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Some Algebraic Approach for the Second Painlevé Equation Using the Optimal Homotopy Asymptotic Method (OHAM)

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Abstract—The study of Painlevé equations has increased during the last years, due to the awareness that these equations and their solutions can accomplish good results both in the field of pure mathematics and in theoretical physics. In this paper we introduced the optimal homotopy asymptotic method (OHAM) approach to propose analytic approximate solutions to the second Painlevé equation. The advantage of this method is that it provides a simple algebraic expression that can be used for further developments while maintaining good performance and fitting closely the numerical solution.

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1. INTRODUCTION

Painlevé equations [1, 2] were found by Painlevé and his colleagues from the consideration of problems for a class of nonlinear second-order differential equations. The main idea consisted in studying the singularities of the solutions to obtain a possible classification of them. Another important idea was the development and definition of new functions.

The study of Painlevé equations has seen a pronounced increase during the last years due to the awareness that these equations and their solutions can accomplish good results both in the field of pure mathematics and in theoretical physics. In general, it has been found that some of them appear naturally from reductions of soliton-type ordinary differential equations (ODEs) [3] from nonlinear partial differential equations (PDEs).

The first three of six Painlevé equations [4–7] are:

$$P_{\mathrm{I}} \to w(z) \ : \quad w'' = 6w^2 + z, \tag{1}$$

$$P_{\mathrm{II}} \to w(z,\alpha) \ : \quad w'' = 2w^3 + zw + \alpha, \tag{2}$$

$$P_{\mathrm{III}} \to w(z,\alpha,\beta,\gamma,\delta) \ : \quad w'' = w^{-1}w'^2 - z^{-1}w' + (\alpha w^2 + \beta)z^{-1} + \gamma w^3 + \delta w^{-1}, \tag{3}$$

where $w = w(z)$, α, β, γ, and δ are constants.

Painlevé equations have connections from various fields of theoretical physics. Particularly, P_I arises from a solution in terms of the traveled wave $z = x - ct$, $u(x,t) = y(z)$ in the Boussinesq equation $u_{tt} = u_{xx} - 6(u^2)_{xx} + u_{xxxx}$, where c is an arbitrary constant and $y(z)$ satisfies $y'' = 6y^2 + (c^2 - 1)y + Az + B$, for some values of the constants of integration A and B. P_{II} can be obtained from the Korteweg–de Vries (KdV) equation [8, 9] $f_t - 6ff_x + f_{xxx} = 0$, making the scaling reduction $z = x(3t)^{-1/3}$ and $f(x,t) = (3t)^{-2/3}(w' + w^2)$; and the modified Korteweg–de Vries (MKdV) equation

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$g_t - 6g^2 g_x + g_{xxx} = 0$, for a scaling reduction $z = x(3t)^{-1/3}$ and $g(x,t) = (3t)^{-1/3}w$, with α as an integration constant, respectively. Finally, the third Painlevé equation arises from sine-Gordon equation $u_{xt} = \sin u$, with the scaling reduction $z = xt$, $u(x,t) = v(z)$, and then $w(z) = exp(-iv)$ satisfies $P_{\rm III}$ with $\alpha = -\beta = 1/2$ and $\gamma = \delta = 0$. Moreover, $P_{\rm II}$ appears in statistical physics in the well-known Tracy–Widom distribution [10] for the probability distribution of the normalized largest eigenvalue of a random Hermitian matrix, in electrostatic theory [11, 12], and as a solution of a radiating particle in Landau–Lifshitz theory [13].

Except $P_{\rm I}$, a Bäcklund transformation relates a Painlevé transcendent of one type either to another of same type but with different parameter values, or to another type. This transformation looks like[1] [4, 14]:

$$-w(z, \alpha \pm 1) = w(z, \alpha) + \frac{2\alpha \pm 1}{2w(z,\alpha)^2 \pm 2w'(z,\alpha) + z}. \tag{4}$$

Here it is obvious that $\alpha \neq \pm\frac{1}{2}$ for the validity of (4), for $\alpha = \pm\frac{1}{2}$ the solutions for different values of α coincide, which is not generally true, so the last equation provides a recursive relationship to obtain several solutions from a first one.

For Eq. (2), there are many known solutions. An example of this is a variety of rational type solutions [15]. Rational solutions of (2) exist for $\alpha = n(\in Z)$ and are generated using the seed solution $w(z, 0) = 0$ and the Bäcklund transformations (4). The task of finding this is much simpler by considering the solution of (2) for $\alpha = 0$ and then using the recurrence relation in (4) to find all the others. In addition to the property of symmetry with respect to α, it is only necessary to consider the case for $\alpha > 0$, thus starting with the seed solution $w(z, 0) = 0$, then the first three solutions are

$$w(z,1) = -\frac{1}{z}, \;\; w(z,2) = \frac{4 - 2z^3}{z(4 + z^3)}, \;\; w(z,3) = \frac{3z^2(160 + 8z^3 + z^6)}{320 - 24z^6 - z^9}. \tag{5}$$

In this case all subsequent solutions are of the order $w(z,n) \sim \mathcal{O}(z^{-1})$.

Other solutions have been explored with a variety of methods given solutions in terms of special functions and asymptotic approximations for real variable (see [16] and references therein). In [17] the author used the method of analytic continuation to find numerical solutions for problem (2). Recently, Dehghan and Shakeri solved problem (2) by means of the Adomian decomposition method (ADM), homotopy perturbation method (HPM), and Legendre tau method (LTM) [18]. Likewise, very recently, the authors of [19] solved this problem using the homotopy analysis method (HAM). Also, the solution of the second Painlevé equation is presented by means of two known techniques in reference [20], sinc-collocation method and variational iteration method (VIM). The application of the perturbation method has limitations relative to the choice of a small parameter to be used, so that in some cases its development for some applications may not be convenient and not directly applicable. This can be corrected with HPM or HAM, however, a small parameter should be considered. The substantial difference with the optimal homotopy asymptotic method (OHAM) [21–25] is that this difficulty is solved from the principle without the need of incorporating any parameter, resulting in a powerful method to solve nonlinear problems. This paper is devoted to studying a class of algebraic-like solutions of $P_{\rm II}$, for the equation written as

$$u(t,\mu): \;\; u'' = 2u^3 + tu + \mu, \;\; u(0) = 1, \;\; u'(0) = 0. \tag{6}$$

In this study, second Painlevé equation (6) is solved with OHAM. For the numerical solution, it is compared with the fourth-order Runge–Kutta method and also other solving methods. This method has been used to solve the first Painlevé equation obtaining very good approximations to the numerical solution [26].

[1] See: `http://dlmf.nist.gov/32.7`.

2. BASIC IDEAS OF OHAM

Consider the following general differential equation:

$$L[u(t)] + g(t) + N[u(t)] = 0, \tag{7}$$

that satisfies the initial/boundary conditions

$$B\left[u(t), \frac{du(t)}{dt}\right] = B\left[u(t), u'(t)\right] = 0, \tag{8}$$

where t denotes the independent variable, $u(t)$ is a function to solve, $g(t)$ is a given function, L, N, and B are linear, nonlinear and boundary operators, respectively.

Applying OHAM to the given problem, a general deformation (homotopy) equation is presented as:

$$(1-\epsilon)\left(L\left[H(t,\epsilon)\right] + g(t)\right) = h(\epsilon)\left[L\left[H(t,\epsilon)\right] + g(t) + N\left[H(t,\epsilon)\right]\right], \tag{9}$$

and

$$B\left[H(t,\epsilon), \frac{\partial H(t,\epsilon)}{\partial t}\right] = 0, \tag{10}$$

where $\epsilon \in [0,1]$ is an embedding parameter, $h(\epsilon)$ is a nonzero auxiliary function for $\epsilon \neq 0$ and $h(0) = 0$, $H(t,\epsilon)$ is an unknown function. Clearly, when $\epsilon = 0$ and $\epsilon = 1$, it holds $H(t,0) = u_0(t)$ and $H(t,1) = u(t)$, respectively.

Thus, as ϵ changes from 0 to 1, the solution $H(t,\epsilon)$ changes from $u_0(t)$ to the solution $u(t)$, where $u_0(t)$ is obtained from Eq. (9) for $\epsilon = 0$:

$$L\left[u_0(t)\right] + g(t) = 0, \quad B\left[u_0(t), u_0'(t)\right] = 0. \tag{11}$$

Now, we propose the auxiliary function $h(\epsilon)$ to be of the form:

$$h(\epsilon) = \epsilon K_1 + \epsilon^2 K_2 + \epsilon^3 K_3 + \cdots + \epsilon^m K_m = \sum_{i=1}^{m} \epsilon^i K_i, \tag{12}$$

where K_i are constants. For actual applications K_i, are finite, say, $i = 1, 2, 3, \ldots, m$.

Expanding $H(t,\epsilon)$ into a Taylor's series about ϵ, we obtain:

$$H(t,\epsilon) = u_0(t) + \sum_{i=1}^{\infty} u_n(t, K_i)\epsilon^n. \tag{13}$$

Substituting (13) into (9), and equating the coefficient of like powers of ϵ, we obtain that the zero-order problem is given by (11), while the first- and second-order problems are given by

$$L\left[u_1(t)\right] = K_1 N_0\left[u_0(t)\right], \quad B\left[u_1(t), u_1'(t)\right] = 0, \tag{14}$$

$$L\left[u_2(t)\right] - (1+K_1)L\left[u_1(t)\right] = K_2 N_0\left[u_0(t)\right] + K_1 N_1\left[u_0(t), u_0(t)\right], \quad B\left[u_2(t), u_2'(t)\right] = 0. \tag{15}$$

It is then possible to write

$$L\left[u_n(t)\right] - L\left[u_{n-1}(t)\right] = K_n N_0\left[u_0(t)\right] + \sum_{i=1}^{n-1} K_i\left[L\left[u_{n-i}(t)\right] + N_{n-i}\left[u_0(t), u_1(t), \ldots, u_{n-1}(t)\right]\right], \tag{16}$$

$$B\left[u_n(t), u_n'(t)\right] = 0. \tag{17}$$

In the last equation $N_m\left[u_0(t), u_1(t), \ldots, u_{n-1}(t)\right]$ is the coefficient of ϵ^m in the expansion of $N\left[H(t,\epsilon)\right]$:

$$N\left[H(t,\epsilon,K_i)\right] = N_0\left[u_0(t)\right] + \sum_{m=1}^{\infty} N_m[u_0(t), u_1(t), \ldots, u_m(t)]\epsilon^m. \tag{18}$$

Here, convergence of series (13) depends upon the constants K_i, $i = 1, 2, 3, \ldots$.

When $\epsilon = 1$, Eq. (13) can be written as

$$\bar{u}(t, K_m) = u_0(t) + \sum_{i=1}^{n} \bar{u}_i(t, K_m), \tag{19}$$

and the sum converges, because in practical applications n is finite in order to find an approximate solution. Substituting (19) into (9), we obtain the residual:

$$R(t, K_m) = L[\bar{u}(t, K_m)] + g(t) + N[\bar{u}(t, K_m)]. \tag{20}$$

If $R = 0$, then $\bar{u}$ yields the exact solution. However, this does not happen in general, especially when dealing with nonlinear problems.

In order to determine K_i, there are various methods like the Ritz method, Galerin's method, and collocation method, or the method of least squares,

$$J(t, K_m) = \int_a^b R^2(t, K_m)dt, \tag{21}$$

with the residual $R = L[\bar{u}] + g(t) + N[\bar{u}]$, and

$$\frac{\partial J(t, K_m)}{\partial K_i} = 0, \tag{22}$$

with a and b properly chosen to locate the desired K_i. Knowing these constants, the approximate solution (of order m) is well defined.

3. APPROXIMATE SOLUTION OF THE SECOND PAINLEVÉ EQUATION USING OHAM

Here we develop a solution for Eq. (6) using OHAM. First we note that in this case we can make the identification

$$L[A] := \frac{d^2}{dt^2}A, \quad g(t) := 0, \quad N[A] := -2A^3 - tA - \mu. \tag{23}$$

The zero order of approximation is given by

$$u_0''(t) = 0, \quad u_0(0) = 1, \quad u_0'(0) = 1, \tag{24}$$

with solution $u_0(t) = 1$. For the first-order problem, we obtain

$$u_1''(t) + K_1[(\mu + 2) + t] = 0, \quad u_1(0) = 0, \quad u_1'(0) = 0, \tag{25}$$

with solution $u_1(t) = -\frac{K_1}{6}t^2[3(\mu + 2) + t]$.

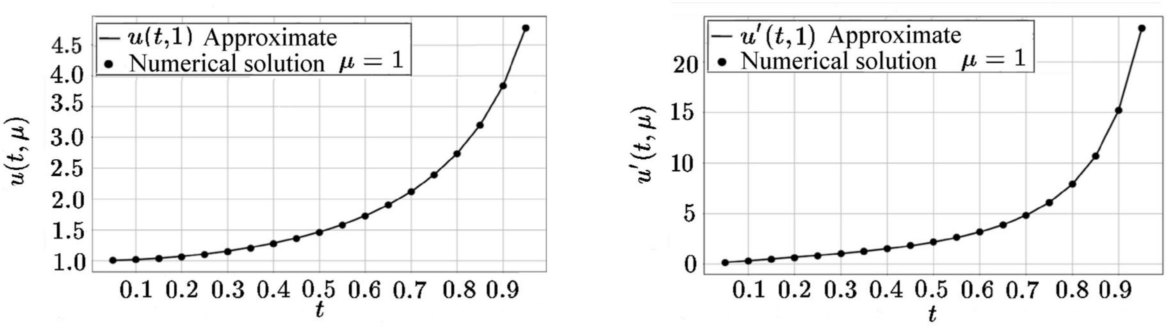

Fig. 1. A plot of the third-order approximation of $u(t,1)$ (left) and $u'(t,1)$ (right) obtained by OAHM for $\mu=1$, compared with numerical solution.

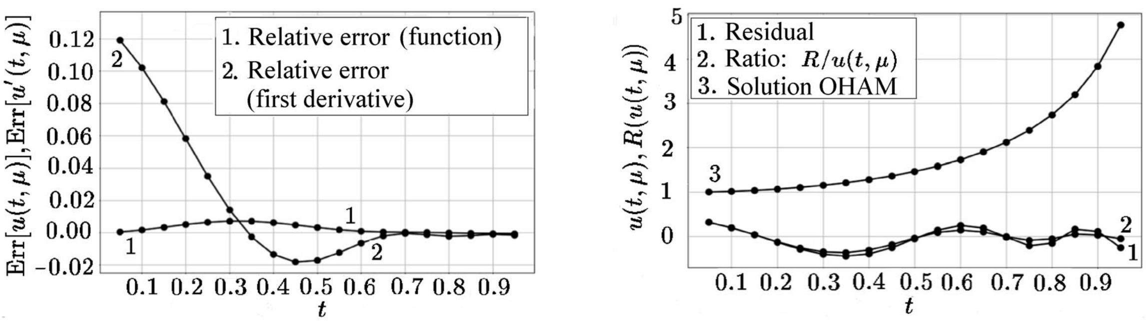

Fig. 2. A plot of relative error for $u(t,1)$ and $u'(t,1)$ with respect to numerical solution $\mu=1$ (left). Residual estimation for the approximate solution with OHAM for Eq. (26) (right).

For the second order of approximation, we can see that $u_2(t)$ is a higher-order polynomial function. Actually, $u_2(t)$ is of the form $u_2(t) = Q_2(t) = \sum_{n=2}^{11} b_n t^n$, with the coefficients b_n expressed in terms of K_1 and K_2, provided by (14) and (15). Higher orders of approximation are given by functions with a higher order of powers in t.

Using this method, after a convenient redefinition of the unknown coefficients K_i for the third order of approximation, solutions of the problem (6) are calculated for the case μ=1, and the numerical results are reported in Tables 1 and 2, and in Figs. 1 and 2. These results are then compared with the results given in [20] obtained by the sinc collocation and variational iteration method (VIM):

$$u(t,\mu=1) = 1 + a_1t^2 + a_2t^6 + a_3t^{10} + a_4t^{14} + a_5t^{20} + a_6t^{28} + a_7t^{30}, \tag{26}$$

with parameters $a_1 = 1.697070917$, $a_2 = 2.609390039$, $a_3 = -1.256388359$, $a_4 = 2.426875319$, $a_5 = -0.4717678848$, $a_6 = -0.1347675836$, and $a_7 = 0.4351224665$.

Figure 1 presents the approximate solution (left) and first derivative (right) of approximate solution with OHAM method in comparison with numerical solution for $u(0) = 1$, $u'(0) = 0$. Our solution is in good according with these. Figure 2 shows the relative error with respect to the numerical solution (left), and in the right side we shown an estimation of residual solution due to (20), we can see that this residual solution of OHAM solution does not exceed 6.5×10^{-2} for $0.05 < t < 0.95$. For large $t > 0.5$ the mean value of this residual is equal to 8.9×10^{-3} in concordance with Tables 1 and 2.

4. CONCLUSIONS

In this paper we introduced the OHAM approach to propose analytic approximate solutions to the second Painlevé equation. The procedure is valid even if the nonlinear equation does not contain small

Table 1. Comparison of the values of $u(t,\mu)$ and $u'(t,\mu)$ by different methods at $\mu = 1$

t	Approximation			First derivative		
	Sinc-collocation	VIM	OAHM	Sinc-collocation	VIM	OAHM
0.05	1.003775662	1.003775569	1.00424272	0.151646889	0.151630056	0.16971198
0.10	1.015243802	1.015243537	1.01697332	0.308080645	0.308099940	0.33957073
0.15	1.034708564	1.034708876	1.03821381	0.471990948	0.471982114	0.51030970
0.20	1.062615730	1.062614651	1.06804971	0.646296100	0.646258916	0.68383199
0.25	1.099569958	1.099567603	1.1067028	0.834541894	0.834535606	0.86377743
0.30	1.146377520	1.146376034	1.15463133	1.041286365	1.041324413	1.05604558
0.35	1.204103691	1.204104479	1.21265429	1.272396475	1.272440758	1.26922961
0.40	1.274150163	1.274152278	1.28209418	1.535574210	1.535576514	1.51491186
0.45	1.358366629	1.358367333	1.36493065	1.841214574	1.841156613	1.80781229
0.50	1.459216534	1.459213319	1.46396018	2.203597241	2.203659640	2.16592221
0.55	1.580028132	1.580020743	1.58297065	2.643787352	2.643721101	2.61108408
0.60	1.725383228	1.725374098	1.7269769	3.191606484	3.191604952	3.17110846
0.65	1.901736804	1.901728548	1.90264234	3.893152468	3.893170320	3.88556235
0.70	2.118441811	2.118431139	2.11915382	4.820785615	4.820621693	4.81887202
0.75	2.389524420	2.389493077	2.39004689	6.093740305	6.092992360	6.08637240
0.80	2.736942571	2.736846427	2.73683163	7.919898095	7.917916630	7.90224947
0.85	3.197020966	3.196770263	3.19593384	10.68888596	10.68432635	10.66978262
0.90	3.834408328	3.833780746	3.83251635	15.20368342	15.19170680	15.18928243
0.95	4.776251311	4.774527172	4.77324886	23.34167691	23.30560345	23.30622457

Table 2. Relative error for $u(t,\mu)$ and $u'(t,\mu)$ with respect to numerical solution at $\mu = 1$ for $0.5 \le t \le 0.95$

t	Our method	Numerical solution	Error, %	Our method	Numerical solution	Error, %
0.50	1.46396018	1.45921345	0.32529381	2.16592221	2.16592221	−0.01712666
0.55	1.58297065	1.58002119	0.18667166	2.61108406	2.61108408	−0.01234942
0.60	1.7269769	1.72537551	0.09281355	3.17110751	3.17110846	−0.00643236
0.65	1.90264234	1.90173279	0.04782699	3.88554687	3.88556235	−0.00197782
0.70	2.11915382	2.11844343	0.03353355	4.81869967	4.81887202	−0.00041477
0.75	2.39004689	2.38952654	0.02177633	6.08486153	6.08637240	−0.00119519
0.80	2.73683163	2.73693549	−0.00379485	7.89117363	7.90224947	−0.00219606
0.85	3.19593384	3.19700418	−0.0334794	10.59948225	10.66978262	−0.00178345
0.90	3.83251635	3.83440022	−0.04913079	14.79386518	15.18928243	−0.00096346
0.95	4.77324886	4.77622801	−0.06237453	21.30160718	23.30622457	−0.00149028

(or large) parameters. The proposed construction of homotopy is different from other approaches in the presence of parameters a_n, which ensure a very rapid convergence of the solutions. In the range $0.05 \leq t \leq 0.95$, the approximate solution is very close to the value of the numerical solution, with errors beginning to grow (smoothly) in the lower range of the variable t. The average error is 2.53×10^{-3} and the maximum error does not exceed 7.20×10^{-3}. Moreover, the derivative for the approximate solution is quite close to the value of the numerical derivatives, i.e., $u'(t = 1) = 40.307$ compared with $u'_{\mathrm{Num}}(t = 1) = 40.378$. In this case the error in $x = 1$ does not exceed 1.7%.

The advantage of the method we have just shown resides in the fact that it provides a simple algebraic expression that can be used for further developments. This allows one to treat the problem as an analytical system and use its solution in a variety of concrete applications, while maintaining good performance and fitting closely the numerical solution.

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