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Semi-Orthogonal Spline-Wavelets with Derivatives and the Algorithm with Splitting

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Abstract—This paper deals with the use of a scalar product with derivatives for constructing semi-orthogonal spline-wavelets. The reduction of supports of such wavelets in comparison with the classical semi-orthogonal wavelets is shown. For splines of the third degree, the algorithm of wavelet-transformation in the form of the solution to a three-diagonal system of linear equations with strict diagonal prevalence has been obtained. The results of numerical experiments on the calculation of derivatives of a discretely given function are presented.

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1. INTRODUCTION

A wavelet is a short or rapidly decaying wave function (splash) whose set of compressions and displacements generates a space of measurable functions on the entire number axis [1, 2]. Owing to compression, wavelets are capable of identifying the difference in the characteristics of the measured signal with different degrees of detail. Owing to displacement, they can analyze the signal properties at various points on the entire examined interval. In the analysis of unsteady signals, the property of wavelet locality ensures a significant advantage over the Fourier transform, which provides only the global information about the properties of the examined signal because the basis functions used thereby (sines and cosines) have infinite supports. As wavelets transform the system of basis functions with distributed parameters to a system with lumped parameters, such a basis is more effective for solving problems of numerical analysis from the viewpoint of conditionality and convergence.

The basis for wavelet construction is the availability of a set of approximating spaces $\cdots V_{L-1} \subset V_L \subset V_{L+1} \cdots$ such that each basis function in V_{L-1} can be expressed as a linear combination of the basis functions in V_L. In particular, such properties are typical for splines, which are smooth functions glued from pieces of polynomials of degree m on a sequence of nested grids. The classical semi-orthogonal wavelets [1] are defined as elements of the space V_L that are orthogonal to the space V_{L-1}. A typical property of semi-orthogonal wavelets, which is sometimes used [3] as a basis for the numerical method of constructing the wavelet-transform, is the fact that the wavelet-decomposition provides the best root-mean-square approximation of splines on a fine grid by means of splines on a sparse grid. This property ensures an advantage in solving the problem of compression of discrete numerical information. However, this advantage is leveled off during differentiation of the resultant spline-wavelet decomposition. Some progress in solving this problem was achieved by constructing spline-wavelets with the increased number of moments equal to zero [4–8] under the condition of increasing the wavelet supports. In our opinion, an optimal compromise between the accuracy of calculating the derivatives and the support length is ensured by the spline-wavelets of the third degree $m = 3$ studied in this paper, which are semi-orthogonal to derivatives of the second order. These wavelets have the following specific feature: owing to their construction, they inherit the property of the best root-mean-square approximation of the second derivatives [9, p. 175] of interpolation splines and, correspondingly, ensure the best root-mean-square deviation of the second derivatives of splines on the fine grid by means of the

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second derivatives of splines on the sparse grid. For example, in the case of splines of the first degree on the segment $[a, b]$ with a uniform grid composed of the nodes $\Delta^L : x_i = a + h\,i,\ i = 0, 1, \ldots, 2^L$, $h = (b-a)/2^L$, and the basis functions $N_i^L(x) = \varphi_1(v - i)\ \forall\, i$, where $v = (x-a)/h$, with the centers at integer numbers, which are generated by compressions and displacements of the function

$$\varphi_1(t) = \begin{cases} 1+t, & -1 \le t \le 0, \\ 1-t, & 0 \le t \le 1, \\ 0, & t \notin [-1, 1], \end{cases}$$

the use of a wavelet with a reduced (as compared to the classical wavelet of the first degree) support $[0, 2] \subset [0, 3]$ [10]:

$$w_1(t) = \varphi_1(2t-1) - \varphi_1(2t-3)$$

leads to semi-orthogonality with respect to the metrics with the first derivatives because

$$\int\limits_{-\infty}^{\infty} w_1'(x-l)\varphi_1'(x-k)\,dx = 0\ \forall\ l, k.$$

Though these wavelets are orthogonal only to constants in the usual sense,

$$\int\limits_{-\infty}^{\infty} w_1(x-l)\,dx = 0, \qquad \int\limits_{-\infty}^{\infty} w_1(x-l)x\,dx \ne 0 \quad \forall\, l,$$

which does not ensure the closeness of discretely given functions to the root-mean-square approximation, nevertheless, fairly acceptable results for the problem of approximating the first derivative are obtained (see Fig. 1).

In the case of cubic splines, wavelets of the third degree $\psi(x-i)\ \forall\, i$ were found [11, 12], for which the conditions of orthogonality to the corresponding basis splines $\varphi(x-j)\ \forall\, j$ with respect to the scalar product with the second derivatives

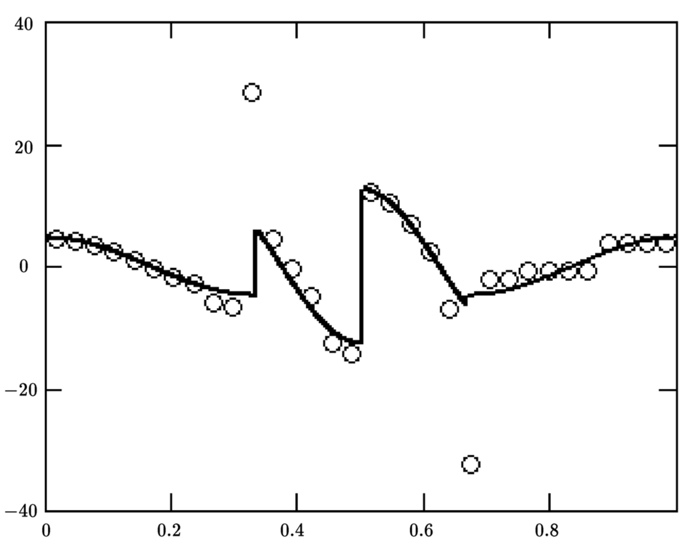

Fig. 1. Wavelet-reconstruction of the first derivative of the spline of the first degree.

$$\int_{-\infty}^{\infty} \psi''(x-i)\varphi''(x-j)\,dx = 0 \;\forall\; i,j$$

are satisfied. It turned out that these wavelets have a very simple structure, in particular, the support is smaller than the supports of the classical semi-orthogonal spline-wavelets of the third degree, namely, $[0,3] \subset [0,7]$. Moreover, they were recognized as being useful for solving differential equations [13] and were numerically implemented as a standard code [14] in the MatLab system. A similar solution [15] was found for the case of cubic Hermite splines-wavelets that are semi-orthogonal with respect to the scalar product with the first derivatives. Moreover, an original method was proposed by the author of the present paper in [16] for even-odd splitting of the system of equations of the wavelet-transform [15] into a parallel solution of two three-diagonal systems of linear equations of a twice smaller order with strict diagonal prevalence. The wavelet-transforms based on the Hermite splines have also some drawbacks: in the problem of processing the measured information, it is necessary first to calculate approximate values of the derivatives at the nodes of the finest grid with acceptable accuracy [17], and only after that can the wavelet-transform algorithms be applied. From the viewpoint of data compression, the number of wavelet-coefficients in this case is much greater than that in methods based on B-splines. Therefore, in Section 2.2, we consider a pioneering idea of using even-odd splitting in the case of the wavelet-transform of usual cubic splines.

It should be noted that even-odd splitting of the wavelet-transform matrix was used in [18] for other wavelets to prove the matrix invertibility; however, there was no clear indication that it can be used for computations in practice.

2. CONSTRUCTION OF SPLINE-WAVELETS OF THE THIRD DEGREE THAT ARE SEMI-ORTHOGONAL TO THE SECOND DERIVATIVES

Let the space V_L be a space of cubic splines of smoothness C^2 on a grid composed of nodes Δ^L, and let the basis functions $N_i^L(x) = \varphi_3(v-i)\;\forall i$ be generated by compressions and displacements of a function of the form [19, p. 23]:

$$\varphi_3(t) = \frac{1}{6}\sum_{j=0}^{4}\binom{4}{j}(-1)^j(t-j)_+^3,$$

where $t_+^n = (\max\{t,0\})^n$. Then these functions satisfy the calibration relation [1, p. 154]:

$$\varphi_3(t) = \frac{1}{8}\sum_{k=0}^{4}\binom{4}{k}\varphi_3(2t-k). \tag{1}$$

To facilitate the construction of wavelets near the ends of a finite segment, we impose the following additional conditions on the functions: $f(a) = f'(a) = f(b) = f'(b) = 0$. The corresponding left-end basis function has the form [11, 12]:

$$\varphi_b(t) = \frac{3}{2}t_+^2 - \frac{11}{12}t_+^3 + \frac{3}{2}(t-1)_+^3 - \frac{3}{4}(t-2)_+^3,$$

and satisfies the calibration relation

$$\varphi_b(t) = \frac{1}{4}\varphi_b(2t) + \frac{11}{16}\varphi_3(2t) + \frac{1}{2}\varphi_3(2t-1) + \frac{1}{8}\varphi_3(2t-2). \tag{2}$$

On any grid Δ^L, $L \geq 2$, a spline of the third degree with zero boundary conditions can be presented as

$$S^L(x) = C_{-1}\varphi_b(v) + \sum_{i=0}^{2^L-4} C_i\varphi_3(v-i) + C_{2^L-3}\varphi_b(2^L - v), \quad a \le x \le b, \tag{3}$$

where the coefficients $C_i \; \forall\, i$ are a solution of, e.g., the following interpolation problem:

$$S^L(x_i) = f(x_i), \quad i = 1, 2, \ldots, 2^L - 1.$$

If the grid Δ^{L-1}, $L \ge 1$, is obtained from Δ^L by means of removing each second node, then the corresponding space V_{L-1} with the basis functions $N_i^{L-1}(x)$ whose supports are twice greater in terms of width and whose centers are at even nodes of the grid Δ^L is nested in V_L. The essence of the wavelet-transform can be formulated as follows: it allows the given function to be hierarchically expanded into a series of rough approximate presentations V_{L-1} and local refining details $W_{L-1} = V_L - V_{L-1}$. Cubic wavelets that are semi-orthogonal to the second derivatives with reduced supports have the following form [11, 12]:

$$\begin{aligned} w_3(t) &= -\frac{3}{7}\varphi_3(2t) + \frac{12}{7}\varphi_3(2t-1) - \frac{3}{7}\varphi_3(2t-2), \\ w_b(t) &= \frac{24}{13}\varphi_b(2t) - \frac{6}{13}\varphi_3(2t). \end{aligned} \tag{4}$$

They satisfy the semi-orthogonality condition

$$\int_0^{2^L} w_3''(x-l)\varphi_3''(x-k)\,dx = \int_0^2 w_b''(x)\varphi_3''(x-k)\,dx = 0, \quad l, k = 0, 1, \ldots, 2^L - 2,$$

$$\int_0^3 w_3''(x-l)\varphi_b''(x)\,dx = \int_0^2 w_b''(x)\varphi_b''(x)\,dx = 0, \quad l = 0, 1, \ldots, 2^L - 2;$$

moreover, the condition of complementarity of dimensions of the resultant spaces $\mathrm{Dim}(V_L) = \mathrm{Dim}(V_{L-1}) + \mathrm{Dim}(W_{L-1})$ is satisfied. These wavelets can be used for solving equations that contain the second derivative of the sought function by the Galerkin method because they approximate the second derivative with the second-order error

$$\int_0^3 w_3''(x)x^m dx = \int_0^2 w_b''(x)x^m dx = 0, \quad m = 0, 1.$$

2.1. Construction of the Constitutive System of Equations

For further considerations, it is convenient to write the basis spline-functions in the form of a single matrix-row $\varphi^L(\cdot) = \left[\varphi_b(\cdot), \varphi_3(\cdot), \varphi_3(\cdot - 1), \ldots, \varphi_3(\cdot - 2^L + 4), \varphi_b(2^L - \cdot)\right]$ and to order the spline coefficients in the form of a vector $C^L = \left[C_{-1}, C_0, \ldots, C_{2^L-3}\right]^\top$. Then Eq. (3) is rewritten as $S^L(x) = \varphi^L(v)C^L$, where $v = (x-a)/h$. In a similar way, we can write the basis wavelet-functions in the form of a matrix-row as

$$\psi^L(\cdot) = \left[w_b(\cdot), w_3(\cdot), w_3(\cdot - 1), \ldots, w_3(\cdot - 2^L + 3), w_b(2^L - \cdot)\right].$$

The corresponding coefficients of the wavelet decomposition at the level L are collected into a vector $D^L = \left[D_{-1}, D_0, \ldots, D_{2^L-2}\right]^\top$. Then, for the decomposition level $L-1$, the functions $\varphi^{L-1}(\cdot)$ and $\psi^{L-1}(\cdot)$ can be written in the form of linear combinations of the functions $\varphi^L(\cdot)$:

$$\varphi^{L-1}(\cdot) = \varphi^L(\cdot)P^L \text{ and } \psi^{L-1}(\cdot) = \varphi^L(\cdot)Q^L,$$

where the blocks of the matrix P^L are composed from the coefficients of relations (1) and (2) because each wide basis function inside the approximation segment can be constructed from five narrow basis functions, each wide basis function at the ends of the interval can be constructed from four narrow basis functions, and the elements of the columns of the matrix Q^L can be constructed from the coefficients of relations (4).

Therefore, the following equalities are valid:

$$\varphi^L(\cdot)C^L = \varphi^{L-1}(\cdot)C^{L-1} + \psi^{L-1}(\cdot)D^{L-1} = \varphi^L(\cdot)P^L C^{L-1} + \varphi^L(\cdot)Q^L D^{L-1}. \tag{5}$$

Thus, the process of obtaining C^L from C^{L-1} and D^{L-1} can be written as

$$C^L = P^L C^{L-1} + Q^L D^{L-1}$$

or, by using the notation for block matrices,

$$C^L = \left[P^L \mid Q^L\right] \left[\frac{C^{L-1}}{D^{L-1}}\right]. \tag{6}$$

The following example shows how it is possible to obtain three basis spline-functions from V_2 and four basis wavelets from W_2 by using seven basis functions from V_3:

$$\left[P^2 \mid Q^2\right] = \left[\begin{array}{ccc|cccc}
\frac{1}{4} & 0 & 0 & \frac{24}{13} & 0 & 0 & 0 \\
\frac{11}{16} & \frac{1}{8} & 0 & -\frac{6}{13} & -\frac{3}{7} & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{12}{7} & 0 & 0 \\
\frac{1}{8} & \frac{3}{4} & \frac{1}{8} & 0 & -\frac{3}{7} & -\frac{3}{7} & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{12}{7} & 0 \\
0 & \frac{1}{8} & \frac{11}{16} & 0 & 0 & -\frac{3}{7} & -\frac{6}{13} \\
0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{24}{13}
\end{array}\right].$$

The reverse process of splitting the coefficients C^L into a rough version C^{L-1} and refining coefficients D^{L-1} implies solving a system of linear equations (6). The solvability of the resultant system is guaranteed by the linear independence of the basis functions. To facilitate the numerical solution of the system of linear equations (6), following [2], we can convert the matrix $\left[P^L \mid Q^L\right]$ to a five-diagonal form by changing the order of unknowns so that the columns of the matrices P^L and Q^L become alternated. However, as is seen from the above-presented example, the derived system of equations has no diagonal prevalence, which can make the wavelet-analysis of large-size data rather difficult.

2.2. Algorithm with Splitting

We propose to perform calculations on the basis of our previously developed procedure [16, 20] of even-odd splitting of wavelet-expansions of the form (5) relating the basis functions of the space of splines on the fine grid, the basis functions on the sparse grid, and the wavelets by finite implicit relations of decomposition. In particular, the following new matrix equality was obtained in [16] for the case of cubic Hermite splines-wavelets [15] by the method of undetermined coefficients:

$$\left[P^L \mid Q^L\right] R^L = G^L. \tag{7}$$

Here the matrix R^L is composed of two simple band matrices, and matrix G^L is a band matrix with five nonzero diagonals. After that, the solution of the system of equations of the form (6) can be written in the matrix form as follows [16]:

$$\left[\frac{C^{L-1}}{D^{L-1}}\right] = \left[P^L \mid Q^L\right]^{-1} C^L = R^L \left(G^L\right)^{-1} C^L. \tag{8}$$

Moreover, after splitting of system (8) over even and odd nodes, the algorithm reduces to solving independently two three-diagonal systems of equations with strict diagonal prevalence, which is preferable from the viewpoint of parallelization and stability of computations.

Similar equalities are also valid for the above-described type of wavelets, which can easily be verified by means of direct calculations, e.g.,

$$\left[P^3 \mid Q^3\right]^{-1} \cdot \left[\begin{array}{cccccccccccc}
60 & -48 & 12 & 0 & & & & & & & & \\
0 & 12 & 0 & 0 & 0 & & & & & & & \\
24 & 0 & 96 & 24 & 24 & 0 & & & & & & \\
0 & 0 & 0 & 12 & 0 & 0 & 0 & & & & & \\
0 & 0 & 24 & 24 & 96 & 24 & 24 & 0 & & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\
 & & & & & & 0 & 24 & 24 & 96 & 0 & 24 \\
 & & & & & & & 0 & 0 & 0 & 12 & 0 \\
 & & & & & & & & 0 & 12 & -48 & 60
\end{array}\right]$$

$$= \left[\begin{array}{cccccccccccc}
24 & 0 & 24 & 0 & 0 & 0 & & & & & & \\
0 & 0 & 24 & 24 & 24 & 0 & 0 & & & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\
 & & & & & & 0 & 0 & 0 & 24 & 0 & 24 \\
\hline
\frac{117}{4} & -26 & \frac{13}{4} & 0 & 0 & 0 & & & & & & \\
7 & 0 & 42 & 7 & 7 & 0 & 0 & & & & & \\
0 & 0 & 7 & 7 & 42 & 7 & 7 & 0 & & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\
 & & & & & & 0 & 7 & 7 & 42 & 0 & 7 \\
 & & & & & & & 0 & 0 & \frac{13}{4} & -26 & \frac{117}{4}
\end{array}\right].$$

Here square 15×15 matrices are located both on the left and on the right, and the right-hand matrix is split into 7×15 and 8×15 blocks. The empty positions of the matrices are zero elements. The dots arranged over the diagonal mean that the rows are repeated the corresponding number of times, being displaced each time by two positions to the right.

The following statement describes the sequence of calculating the wavelet-coefficients on the basis of the known coefficients of the spline-decomposition on an arbitrary grid Δ^L, $L \geq 3$.

Theorem. *Let the values of the coefficients C_i at odd nodes be recalculated from the solution of a three-diagonal system of linear equations of the form*:

$$\begin{bmatrix} 5 & 1 & & & & \\ 2 & 8 & 2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & & 2 & 8 & 2 \\ & & & & 1 & 5 \end{bmatrix} \begin{bmatrix} C_{-1} \\ C_1 \\ \vdots \\ C_{2^L-3} \end{bmatrix} := \begin{bmatrix} C_{-1} + 4C_0 \\ C_1 - 2C_2 \\ C_3 - 2C_2 - 2C_4 \\ \vdots \\ C_{2^L-5} - 2C_{2^L-6} \\ C_{2^L-3} + 4C_{2^L-4} \end{bmatrix}. \tag{9}$$

As it is usually done in algorithmic languages, the symbol " := " is used here to indicate that the left-hand part of the system contains new values of the coefficients C_i, whereas the right-hand part contains their known values.

Then the vector of the spline-coefficients of size $\left(2^{L-1} - 1\right)$ on the sparse grid Δ^{L-1} is the result of multiplication of the matrix of size $\left(2^{L-1} - 1\right) \times \left(2^L - 1\right)$:

$$\begin{bmatrix} 2 & 0 & 2 & & & & & \\ & & 2 & 2 & 2 & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & 2 & 2 & 2 & \\ & & & & & 2 & 0 & 2 \end{bmatrix}$$

by the modified vector of the spline-coefficients of size $\left(2^L - 1\right)$ at the nodes of the fine grid Δ^L, whereas the vector of the wavelet-coefficients of size 2^{L-1} is equal to the same product with the matrix of size $2^{L-1} \times \left(2^L - 1\right)$:

$$\frac{1}{12} \begin{bmatrix} \frac{117}{4} & -26 & \frac{13}{4} & & & & & & & \\ 7 & 0 & 42 & 7 & 7 & & & & & \\ & & 7 & 7 & 42 & 7 & 7 & & & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & & 7 & 7 & 42 & 0 & 7 \\ & & & & & & \frac{13}{4} & -26 & \frac{117}{4} \end{bmatrix}.$$

The dots arranged along the diagonal mean that the previous row is repeated the corresponding number of times, being shifted by two positions to the right.

Proof. The theorem is proved by direct verification of the scheme of splitting (7), (8). For example, let us multiply the second row of the matrix $\left[P^L \mid Q^L\right]$ by the first three columns of the matrix R^L composed of the above-constructed matrices:

$$\frac{11}{16}\cdot 2-\frac{6}{13}\cdot\frac{117}{4\cdot 12}-\frac{3}{7}\cdot\frac{7}{12}=0,\qquad \frac{6}{13}\cdot\frac{26}{12}=1,\qquad \frac{11}{16}\cdot 2-\frac{6}{13}\cdot\frac{13}{4\cdot 12}-\frac{3}{7}\cdot\frac{42}{12}=0.$$

These equalities mean that the spline coefficient C_0 remains unchanged in the course of the transformation $G^LC^L := C^L$ with the matrix G^L obtained by formula (7). The same manipulations with the first row of the matrix $\left[P^L \mid Q^L\right]$ yield the values

$$\frac{1}{4}\cdot 2+\frac{24}{13}\cdot\frac{117}{4\cdot 12}=5,\qquad -\frac{24}{13}\cdot\frac{26}{12}=-4,\qquad \frac{1}{4}\cdot 2+\frac{24}{13}\cdot\frac{13}{4\cdot 12}=1.$$

Moving the unchanged value of C_0 to the right-hand side of the resultant equation, we obtain the first equation of system (9) with respect to the changing odd coefficients of the spline on the fine grid. Similar manipulations with the fourth row of the matrix $\left[P^L \mid Q^L\right]$ ensure justification of the fact that the coefficient of the spline C_2 also remains unchanged in the course of modification with the matrix G^L. The same manipulations with the third row of the matrix $\left[P^L \mid Q^L\right]$ yield the values

$$\frac{1}{2}\cdot 2+\frac{12}{7}\cdot\frac{7}{12}=2,\quad 0,\quad \frac{1}{2}\cdot 2+\frac{1}{2}\cdot 2+\frac{12}{7}\cdot\frac{42}{12}=8,\quad \frac{1}{2}\cdot 2+\frac{12}{7}\cdot\frac{7}{12}=2,\quad \frac{1}{2}\cdot 2+\frac{12}{7}\cdot\frac{7}{12}=2.$$

Moving the unchanged value of C_2 to the right-hand side of the resultant equation, we obtain the second equation of system (9) with respect to the changing odd coefficients of the spline on the fine grid, etc. Thus, indeed, the theorem conditions can be written in matrix form as Eq. (8). Therefore, they yield the solution of system (6). □

The number of arithmetic operations required for solving system (9) by the sweeping method is $3\cdot(2^{L-1}-1)$ additions, $3\cdot(2^{L-1}-1)$ multiplications, and $2\cdot(2^{L-1}-1)+1$ divisions [19, p. 337]. Calculating the right-hand sides of the equations requires $2\cdot(2^{L-1}-2)$ "short" displacement multiplications and $2\cdot(2^{L-1}-2)$ additions; obtaining the spline-coefficients at the nodes of the sparse grid requires $2\cdot(2^{L-1}-1)$ additions. The most computationally expensive part of the algorithm is the calculation of the wavelet-coefficients: $5\cdot(2^{L-1}-4)+14$ multiplications and $4\cdot(2^{L-1}-4)+10$ additions. If we make no differences between the arithmetic operations, then the total number of such operations for one step of the wavelet decomposition is $23\cdot 2^{L-1}-29$. Taking into account that $L=3$ at the last stage of decimation, we obtain the number of arithmetic operations for calculating the total set of the wavelet coefficients: $23\cdot 2^L-29L-34$. As compared to the earlier known fast algorithm [11–13] of the discrete wavelet transform based on solving a sequence of interpolation problems, this algorithm allows the wavelet decomposition coefficients to be obtained in a different way with a comparable number of operations. The advantages of the new algorithm are its stability and possibility of parallelization because one system of linear equations (instead of two systems in the known algorithm) with a matrix possessing strict diagonal prevalence is solved at each step.

3. EXAMPLES

Let us consider the test function in the form of the Harten function [17] defined on the segment $[0,1]$:

$$f(x)=\begin{cases}\dfrac{1}{2}\sin(3\pi x), & x\le\dfrac{1}{3},\\ \left|\sin(4\pi x)\right|, & \dfrac{1}{3}\le x\le\dfrac{2}{3},\\ -\dfrac{1}{2}\sin(3\pi x), & x>\dfrac{2}{3}.\end{cases}$$

This is a piecewise-smooth function equal to zero at the points $x = 0$ and 1. It may have discontinuities of the first kind at the points $x = 1/3$ and $2/3$ and an inflection (discontinuity of the first derivative) at the point $x = 1/2$. Its first and second derivatives are also piecewise-smooth functions. Let us make an attempt to calculate the second derivative of the Harten function by using the spline-wavelets of the third degree studied in this paper that are semi-orthogonal to the second derivatives.

3.1. Example of Spline-Wavelets of the First Degree That are Semi-Orthogonal to the First Derivatives

Let us assume that

$$\psi_1^L(x) = \frac{1}{2} 2^{-L/2} w_1(2^L x + 1),$$

$$\psi_i^L(x) = \frac{\sqrt{2}}{4} 2^{-L/2} w_1(2^L x + 2 - i), \quad i = 2, 3, \dots, 2^L.$$

Then $\psi_i^L(x)$ are normalized so that $\left\| \frac{d}{dx} \psi_i^L(x) \right\|_{L_2(0,1)} = 1$ for $i = 1, 2, \dots, 2^L$. Based on the consideration of the complementarity of dimensions, we remove the basis functions at the last node from the spaces of splines and wavelets. This procedure is supported by the property of setting the approximated function to zero at the right end of the interval. A different situation was discussed in [21].

Starting from the upper level of the resolution $L = 5$, i.e., at the number of grid steps $2^L = 32$, on the interval $0 \le x \le 1$ with a step length $h = 2^{-L} = 0.031$, we find the coefficients of the normalized wavelet-basis for

$$\begin{aligned} L &= 5 : D^4 = [-8.373, -11.77, -11.58, -11.34, -11.13, -11.01, -6.767, -5.971, -5.642, \\ &\qquad -5.312, -4.516, 0.8311, 0.7162, 0.5014, 0.259, 0.07071]^\top; \\ L &= 4 : D^3 = [3.314, 5.061, 5.722, 1.088, 2.745, 4.402, 1.036, 0.3745]^\top; \\ L &= 3 : D^2 = [-8, -9.701, -5.044, 1.613]^\top; \\ L &= 2 : D^1 = [3.14 \cdot 10^{-16}, 1.414]^\top; \end{aligned}$$

$L = 1$: at the last (roughest) level, there remains only one wavelet-coefficient $D^0 = -4.898 \cdot 10^{-16}$ and one spline decomposition coefficient $C^0 = 0$ at the left end of the interval.

The circles in Fig. 1 show the results of the reconstruction of the first derivative of the spline of the first degree $\left(S^5\right)'(x_i + h/2) = (C_{i+1}^5 - C_i^5)/h$ under the condition of setting ten wavelet-coefficients with the absolute values smaller than 1.414 to zero. The solid curve is the first derivative of the original function. In this case, the compression coefficient $K = 32/22 \approx 1.455$ is reached. The finite-difference approximation of the first derivative, which is not shown in the figure, coincides with the curve on smooth segments and is little different from the circles at the points where the function is discontinuous.

3.2. Example of Spline-Wavelets of the Third Degree That are Semi-Orthogonal to the Second Derivatives

In the case of cubic splines, the normalized wavelets have the form

$$\psi_1^L(x) = 0.083 \cdot 8^{-L/2} w_b(2^L x), \qquad \psi_{2^L}^L(x) = 0.083 \cdot 8^{-L/2} w_b(1 - 2^L x),$$

$$\psi_i^L(x) = 0.097 \cdot 8^{-L/2} w_3(2^L x + 1 - i), \quad i = 2, 3, \dots, 2^L - 1,$$

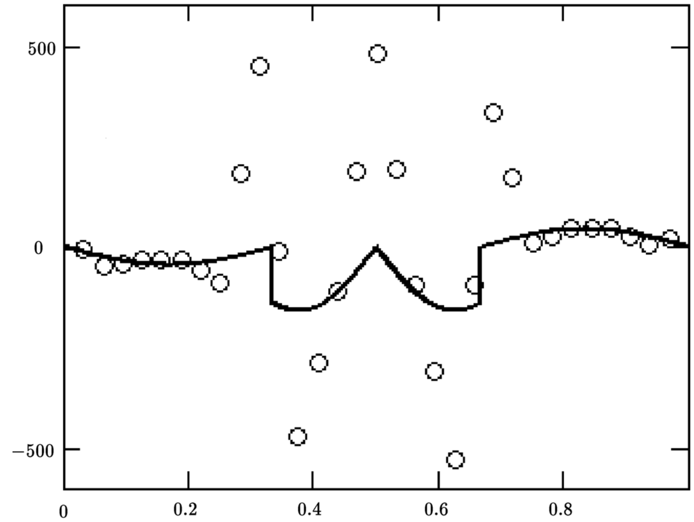

Fig. 2. Wavelet-reconstruction of the second derivative of the spline of the third degree.

where $\left\| \dfrac{d^2}{dx^2} \psi_i^L(x) \right\|_{L_2(0,1)} = 1$ for $i = 1, 2, \ldots, 2^L$.

As the first derivative of the Harten function at the points $x = 0$ and 1 is not equal to zero, we subtract the values of the cubic interpolation polynomial $f'(0)x(1-x)^2 - f'(1)(1-x)x^2$ with subsequent addition of this polynomial to the wavelet-synthesis results. Instead of the original coefficients of decomposition in the basis of B-splines, we use the function values that are little different from them. This trick is very popular in the literature all over the world; it is called a "Wavelet Crime" [22, 23].

Starting from the upper level of the resolution $L = 5$, we find the coefficients of the normalized wavelet-basis for

$L = 5:\ D^4 = [0.8956, -0.4546, 1.546, -5.021, 19.17, 115.5, -37.04, 36.8, 38.49, -42.12, 134.1, 9.012, -2.529, 0.4743, -0.07706, -0.7798]^\top$;

$L = 4:\ D^3 = [-50, 248.7, -1151, 387, 234.4, -915.1, 201.3, -54.04]^\top$;

$L = 3$: at the roughest level of decomposition, there remain four wavelet-coefficients $D^2 = [1849, 6881, 1.009 \cdot 10^4, -184.5]^\top$ and three spline-coefficients $C^2 = [-239.1, 242.9, 100.3]^\top$.

Figure 2 shows the results of the reconstruction of the second derivative of the spline of the third degree $\left(S^5\right)''(x_i) = (C_{i+1}^5 - 2C_i^5 + C_{i-1}^5)/h^2$ under the condition of setting 19 wavelet-coefficients with the absolute values smaller than 185 to zero. In this case, the compression coefficient $K = 31/12 \approx 2.583$ is reached. The circles clearly demonstrate the alternance behavior of the second derivative of the spline, which is similar on segments where the function is smooth to the behavior of the broken line of the best root-mean-square approximation of the second derivative (solid curve) with inflections at spline nodes that remain after reconstruction. The finite-difference approximation of the second derivative, which is not shown in the figure, is little different from the curve, demonstrating splashes up to $\pm 10^3$ at points where the function and its first derivative are discontinuous.

4. CONCLUSIONS

A pioneering application of the author's procedure of even-odd splitting of the constitutive systems of wavelet decompositions to the basis of B-splines is considered. The procedure for the case of the Hermite wavelets is adjusted for approximation of functions that do not require setting the values of the derivatives, which is important for practice. Examples and calculations demonstrating the root-mean-square approximation of the second derivative of the given function with the use of the second derivative of the spline on a sparse grid are presented. The advantage of the proposed procedure over other methods of calculating derivatives is the possibility of adaptive selection of the nodes of the approximating spline on the basis of the coefficients of its wavelet decomposition. Extension of the proposed method to splines of higher degrees and smoothness may offer new possibilities of developing stable algorithms of spline-wavelets construction and application.

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REFERENCES

1. Chui, C.K., *An Introduction to Wavelets*, Academic Press, 1992.
2. Stollnitz, E.J., DeRose, T.D., and Salesin, D.H., *Wavelets for Computer Graphics, The Morgan Kaufmann Series in Computer Graphics*, 1996.
3. Lyche, T., Mørken, K., and Quak, E., Theory and Algorithms for Non-Uniform Spline Wavelets, in *Multivariate Approximation and Applications*, Dyn, N., Leviatan, D., Levin, D., and Pinkus, A., Eds., Cambridge: Cambridge University Press, 2001, pp. 152–187.
4. Cohen, A., Daubechies, I., and Feauveau, J.C., Biorthogonal Bases of Compactly Supported Wavelets, *Comm. Pure Appl. Math.*, 1992, vol. 45, pp. 485–560.
5. Koro, K. and Abe, K., Non-Orthogonal Spline Wavelets for Boundary Element Analysis, *Engin. An. Bound. El.*, 2001, vol. 25, pp. 149–164.
6. Bittner, K., A New View on Biorthogonal Spline Wavelets, Preprint of Ulm Universität, Ser.: 2005-03, Ulm, 2005.
7. Ĉerná, D. and Finêk, V., Construction of Optimally Conditioned Cubic Spline Wavelets on the Interval, *Adv. Comp. Math.*, 2011, vol. 34, pp. 219–252.
8. Ĉerná, D. and Finêk, V., Cubic Spline Wavelets with Complementary Boundary Conditions, *Appl. Math. Comput.*, 2012, vol. 219, pp. 1853–1865.
9. De Boor, C., *A Practical Guide to Splines*, Springer, 1978.
10. Cohen, A. and Daubechies, I., A New Technique to Estimate the Regularity of Refinable Functions, *Rev. Math. Iberoam.*, 1996, vol. 12, iss. 2, pp. 527–591.
11. Wang, J., Cubic Spline Wavelet Bases of Sobolev Spaces and Multilevel Interpolation, *Appl. Comput. Harm. An.*, 1996, vol. 3, iss. 2, pp. 154–163.
12. Cai, W. and Wang, J., Adaptive Multiresolution Collocation Methods for Initial Boundary Value Problems of Nonlinear PDEs, *SIAM J. Num. An.*, 1996, vol. 33, iss. 3, pp. 937–970.
13. Kumar, V. and Mehra, M., Cubic Spline Adaptive Wavelet Scheme to Solve Singularly Perturbed Reaction Diffusion Problems, *Int. J. Wavelets, Multiresol. Inform. Proc.*, 2007, vol. 5, iss. 2, pp. 317–331.
14. Mehra, M. and Goyal, K., Algorithm 929: A Suite on Wavelet Differentiation Algorithms, *ACM Transact. Math. Soft.*, 2013, vol. 39, iss. 4, art. no. 27.
15. Jia, R.Q. and Liu, S.T., Wavelet Bases of Hermite Cubic Splines on the Interval, *Adv. Comput. Math.*, 2006, vol. 25, pp. 23–39.
16. Shumilov, B.M., Algorithm with Splitting for the Wavelet-Transform of the Cubic Hermite Splines, *Vest. Tomsk Gos. Univ., Ser. Mat. Mekh.*, 2010, no. 4, pp. 45–55.
17. Arandiga, F., Baeza, A., and Donat, R., Discrete Multiresolution Based on Hermite Interpolation: Computing Derivatives, *Comm. Nonlin. Sci. Num. Simul.*, 2004, vol. 9, pp. 263–273.
18. Ĉerná, D. and Finêk, V., Cubic Spline Wavelets with Short Support for Fourth-Order Problems, *Appl. Math. Comput.*, 2014, vol. 243, pp. 44–56.
19. Zav'yalov, Yu.S., Kvasov, B.I., and Miroshnichenko, V.L., *Metody splain-funktsii* (Methods of Spline Functions), Moscow: Nauka, 1980.
20. Shumilov, B.M., Cubic Multiwavelets Orthogonal to Polynomials and Algorithm with Splitting, *Sib. Zh. Vych. Mat.*, 2013, vol. 16, no. 3, pp. 283–297.
21. Shumilov, B.M., Algorithms with Splitting for the Wavelet Transform of Splines of the First Degree on Nonuniform Grids, *Zh. Vych. Mat. Mat. Fiz.*, 2016, vol. 56, no. 7, pp. 39–50.
22. Bultheel, A. and Huybrechs, D., Wavelets with Applications in Signal and Image Processing, 2011; http://people.cs.kuleuven.be/adhemar.bultheel/WWW/WAVE/wavelets2011.pdf (ch. 6, p. 6.5: The wavelet crime).
23. Strang, G. and Nguyen, T., *Wavelets and Filter Banks*, Cambridge: Wellesley-Cambridge, 1996.