., Sharp a Posteriori Error Estimates for Optimal Control Governed by Parabolic Integro-Differential Equations, *J. Sci. Comput.*, 2015, vol. 65, no. 1, pp. 1–33.
33. Xiong, C. and Li, Y., A Posteriori Error Estimates for Optimal Distributed Control Governed by the Evolution Equations, *Appl. Num. Math.*, 2011, vol. 61, no. 2, pp. 181–200.