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Numerical Solution of Second-Order One-Dimensional Hyperbolic Equation by Exponential B-Spline Collocation Method

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Abstract—In this paper, we propose a method based on collocation of exponential B-splines to obtain numerical solution of a nonlinear second-order one-dimensional hyperbolic equation subject to appropriate initial and Dirichlet boundary conditions. The method is a combination of B-spline collocation method in space and two-stage, second-order strong-stability-preserving Runge—Kutta method in time. The proposed method is shown to be unconditionally stable. The efficiency and accuracy of the method are successfully described by applying the method to a few test problems.

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1. INTRODUCTION

We consider the following nonlinear one-dimensional hyperbolic equation:

$$u_{tt} + 2\alpha u_t + \beta^2 u = u_{xx} + g(x,t) + f(u), \quad a < x < b, \quad t > 0, \tag{1.1}$$

subject to the initial conditions

$$u(x,0) = \phi_1(x), \quad u_t(x,0) = \phi_2(x), \quad a \le x \le b, \tag{1.2}$$

and the Dirichlet boundary conditions

$$u(a,t) = \psi_1(t), \quad u(b,t) = \psi_2(t), \quad t \ge 0, \tag{1.3}$$

where $\alpha > 0$ and $\beta \ge 0$ are constants. If $\alpha > 0$, $\beta > 0$, Eq. (1.1) is referred to as a telegraphic equation and $g(x,t)$ is an arbitrary external forcing function. However, for $\alpha > 0$, $\beta = 0$, it represents a damped wave equation. The numerical solution of damped wave equation is of great importance in wave phenomenon. For $f(u) = 0$, (1.1) represents linear second-order hyperbolic equation.

In the past few years, several methods [1—12] have been developed for solving second-order one-dimensional hyperbolic equations subject to initial and Dirichlet boundary conditions. In [1], an unconditionally stable explicit difference scheme is discussed for solving a telegraphic equation. Mohanty et al. [2—5] have given various finite difference methods for the solution of one-dimensional hyperbolic equations. In [6, 7], Mittal et al. have presented the differential quadrature method and the collocation method based on cubic B-spline basis functions for the solution of a telegraphic equation. In [8], the

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author uses collocation points and radial basis function. Parameters spline methods for the solution of a telegraphic equation are discussed in [11]. Dosti and Nazemi [9, 10] discussed a quartic B-spline collocation method and a cubic B-spline quasi-interpolation method for solving a linear telegraphic equation. In [13], Kharenko et al. proposed methods including collocation and least squares method to obtain a numerical solution of nonlinear second-order hyperbolic partial differential equations.

Until now, some types of spline were developed among which, major emphasis is given on the use of polynomial splines. We, in this paper, discuss an exponential B-spline collocation method. The exponential splines and exponential B-splines are defined to be more general splines and B-splines by McCartin [17, 18]. McCartin states that in some cases polynomial splines can and do produce spurious oscillations in the interpolant. For example, in combustion calculations it could produce an unrealistic detonation, or in computational aerodynamics it could result in the generation of a nonphysical shock wave. To overcome these difficulties, exponential splines were introduced [19]. The use of exponential splines is not very common in finding numerical solutions to differential equations. Very recently, Mohammadi [14] and Ersoy and Dag [15] used exponential B-splines to obtain the solution of convection-diffusion equations and Korteweg–de Vries equation, respectively.

In the present paper, Eq. (1.1) is first converted into a system of partial differential equations. Then, the collocation of exponential B-splines is used to approximate the spatial derivatives. The resulting system of ordinary differential equations is solved by using a well-known two-stage, second-order strong-stability-preserving Runge–Kutta method (SSPRK(2,2)) [16].

The organization of this paper is as follows. In Section 2, some details about the exponential B-spline collocation method are given. In Section 3, numerical method for solving (1.1) is discussed. The method is shown to be unconditionally stable in Section 4. In Section 5, numerical examples are given to illustrate the usefulness of the proposed method and finally, concluding remarks are given in Section 6.

2. EXPONENTIAL B-SPLINE COLLOCATION METHOD

We consider a set of knots $a = x_0 < x_1 < \ldots < x_{N-1} < x_N = b$ as a uniform partition of the solution domain $a \le x \le b$ with a spacing $h = x_l - x_{l-1} = \dfrac{b-a}{N}$ for $l = 1, 2, \ldots, N-1, N$. The exponential B-splines $B_l(x)$ at the above defined knots along with additional knots x_{-1} and x_{N+1} can be defined as:

$$B_l(x) = \begin{cases} a\Big((x_{l-2} - x) - \dfrac{1}{p}\,(\sin h(p(x_{l-2} - x)))\Big), & x \in [x_{l-2}, x_{l-1}), \\ b_1 + b_2(x_l - x) + b_3 \exp(p(x_l - x)) + b_4 \exp(-p(x_l - x)), & x \in [x_{l-1}, x_l), \\ b_1 + b_2(x - x_l) + b_3 \exp(p(x - x_l)) + b_4 \exp(-p(x - x_l)), & x \in [x_l, x_{l+1}), \\ a\Big((x - x_{l+2}) - \dfrac{1}{p}\,(\sin h(p(x - x_{l+2})))\Big), & x \in [x_{l+1}, x_{l+2}), \\ 0 & \text{otherwise}, \end{cases} \tag{2.1}$$

where

$$a = \frac{p}{2(phc - s)}, \quad b_1 = \frac{phc}{(phc - s)}, \quad b_2 = \frac{p}{2}\left[\frac{c(c-1) + s^2}{(phc - s)(1 - c)}\right],$$

$$b_3 = \frac{1}{4}\left[\frac{\exp(-ph)(1-c) + s(\exp(-ph) - 1)}{(phc - s)(1 - c)}\right], \quad b_4 = \frac{1}{4}\left[\frac{\exp(ph)(c-1) + s(\exp(ph) - 1)}{(phc - s)(1 - c)}\right],$$

$$s = \sin h(ph), \quad c = \cos h(ph),$$

where p is a free parameter. Existence of the parameter p yields different shapes of the spline functions. The set of functions $\{B_{-1}, B_0, B_1, \ldots, B_{N-1}, B_N, B_{N+1}\}$ forms basis for the functions defined over the domain $[a, b]$. Additional knots outside the problem domain are necessary to define all the exponential splines.

Table 1. Values of exponential B-spline and its derivatives

x	x_{l-2}	x_{l-1}	x_l	x_{l+1}	x_{l+2}
$B_l(x)$	0	$\dfrac{s-ph}{2(phc-s)}$	1	$\dfrac{s-ph}{2(phc-s)}$	0
$B_{x_l}(x)$	0	$\dfrac{p(c-1)}{2(phc-s)}$	0	$-\dfrac{p(c-1)}{2(phc-s)}$	0
$B_{xx_l}(x)$	0	$\dfrac{p^2s}{2(phc-s)}$	$-\dfrac{p^2s}{(phc-s)}$	$\dfrac{p^2s}{2(phc-s)}$	0

An approximate solution $U(x,t)$ to the analytical solution $u(x,t)$, using exponential B-spline collocation method, can be written as

$$U(x,t)=\sum_{l=-1}^{l=N+1} c_l(t)B_l(x), \tag{2.2}$$

where $c_l(t)$ are time-dependent parameters to be determined from boundary conditions and the collocation method. The first and second spatial derivatives can be written as

$$U_x(x,t)=\sum_{l=-1}^{l=N+1} c_l(t)B_{xx_l}(x), \tag{2.3}$$

$$U_{xx}(x,t)=\sum_{l=-1}^{l=N+1} c_l(t)B_{xx_l}(x). \tag{2.4}$$

The values of $B_l(x)$ and its first and second derivatives at various knots are tabulated in Table 1.

Using Eqs. (2.2)–(2.4) and Table 1, we obtain approximate values of $U(x,t)$ and its spatial derivatives in terms of the time parameters c_l as

$$\left.\begin{aligned}
U(x_l,\cdot)&=m_1c_{l-1}+c_l+m_1c_{l+1},\\
U_x(x_l,\cdot)&=m_2(c_{l+1}-c_{l-1}),\\
U_{xx}(x_l,\cdot)&=m_3(c_{l-1}-2c_l+c_{l+1}),
\end{aligned}\right\} \tag{2.5}$$

where

$$m_1=\frac{s-ph}{2(phc-s)},\quad m_2=\frac{p(c-1)}{2(phc-s)},\quad m_3=\frac{p^2s}{2(phc-s)}.$$

3. NUMERICAL METHOD

Equation (1.1) is equivalent to the following system of equations:

$$\left.\begin{aligned}
u_t&=v,\\
v_t&=u_{xx}-2\alpha v-\beta^2u+g+f(u).
\end{aligned}\right\} \tag{3.1}$$

By using (2.2) the approximate value of $U_t(x,t)$ can be written as follows:

$$U_t(x,t) = \sum_{l=-1}^{l=N+1} \dot{c}_l(t) B_l(x), \tag{3.2}$$

where $\dot{c}_l(t)$ is the derivative of $c_l(t)$ with respect to t.

Using basis functions (2.1) and Table 1 in (3.2), we get the values of $U_t(x,t)$ as

$$U_t(x_l,t) = m_1\dot{c}_{l-1} + \dot{c}_l + m_1\dot{c}_{l+1}, \quad l = 0,1,\dots,N, \tag{3.3}$$

and

$$\begin{gathered}\dot{v}_l = \sum_{i=-1}^{N+1} c_i(t) B_{xx_i}(x_l) - 2\alpha v_l - \beta^2 \sum_{i=-1}^{N+1} c_i(t) B_i(x_l) + g_l \\ +f\left(\sum_{i=-1}^{N+1} c_i(t) B_i(x_l)\right), \quad l = 0,1,\dots,N,\end{gathered} \tag{3.4}$$

where v_l denotes $v(x_l,t)$ for $l = 0,1\dots,N$. Finally, using Eqs. (3.3) and (3.4) and Table 1, we get

$$m_1\dot{c}_{l-1} + \dot{c}_l + m_1\dot{c}_{l+1} = v_l, \quad l = 0,1,\dots,N, \tag{3.5}$$

$$\begin{aligned}\dot{v}_l &= m_3(c_{l-1} - 2c_l + c_{l+1}) - 2\alpha v_l - \beta^2(m_1c_{l-1} + c_l + m_1c_{l+1}) + g_l \\ &+f(m_1c_{l-1} + c_l + m_1c_{l+1}), \quad l = 0,1,\dots,N.\end{aligned} \tag{3.6}$$

These are $2(N+1)$ equations in $2(N+3)$ unknowns. To eliminate extra unknowns, we make use of boundary conditions

$$U(x_0,t) = \psi_1(t), \quad U(x_N,t) = \psi_2(t),$$

and (2.5) to obtain

$$c_{-1} = \frac{\psi_1 - c_0 - m_1c_1}{m_1}, \tag{3.7}$$

$$c_{N+1} = \frac{\psi_2 - c_N - m_1c_{N-1}}{m_1}. \tag{3.8}$$

Eliminating c_{-1} and c_{N+1} from Eqs. (3.6)–(3.8) for $l = 0, N$, we obtain

$$c_0 = \frac{m_1}{m_3(1+2m_1)}\left(\left(\frac{m_3}{m_1} - \beta^2\right)\psi_1 - 2\alpha\dot{\psi}_1 - \ddot{\psi}_1 + g_0 + f(\psi_1)\right) = w_0 \tag{3.9}$$

and

$$c_N = \frac{m_1}{m_3(1+2m_1)}\left(\left(\frac{m_3}{m_1} - \beta_N^2\right)\psi_2 - 2\alpha\dot{\psi}_2 - \ddot{\psi}_2 + g_N + f(\psi_2)\right) = w_N. \tag{3.10}$$

Hence, the problem now reduces to solving

$$\boldsymbol{A}\dot{\boldsymbol{c}} = \boldsymbol{F}, \tag{3.11}$$

where,

$$\boldsymbol{A} = \begin{bmatrix} 1 & m_1 & \dots & \dots & 0 \\ m_1 & 1 & m_1 & \dots & 0 \\ & \ddots & \ddots & \ddots & \\ & & m_1 & 1 & m_1 \\ 0 & \dots & \dots & m_1 & 1 \end{bmatrix}, \quad \dot{\boldsymbol{c}} = \begin{bmatrix} \dot{c}_1 \\ \dot{c}_2 \\ \vdots \\ \dot{c}_{N-2} \\ \dot{c}_{N-1} \end{bmatrix}, \quad \boldsymbol{F} = \begin{bmatrix} v_1 - m_1\dot{w}_0 \\ v_2 \\ \vdots \\ v_{N-2} \\ v_{N-1} - m_1\dot{w}_N \end{bmatrix},$$

and

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \vdots \\ \dot{v}_{N-2} \\ \dot{v}_{N-1} \end{bmatrix} = \begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_{N-2} \\ G_{N-1} \end{bmatrix}, \tag{3.12}$$

where,

$$\begin{aligned} G_l = {} & m_3(c_{l-1} - 2c_l + c_{l+1}) - 2\alpha v_l - \beta^2(m_1c_{l-1} + c_l + m_1c_{l+1}) + g_l \\ & + f(m_1c_{l-1} + c_l + m_1c_{l+1}), \quad l = 1, 2, \dots, N-1. \end{aligned}$$

Vector $\dot{\boldsymbol{c}}$ is computed by using tri-diagonal solver at each time level to obtain a system of N first-order ordinary differential equations. Then, these equations along with the equations in (3.12) are solved by using an optimal two-stage, second-order SSPRK(2,2) method. c_0, c_N and hence c_{-1}, c_{N+1} are obtained from (3.7)–(3.10). Hence, the approximate solution $U(x,t)$ is completely known.

To initiate the computation, we need initial vectors $\boldsymbol{c}^0$ and $\boldsymbol{v}^0$, which can be determined by using initial conditions (1.2):

$$U(x_l, 0) = \phi_1(x_l), \quad l = 0, 1, \dots, N, \tag{3.13}$$

and

$$v(x_l, 0) = \phi_2(x_l), \quad l = 0, 1, \dots, N. \tag{3.14}$$

Using (2.5) in (3.13) gives $(N+1)$ equations in $(N+1)$ unknowns, which can be written in matrix form as

$$\begin{bmatrix} 1 & 2m_1 & \dots & \dots & 0 \\ m_1 & 1 & m_1 & \dots & 0 \\ & \ddots & \ddots & \ddots & \\ & & m_1 & 1 & m_1 \\ 0 & \dots & \dots & 2m_1 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ \vdots \\ c_{N-1} \\ c_N \end{bmatrix} = \begin{bmatrix} \phi_1(x_0) + \frac{m_1}{m_2}\phi_{1x}(x_0) \\ \phi_1(x_1) \\ \vdots \\ \phi_1(x_{N-1}) \\ \phi_1(x_N) - \frac{m_1}{m_2}\phi_{1x}(x_N) \end{bmatrix}, \tag{3.15}$$

and using (3.14) gives $(N+1)$ equations in $(N+1)$ unknowns:

$$\begin{bmatrix} v_0^0 \\ v_1^0 \\ \vdots \\ v_{N-1}^0 \\ v_N^0 \end{bmatrix} = \begin{bmatrix} \phi_2(x_0) \\ \phi_2(x_1) \\ \vdots \\ \phi_2(x_{N-1}) \\ \phi_2(x_N) \end{bmatrix}. \tag{3.16}$$

4. STABILITY ANALYSIS

In this section, we discuss stability of the method discussed in the previous section by the matrix method. For studying stability we take $f(u) = 0$ and combine Eqs. (3.11) and (3.12) as

$$\mathcal{A}\dot{\boldsymbol{C}} = \mathcal{B}\boldsymbol{C} + \mathcal{F}, \tag{4.1}$$

where

$$\mathcal{A} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{P} & -2\alpha\boldsymbol{I} \end{bmatrix}, \quad \boldsymbol{C} = [c_1, \ldots, c_{N-1}, v_1, \ldots, v_{N-1}]',$$

and $\mathcal{F}$ is a known vector of order $2(N-1)$, $\boldsymbol{0}$ and $\boldsymbol{I}$ are null and unit matrices, respectively, of order $N-1$ and

$$\boldsymbol{P} = m_3\boldsymbol{P}_1 - \beta^2\boldsymbol{A},$$

where

$$\boldsymbol{P}_1 = \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}.$$

Using the expansions of $\sin h(ph)$, $\cos h(ph)$, we deduce, $0 < m_1 < \frac{1}{2}$ and $m_3 > 0$, $\forall p, h > 0$. In light of this, we see that $\boldsymbol{A}$ is a strictly diagonally dominant matrix and, hence, is invertible. So that, we have

$$\dot{\boldsymbol{C}} = (\mathcal{A}^{-1}\mathcal{B})\boldsymbol{C} + \mathcal{A}^{-1}\mathcal{F}, \tag{4.2}$$

where,

$$\mathcal{A}^{-1}\mathcal{B} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{A}^{-1} \\ \boldsymbol{P} & -2\alpha\boldsymbol{I} \end{bmatrix}.$$

For proving the stability of the system (4.1), we need to prove that the eigenvalues $\boldsymbol{\Lambda}$ of the coefficient matrix $\mathcal{A}^{-1}\mathcal{B}$ have negative real part.

Now, matrices $\boldsymbol{P}_1$ and $\boldsymbol{A}$ have the same base of eigenvectors. Matrix $\boldsymbol{A}$ is a Hermitian matrix with all diagonal entries positive and, hence, all the eigenvalues of $\boldsymbol{A}$ are real and positive. Further, matrix $\boldsymbol{P}_1$ being real symmetric negative definite matrix, has negative eigenvalues. Hence, eigenvalues of matrix $\boldsymbol{P}$ are real and negative.

Let $X = [X_1, X_2]'$ be the eigenvector corresponding to eigenvalue $\boldsymbol{\Lambda}$. Then we have

$$\begin{bmatrix} \mathbf{0} & \boldsymbol{A}^{-1} \\ \boldsymbol{P} & -2\alpha\boldsymbol{I} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \Lambda \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}. \tag{4.3}$$

From (4.3), we can write

$$\left.\begin{aligned} &\boldsymbol{A}^{-1}X_2 = \Lambda X_1, \\ &\boldsymbol{P}X_1 - 2\alpha X_2 = \Lambda X_2. \end{aligned}\right\} \tag{4.4}$$

Then we get

$$\boldsymbol{P}\boldsymbol{A}^{-1}X_2 = \Lambda(\Lambda + 2\alpha)X_2, \tag{4.5}$$

which implies that $\Lambda(\Lambda + 2\alpha)$ is an eigenvalue of $\boldsymbol{P}\boldsymbol{A}^{-1}$. Let $\boldsymbol{\Lambda} = x + iy$, where x and y are real numbers. Then, we have $(x + iy)(x + iy + 2\alpha)$ is real and negative, which provides

$$y(x + \alpha) = 0, \quad x(x + 2\alpha) - y^2 < 0.$$

From the above equations, we get the solutions as:

(i) y is arbitrary real number and $x + \alpha = 0 \Rightarrow x$ is negative real number, since α is real and positive.

(ii) $y = 0 \Rightarrow x(x + 2\alpha) < 0 \Rightarrow (x + \alpha)^2 < \alpha^2 \Rightarrow x$ is negative, since α is positive.

Hence, since the real part of eigenvalues of the coefficient matrix $\mathcal{A}^{-1}\mathcal{B}$ is negative, the proposed method is unconditionally stable.

5. NUMERICAL EXPERIMENTS

In this section, we present the numerical results of the present method when applied to a few test problems. We also compare our results with results obtained by other existing methods. The accuracy of the presented method is measured using L_∞ errors:

$$L_\infty = \|u - U\|_\infty = \max_i |u_i - U_i|,$$

where u and U represent the analytical and approximate solutions, respectively. Order of convergence of the method is obtained by using the formula

$$\frac{\log\left(\frac{e_{h1}}{e_{h2}}\right)}{\log\left(\frac{h_1}{h_2}\right)},$$

where e_{h1} and e_{h2} are L_∞ errors for grid sizes h_1 and h_2, respectively. We performed our computations using MATLAB 12 software on a laptop with Intel Pentium processor, 2.0 GHz CPU and 2 GB RAM.

Example 1. We consider the following telegraphic equation:

$$u_{tt} + 2\alpha u_t + \beta^2 u = u_{xx} + (2 - 4t + t^2 + 4\alpha t - 2\alpha t^2 + \beta^2 t^2)(x - x^2)e^{-t} + 2t^2 e^{-t}$$

Table 2. L_∞ errors, $\Delta t = 0.001$, $h = 0.01$, $\alpha = 0.5$, $\beta = 1$

t, s	Proposed method	Mittal and Bhatia [7]
1	7.6936e−06	5.9153e−05
2	2.0453e−06	1.7864e−05
3	9.3772e−06	1.4309e−05
4	2.4189e−06	1.3529e−05
5	4.8353e−06	5.2032e−06

Table 3. L_∞ errors, $\Delta t = 0.4h$ and $t = 2$ for different values of p

h	$p = 0.1$	$p = 0.5$	$p = 1$	$p = 2$	$p = 10$
1/8	1.3234e−04	9.2671e−05	3.1116e−05	5.2352e−04	1.4200e−02
1/16	3.2447e−05	2.2483e−05	8.6799e−06	1.3298e−04	4.0000e−03
1/32	7.8474e−06	5.3496e−06	2.4553e−06	3.3664e−05	1.0000e−03
1/64	1.9250e−06	1.2996e−06	6.5468e−07	8.9042e−06	2.5803e−04

subject to initial conditions:

$$u(x, 0) = 0, \quad u_t(x, 0) = 0, \quad 0 \le x \le 1,$$

and boundary conditions:

$$u(0, t) = 0, \quad u(1, t) = 0, \quad t \ge 0.$$

The analytical solution of this example is given as $u(x, t) = (x - x^2)t^2e^{-t}$. In this example, we solve a telegraphic equation for $\Delta t = 0.001$, $h = 0.01$, $\alpha = 0.5$, $\beta = 1$, $p = 2$, and the L_∞ errors are compared with the errors obtained in [7] (see Table 2). It can clearly be seen that the numerical solutions produced by our method are more accurate than [7]. Further, for $\alpha = 1$ and $\beta = 1$ we compute errors for different values of p at $t = 2$ (Table 3). We observe that the error is least when $p = 1$, however, there is no remarkable change in the order of accuracy.

Example 2. Consider the following telegraphic equation:

$$u_{tt} + 2\alpha u_t + \beta^2 u = u_{xx} + (2 - 2\alpha + \beta^2)e^{-t}\sin(x)$$

subject to initial conditions:

$$u(x, 0) = \sin(x), \quad u_t(x, 0) = -\sin(x), \quad 0 \le x \le \pi,$$

and boundary conditions:

$$u(0, t) = 0, \quad u(\pi, t) = 0, \quad t \ge 0.$$

The analytical solution of this example is given as $u(x, t) = e^{-t}\sin(x)$. L_∞ errors are tabulated in Table 4 for $h = 0.02$, $\Delta t = 0.0001$ and for $\alpha = 4$, $\beta = 2$, $p = 1$ at different time levels. The results are compared with the results obtained by Dosti and Nazemi [9]. We also compare our results with results obtained by Dosti and Nazemi in [10] for $h = 0.02$, $\Delta t = 0.001$ at different time levels (Table 5). Our results are better in comparison with the results obtained in [9, 10].

Table 4. L_∞ errors, $h = 0.02$, $\Delta t = 0.0001$ for $\alpha = 4$, $\beta = 2$

t, s	Proposed method	Dosti and Nazemi [9]
0.4	2.3010e−05	2.9000e−03
0.8	6.7857e−06	3.2000e−03
1.2	3.1884e−06	2.8000e−03
1.6	1.1679e−06	2.3000e−03
2	2.3203e−07	1.8000e−03

Table 5. L_∞ errors with $h = 0.02$ and $\Delta t = 0.001$ for $\alpha = 4$, $\beta = 2$

t, s	Proposed Method	Dosti and Nazemi [10]
0.5	9.7967e−05	1.0676e−03
1	6.8394e−05	7.1563e−04
1.5	4.6283e−05	4.8126e−04
2	3.1320e−05	2.8398e−04

Table 6. L_∞ errors, $\alpha = 1$, $\beta = 1$ for $p = 1$

h	L_∞	Order of convergence
1/8	1.1300e−02	—
1/16	3.3000e−03	1.8
1/32	9.0441e−04	1.9
1/64	2.1867e−04	2.0

Example 3. We consider the following nonlinear problem:

$$u_{tt} = u_{xx} - 2\alpha u_t - \beta^2 u - \exp(u) + \cos h(x)(\beta^2 t^2 - t^2 + 4\alpha t + 2) + \exp(t^2 \cos h(x)), \quad 0 \le x \le 1,$$

subject to initial conditions:

$$u(x, 0) = 0, \quad u_t(x, 0) = 0, \quad 0 \le x \le 1,$$

and boundary conditions:

$$u(0, t) = t^2, \quad u(1, t) = t^2 \cos h(1), \quad t \ge 0.$$

The analytical solution of this example is given as $u(x, t) = t^2 \cos h(x)$. In this example, we consider a nonlinear telegraphic equation. We compute L_∞ errors at $t = 1$ for $p = 1$, $\alpha = 1, \beta = 1$ and take $\Delta t = 0.4h$. The results are tabulated in Table 6. In Table 7, we choose $\alpha = 1$, $\beta = 0.5$ and compute errors for different values of p at $t = 1$. We observe that the accuracy is not much affected with different choices of p.

Example 4. We consider the telegraphic equation in general form [4]:

$$u_{tt} + (\alpha + \beta)u_t + \alpha\beta u = c^2 u_{xx} + (1 - \alpha - \beta + \alpha\beta - c^2)(e^{-t} \sin h(x)), \quad 0 \le x \le 1,$$

Table 7. L_∞ errors, $\alpha = 1$, $\beta = 0.5$, $t = 1$ for different values of p

h	$p = 0.1$	$p = 0.5$	$p = 1$	$p = 2$	$p = 10$
1/8	1.2300e−02	1.2300e−02	1.2300e−02	1.2400e−02	1.5900e−02
1/16	3.5000e−03	3.6000e−03	3.6000e−03	3.6000e−03	4.4000e−03
1/32	9.6083e−04	9.6113e−04	9.6208e−04	9.6589e−04	1.1000e−03
1/64	2.4922e−04	2.4927e−04	2.4940e−04	2.4995e−04	2.7713e−04

Table 8. L_∞ errors, $t = 5$

h	$\alpha = 3\pi, \beta = \pi$	$\alpha = \pi, \beta = \pi$	Order of convergence
1/16	3.0127e−05	1.6741e−05	—
1/32	9.0791e−06	4.6353e−06	1.9
1/64	2.5120e−06	1.2241e−06	1.9
1/128	6.6200e−07	3.1151e−07	2.0

Table 9. L_∞ errors, $t = 5$, $\alpha = 12$, $\beta = 6$ for $p = 0.1$

h	L_∞	Order of convergence
1/8	8.0006e−04	—
1/16	1.7892e−04	2.2
1/32	4.6840e−05	1.9
1/64	1.2211e−05	1.9

Table 10. L_∞ errors, $t = 5$, $\alpha = 10$, $\beta = 5$ and different values of p

h	$p = 0.1$	$p = 0.5$	$p = 1$	$p = 2$	$p = 10$
1/8	7.3086e−04	7.3045e−04	7.2919e−04	7.2417e−04	5.7892e−04
1/16	1.6646e−04	1.6641e−04	1.6626e−04	1.6564e−04	1.4637e−04
1/32	4.1624e−05	4.1617e−05	4.1598e−05	4.1522e−04	3.9092e−05
1/64	1.0508e−05	1.0507e−05	1.0505e−05	1.0495e−05	1.0194e−05

subject to initial conditions:

$$u(x, 0) = \sin h(x), \quad u_t(x, 0) = -\sin h(x), \quad 0 \leq x \leq 1,$$

and boundary conditions:

$$u(0, t) = 0, \quad u(1, t) = e^{-t} \sin h(1), \quad t \geq 0.$$

The analytical solution of this example is given as $u(x, t) = e^{-t} \sin h(x)$. In this problem, we choose different values of α, β and take $p, c = 1$. L_∞ errors are tabulated in Table 8 at $t = 5$ with $\Delta t = 0.4h$. We show an error plot for different grid sizes at $t = 5$ for $\alpha = \pi$, $\beta = \pi$ in Fig. 1.

Example 5. We consider the following problem:

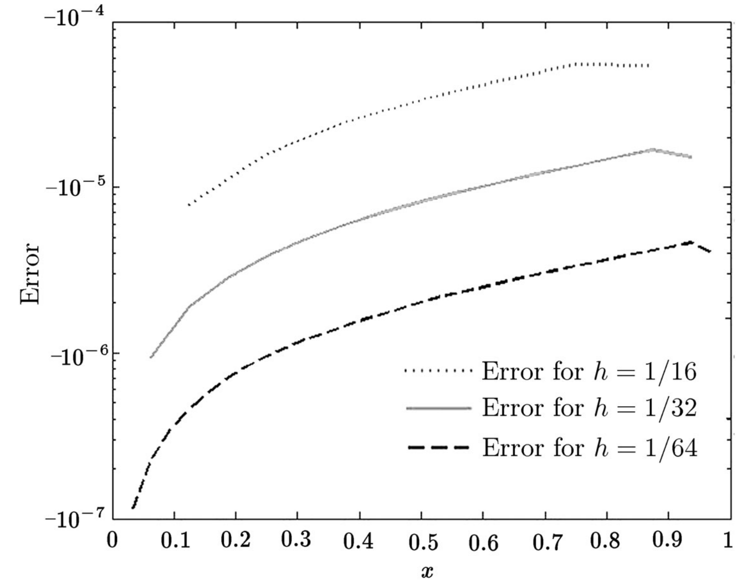

Fig. 1. Error plot for different grid sizes at $t = 5$ for $\alpha = \pi$, $\beta = \pi$.

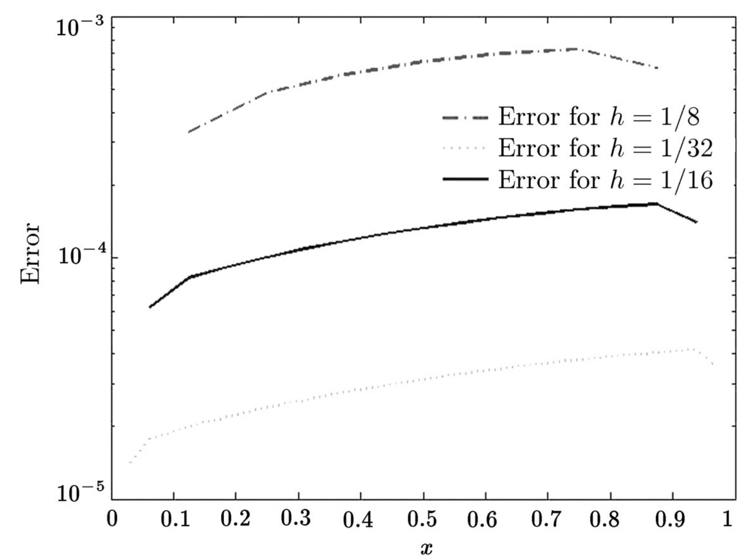

Fig. 2. Error plot for different grid sizes at $t = 5$ for $\alpha = 10$, $\beta = 5$.

$$u_{tt} = u_{xx} - 2\alpha u_t - \beta^2 u + (\beta^2 - 2\alpha)e^{x-t}, \quad 0 \le x \le 1,$$

subject to initial conditions:

$$u(x, 0) = e^x, \quad u_t(x, 0) = -e^x, \quad 0 \le x \le 1,$$

and boundary conditions:

$$u(0, t) = e^{-t}, \quad u(1, t) = e^{1-t}, \quad t \ge 0.$$

The analytical solution of this example is given as $u(x, t) = e^{x-t}$. In this problem, we obtain L_∞ errors at $t = 5$ for $p = 0.1$, $\Delta t = 0.4h$ and register them in Table 9. In Table 10, we experiment with different values of p. We observe that L_∞ errors and, hence, the order of accuracy of the method is not affected by choosing different p. We show an error plot for different grid sizes at $t = 5$ for $\alpha = 10$, $\beta = 5$ in Fig. 2.

6. CONCLUSION

In this paper, the exponential B-spline collocation method has been developed to solve an nonlinear one-dimensional hyperbolic equation of second order. The equation is first converted into a system of partial differential equations; then the exponential B-spline collocation method is applied to obtain a system of first-order ordinary differential equations, which is then solved by SSPRK(2,2) method. The method is shown to be unconditionally stable by matrix stability analysis. To show the efficiency and accuracy of the method, it is applied to a few test problems and the results are found to be better in comparison with the other known works. The proposed method is efficient and can easily be applied to solve various linear and nonlinear partial differential equations.

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